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LINEAR THEORY OF SHELLS

Parts I and II

K. F. Chernykh

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FROM THE AUTHOR

This book, presented to the reader, considers the problem of the linear theory of thin shells. The initial fundamental works associated with the names of Aron, Basset and Love, pertain to the end of the last century. Practical requirements (associated basically with the necessity of designing thin-walled domes) led to further development of the theory during the period 1910-1925. The works published during this period are characterized by a clear understanding of the special features associated with thin shells. There is a clear separation of the state of stress int the membrane effect and the edge effect. During this period some methods were proposed and applied to the solution of specific problems. The work of Love (ref. 25), which sums up the work carried out during this period, has not lost its significance, even today.

In view of the rapid progress in engineering, the theory of shells was developed with particular intensity during the period 1935-1948. New general approaches to the solution of an ever-increasing class of problems have been sought. Close association has been established with the theory of elasticity and with structural mechanics. The mathematical aspects of the problem have been studied in detail. A great contribution to the development of the theory of shells has been made by Soviet specialists in mechanics, among whom we should mention, first of all, V. Z. Vlasov, A. L. Gol'denveyzer, A. I. Lur'ye, Kh. M. Mushtari and V. V. Novozhilov. The basic results obtained during this period of time are presented in a series of monographs (refs. 3, 11, 23 and 31).

The linear theory of shells presented in this book has a series of special features:

1. The relationships of the theory of shells are initially established from the equations of the three-dimensional theory of elasticity, referred to a triorthogonal system of coordinates fixed with respect to the middle surface of the shell. The basic presentation is carried out using the lines of principal curvatures.

2. A complex notation has been adopted. At the present time the V. V. Novozhilov complex method is becoming more and more popular, both in our country and abroad (after the publication of ref. 60). This concept is developed further in the present book. A wider circle of questions and problems is considered, revealing the advantages of the complex method.

3. The membrane theory is also presented in a special manner. In the first part, the membrane solution is considered as a particular solution of the general moment problem. Specific examples are used to illustrate how the limited arbitrariness associated with the membrane solution together with the realization of suitable boundary conditions can be used (in a series of cases)

to achieve a state of stress which is close to that in a membrane. A systematic presentation of the problems associated with the membrane theory is carried out at the end of the second part. This makes it possible to develop the necessary apparatus and to deal with a reader who is already prepared.

We shall pause briefly to consider the contents of the first part. The first chapter is concerned with the presentation of problems associated with the geometry of the shell.

The second chapter considers the deformations of the shell. By analyzing the expressions of the theory of elasticity for strain components and angular displacements, it is shown that the geometric hypothesis of Kirchhoff follows, in a natural way, from the properties of thin-walled and flexible shells. In conjunction with the usual strain components four quantities are introduced which characterize the elongation and flexure of the boundary element. The displacement vector is expressed in terms of these quantities (as in the three-dimensional theory of elasticity) by means of one curvilinear integral. In addition, these quantities make it possible to introduce the concept of deformation boundary conditions, which are formulated in terms of the conventional strain components.

In the third chapter the principle of possible displacements is used to derive equilibrium equations and boundary conditions in terms of forces and moments.

The fourth chapter is the principal chapter of the first part. It presents a generalization for the nonhomogeneous problem of the Lur'ye-Gol'denveyzer stress functions. It is shown that the stress components of the boundary element introduced in the second chapter correspond, according to the static-geometric analogy, to the edge forces and moments. In this way a final form is given to the concept of static-geometric analogy, which establishes a correspondence between the geometric and static relationships of the theory of shells. Expressions are obtained for the vector of the stress functions in terms of the boundary forces and moments. For shells whose middle surface is a multiconnected region, the ambiguity of the stress functions is associated (as in the case of the two-dimensional problem) with the lack of self-balancing of edge loading along specific contours. Subsequently the concept of complex analogy is introduced. According to this concept, corresponding complex relationships can be associated with each static and geometric quantity (relationship). Various complex relationships are introduced and simplified. The V. V. Novozhilov version is examined in detail. An even simpler version is proposed. The conditions for complex junction are introduced, which make it possible to combine the solutions for two shells of equal thickness when these are joined (continuously or at an angle) without the preliminary separation of the real and imaginary parts.

Chapter V presents the theory which has been developed for shells of revolution. The new feature here is the further development of the case when loading is antisymmetric (wind loading). Equations are obtained which are convenient for practical application. Compliance coefficients are

introduced. Specific forms of shells of revolution are considered: cylindrical, conic, plates. Numerous examples are presented which are both of theoretical and practical interest.

Chapter VI considers cylindrical shells of the general form. A more compact solution is given for an oval cylindrical shell. The asymptotic investigation of the basic complex equation is carried out in a simpler and more illustrative manner than in analogous investigations utilizing real equations.

A list of references is presented which gives more information on the problems presented in the first part. In selecting the references, the author has made an effort to specify those which present the problems covered in the book in more detail or from a different point of view. Furthermore, it was the author's desire to help the reader who is not too well acquainted with the existing literature by selecting the particular literature pertinent to the specific problems.

Thus, the first part presents the traditional problems of the theory, augmented by the results obtained by the author.

The second part, which is being prepared for publication,[1] considers more specialized problems of the linear theory of shells: concentrated forces and discontinuous solutions, toroidal shells, stress concentration near holes, ambiguous displacements, thermal stresses, the principles of the theory of shells of medium thickness and several other problems. The second volume also gives a short historical survey relative to the development of the theory of thin shells and discusses its present state. Finally, a rather detailed list of references is presented, relative to the specific problems considered.

The author wishes to express his gratitude to V. A. Gavrilov, V. Ya. Pavilaynen, V. V. Ponyatovskiy and particularly to the editor V. A. Shamina, co-workers in the laboratory for the theory of shells of the Institute of Mathematics and Mechanics of the Leningrad University, for the assistance in preparing the manuscript for publication.

The book was prepared taking into account the critical remarks and advice of my teacher V. V. Novozhilov. It would be difficult to overemphasize the assistance which he has rendered me.

[1] See Part II.

INTRODUCTION

In the theory of elasticity a shell is a body one of whose dimensions (thickness) is small compared with the other two dimensions. We shall be limited to shells which are made up of a homogeneous isotropic material that follows Hooke's law. In addition, we shall assume that the displacements of points on the shell are small compared with the thickness of the shell. These limitations make it possible to develop a simple version of the linear theory of shells, which is appropriately called the classical theory.

Of course, when we do this it is impossible to consider a whole series of practical problems concerning the stability of a shell, its carrying capacity, etc. However, these problems differ substantially from the problems of the linear (classical) theory of shells, both in their formulation and in the methods of solution. They are considered in special monographs.

The great practical value of the linear theory is explained by the fact that in many problems the limitations stipulated above are not significant.

The light weight of shells together with their high strength makes them irreplaceable in the most diverse branches of technology: aircraft construction, shipbuilding, chemical machine construction, erection of structures, etc. The wide application of shells in practice has stimulated the development of sufficiently simple and effective methods for designing them. Two basic approaches to the development of the theory of shells are known.

In the first approach the shell is considered to be a three-dimensional elastic body. The solution of corresponding equations of the theory of elasticity are sought by expanding all of the quantities into power series of the distance between a point on the shell and the middle surface (or by using some system involving the functions of this variable). Aside from its awkwardness, this method which is accurate in principle has one other shortcoming: to date the scope of its applicability has not been determined nor has the convergence of the utilized series been established.

The second approach which is associated with considerable approximation involves the adoption of certain static-geometric hypotheses (propositions), which reduce the three-dimensional problem to a simpler problem involving the equilibrium and deformation of the middle shell surface.

The simplest version of the theory is obtained if we adopt the Kirchhoff hypothesis in which case:

(a) in considering the deformation picture of the shell we can assume that the fibers of the shell which are perpendicular to the middle surface

before deformation remain rectilinear and perpendicular to the deformed surface after deformation and retain their original length;

(b) the normal stresses parallel to the middle surface may be neglected compared with similar stresses in areas perpendicular to the mean surface.

The approximate theory formulated on the basis of Kirchhoff's hypothesis is called the theory of thin shells. The theories which utilize the first of the two approaches are called the theories of thick shells. The same shell can be analyzed using the theory of thick shells and less accurately, but in a much simpler manner, by using the Kirchhoff theory. The results obtained by the theory of thin shells will become more accurate as the relative thickness of the shell decreases (the ratio h/R where R is the minimum linear length of the middle surface, while h is the thickness of the shell).

At the present time it is well established that the Kirchhoff theory produces results in which the order of magnitude of the error does not exceed the ratio h/R. Proceeding from this and assigning the permissible value of error, it is possible to establish a quantitative criterion for the thinness of walls. Thus V. V. Novozhilov, proceeding from the error commonly encountered in engineering calculations, has proposed that a shell be considered thin, if $h/R < 1/20$. This number is, of course, approximate. The difficulties associated with the utilization of the theory of thick shells necessitate the application (particularly in the design of turbines) of the Kirchhoff theory, even for large values of the relative thickness $h/R = 1/5$ to $1/3$.

It should be pointed out that at the present time there is an intense development of theories of shells with medium thickness for such shells. These theories are also approximate; however, in their formulation the Kirchhoff hypotheses are replaced by certain more general propositions. Below we shall examine the theory of Kirchhoff. In this connection, in line with the accuracy of the initial equations, quantities of the order of h/R with respect to unity will be neglected in the relationships. In addition, while seeking the approximate solutions of various equations, we shall not strive to achieve high accuracy.

Furthermore, the book will not consider such problems as the allowable values of stresses and strains, the required factors of safety, etc. This is associated with the fact that problems of this type are considered in a special manner in each branch of engineering, based on accumulated experience and on standard practices. In the course of this, the most diverse factors are taken into account: the fabrication technology, the properties of materials which are used, requirements for the safety and reliability of a structure, the accuracy with which the external load parameters are determined, the statistical analysis of operation, failure of similar structures and many other factors. An attempt to consider these questions would substantially sidetrack our effort.

The book utilizes a binary numbering of the equations. The first number indicates the number of the section in the chapter, while the second number gives the number of the equation in the section. When references are made to

equations from other chapters a Roman numeral is added which specifies the number of the chapter. The list of references serves as an indication of works which consider the general problems of designing shells of a specific type. References which consider special problems (stress concentration, toroidal shells, temperature stresses, etc.) are included in the list of references provided at the end of the second part of the book.

CHAPTER I. SHELL GEOMETRY

From the geometric point of view a shell is determined by its bounding faces and, if it is not closed, also by its lateral surfaces. A surface which is at an equal distance from the faces is called the middle surface. Let us construct a perpendicular at an arbitrary point on the middle surface. The length of the segment between the faces determines the thickness of the shell (h). In the general case the thickness of the shell is a variable quantity. In developing a general theory we shall consider a class of shells with constant thickness, since it is of the greatest practical value. If a shell is not closed, we shall assume that its lateral surface may be formed by displacing the perpendicular to the mean surface along some contour which bounds part of the middle surface. /11*

Thus the geometry of a shell is determined entirely by assigning a mean surface, a bounding contour and thickness. Following the accepted procedure we shall consider a shell by utilizing an orthogonal system of coordinates, fixed with respect to the middle surface of the shell. We begin our presentation by considering the geometry of the middle surface.

1. Curvilinear Coordinates Fixed With Respect to the Surface

In a Cartesian system of coordinates a surface of smooth form may be specified by means of three equations

$$x = x(\alpha, \beta), \quad y = y(\alpha, \beta), \quad z = z(\alpha, \beta), \tag{1.1}$$

in which x (α, β), y (α, β) and z (α, β) are continuous functions of two parameters α and β with the required number of derivatives.

We limit the region where these parameters vary in such a way that each point on the surface corresponds to one completely determined pair of values (α, β). Then α and β may be called the curvilinear coordinates of the surface in question. /12

By assigning a series of constant values to the coordinate β in (1.1) and varying the coordinate α, we obtain a family of curves which we shall call the coordinate α-lines. In a same manner we can obtain the β-lines.

*Numbers given in margin indicate pagination in original foreign text.

The three scalar equations (1.1) may be replaced by one vector equation

$$\vec{r} = \vec{r}(\alpha, \beta). \tag{1.2}$$

The vectors $\frac{\partial \vec{r}}{\partial \alpha}$, $\frac{\partial \vec{r}}{\partial \beta}$ are directed, respectively, along the α- and β-lines (fig. 1). By introducing the designations

$$\left|\frac{\partial \vec{r}}{\partial \alpha}\right| = A \text{ and } \left|\frac{\partial \vec{r}}{\partial \beta}\right| = B$$
$$\left(A = \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2}, \quad B = \sqrt{\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right) + \frac{\partial z}{\partial \beta}}\right), \tag{1.3}$$

we obtain the following equations for unit vectors which are tangent to the coordinate lines

$$\vec{e}_\alpha = \frac{1}{A}\frac{\partial \vec{r}}{\partial \alpha}, \quad \vec{e}_\beta = \frac{1}{B}\frac{\partial \vec{r}}{\partial \beta}. \tag{1.4}$$

For the third vector we select a unit vector $\vec{n}$ which is normal to the surface and orthogonal with respect to the first two vectors.

For two adjoining points on the surface having coordinates (α, β) and $(\alpha + d\alpha, \beta + d\beta)$ the vector which connects them is written in the following manner

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \alpha} d\alpha + \frac{\partial \vec{r}}{\partial \beta} d\beta. \tag{1.5}$$

Inasmuch as

$$\frac{\partial \vec{r}}{\partial \alpha} \cdot \frac{\partial \vec{r}}{\partial \beta} = AB(\vec{e}_\alpha \cdot \vec{e}_\beta) = AB\cos\chi,$$

(see 1.4), it follows that

$$ds^2 = |d\vec{r}|^2 = A^2 d\alpha^2 + 2AB\cos\chi\, d\alpha\, d\beta + B^2 d\beta^2. \tag{1.6}$$

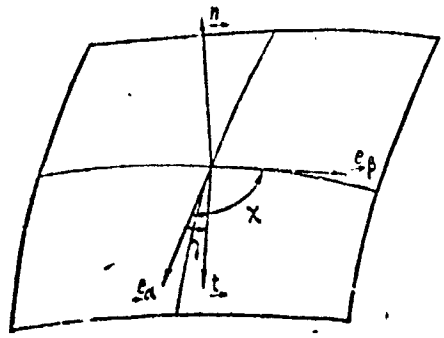

Figure 1

In the above equation the symbol χ designates the angle between the coordinate lines (fig. 1). In the future we shall consider exclusively the orthogonal system of coordinates for which $\cos\chi = 0$ $\left(\chi = \frac{\pi}{2}\right)$. /13

The expression which is contained in the right side of equality (1.6) is called the first quadratic form of the surface. It is proved in courses in differential geometry (see, for example ref. 17) that the first quadratic form does not change when the surface is subjected to flexure without extension.

Assuming successively that $d\beta = 0$ and $d\alpha = 0$, we obtain the equalities

$$ds_\alpha = Ad\alpha, \quad ds_\beta = Bd\beta, \tag{1.7}$$

which establish the geometric meaning of the quantities A and B known as the Lamé parameters. The Lamé parameters are scale factors which relate the increments in the arcs of the coordinate lines to the increments of the corresponding curvilinear coordinates.

From equation (1.5) it follows that the unit vector tangent to the curve Γ lying on the surface, is equal to

$$\vec{t} = \frac{d\vec{r}}{ds} = \frac{\partial\vec{r}}{\partial\alpha}\frac{d\alpha}{ds} + \frac{\partial\vec{r}}{\partial\beta}\frac{d\beta}{ds}. \tag{1.8}$$

The well-known Frenet theorem (ref. 17) relates the unit vector of the principal normal considered on the surface of the curve ($\vec{m}$), its spatial curvature ($1/\rho$) and the unit vector of the tangent ($\vec{t}$) by means of the following differential expression

$$\frac{\vec{m}}{\rho} = \frac{d\vec{t}}{ds} = \frac{\partial^2\vec{r}}{\partial\alpha^2}\left(\frac{d\alpha}{ds}\right)^2 + 2\frac{\partial^2\vec{r}}{\partial\alpha\partial\beta}\frac{d\alpha}{ds}\frac{d\beta}{ds} + \frac{\partial^2\vec{r}}{\partial\beta^2}\left(\frac{d\beta}{ds}\right)^2 + \frac{\partial\vec{r}}{\partial\alpha}\frac{d^2\alpha}{ds^2} + \frac{\partial\vec{r}}{\partial\beta}\frac{d^2\beta}{ds^2}. \tag{1.9}$$

Taking the scalar product of the left and right sides and $\vec{n}$, we obtain (fig. 2)

$$\frac{\cos\varphi}{\rho} = \frac{b_{\alpha\alpha}d\alpha^2 + 2b_{\alpha\beta}d\alpha d\beta + b_{\beta\beta}d\beta^2}{ds^2}, \tag{1.10}$$

where

$$\begin{aligned} b_{\alpha\alpha} &= \frac{\partial^2\vec{r}}{\partial\alpha^2}\cdot\vec{n} = A^2\frac{\partial^2\vec{r}}{A\partial\alpha A\partial\alpha}\cdot\vec{n}, \\ b_{\beta\beta} &= \frac{\partial^2\vec{r}}{\partial\beta^2}\cdot\vec{n} = B^2\frac{\partial^2\vec{r}}{B\partial\beta B\partial\beta}\cdot\vec{n}, \\ b_{\alpha\beta} &= \frac{\partial^2\vec{r}}{\partial\alpha\partial\beta}\cdot\vec{n} = AB\frac{\partial^2\vec{r}}{A\partial\alpha B\partial\beta}\cdot\vec{n} = AB\frac{\partial^2\vec{r}}{B\partial\beta A\partial\alpha}\cdot\vec{n} = \frac{\partial^2\vec{r}}{\partial\beta\partial\alpha}\cdot\vec{n} = b_{\beta\alpha}. \end{aligned} \tag{1.11}$$

In deriving the last equalities we utilized the obvious conditions of orthogonality /14

$$\frac{\partial\vec{r}}{\partial\alpha}\cdot\vec{n} = \frac{\partial\vec{r}}{\partial\beta}\cdot\vec{n} = 0. \tag{1.12}$$

We introduce the designation

$$\frac{\cos\varphi}{\rho} = -\frac{1}{R}. \tag{1.13}$$

Then equality (1.10) may be written in the following form

$$-\frac{1}{R} = \frac{b_{\alpha\alpha}d\alpha^2 + 2b_{\alpha\beta}d\alpha d\beta + b_{\beta\beta}d\beta^2}{ds^2}. \tag{1.14}$$

The quantity 1/R is called the normal curvature of the surface in a given direction determined by the quantities ($d\alpha$, $d\beta$). From (1.13) and figure 2 it is easy to see that the normal curvature is the projection of the spatial curvature vector of the curve $\frac{\vec{m}}{\rho}$ along the negative direction of the normal to the surface at the point in question. The quantity R, on the other hand, is the radius vector of the plane curve formed by the intersection of the surface by a plane passing through the normal to the surface in the given direction. The expression which is in the numerator of the first part of relationship

(1.14) is called the second quadratic form of the surface. Unlike the first quadratic form it varies substantially when the surface is subjected to flexure. Thus the ratio of the first to the second quadratic form determines the normal curvature of the curved line on the surface.

If we limit ourselves to systems with orthogonal coordinates, we can always select a direction for measuring the coordinates such that the unit orthogonal vectors $\{\vec{e}_\alpha, \vec{e}_\beta, \vec{n}\}$ form a right hand coordinate system. In this case the following equalities will be valid

$$[\vec{e}_\alpha, \vec{e}_\beta] = \vec{n}, \quad [\vec{e}_\beta, \vec{n}] = \vec{e}_\alpha, \quad [\vec{n}, \vec{e}_\alpha] = \vec{e}_\beta. \tag{1.15}$$

Here and below the symbol $[\vec{a}, \vec{b}]$ will be used to designate the vector product of vectors $\vec{a}$ and $\vec{b}$.

Let us designate by γ' the angle between $\vec{e}_\alpha$ and $\vec{t}$, measured in the positive direction. We can see from figure 3 that

$$\frac{Ad\alpha}{ds} = \frac{ds_\alpha}{ds} = \cos\gamma', \quad \frac{Bd\beta}{ds} = \frac{ds_\beta}{ds} = \sin\gamma'. \tag{1.16}$$

Furthermore, by utilizing the obvious identity

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial\vec{r}}{\partial\alpha}\cdot\vec{n}\right) = 0 = \frac{\partial^2\vec{r}}{\partial\alpha^2}\cdot\vec{n} + \frac{\partial\vec{r}}{\partial\alpha}\cdot\frac{\partial\vec{n}}{\partial\alpha} \tag{1.17}$$

and analogous identities, expressions (1.11) may be rewritten in the following form /15

$$\begin{aligned} b_{\alpha\alpha} &= -\frac{\partial\vec{r}}{\partial\alpha}\cdot\frac{\partial\vec{n}}{\partial\alpha} = -A\frac{\partial\vec{n}}{\partial\alpha}\cdot\vec{e}_\alpha, \\ b_{\beta\beta} &= -\frac{\partial\vec{r}}{\partial\beta}\cdot\frac{\partial\vec{n}}{\partial\beta} = -B\frac{\partial\vec{n}}{\partial\beta}\cdot\vec{e}_\beta, \\ b_{\alpha\beta} &= b_{\beta\alpha} = -A\frac{\partial\vec{n}}{\partial\beta}\cdot\vec{e}_\alpha = -B\frac{\partial\vec{n}}{\partial\alpha}\cdot\vec{e}_\beta. \end{aligned} \tag{1.18}$$

By using the resulting expressions, equality (1.14) may be written in the form

$$-\frac{1}{R} = \frac{b_{\alpha\alpha}}{A^2}\cos^2\gamma' + \frac{b_{\alpha\beta}}{AB}2\sin\gamma'\cos\gamma' + \frac{b_{\beta\beta}}{B^2}\sin^2\gamma'. \tag{1.19}$$

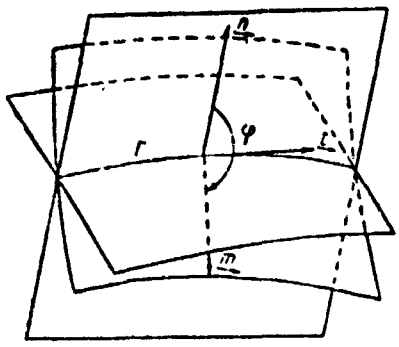

Figure 2

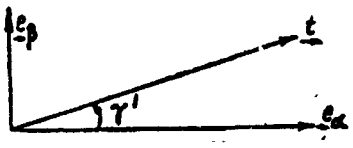

Figure 3

Assuming successively that $\gamma' = 0$ and $\gamma' = \frac{\pi}{2}$ we find

$$b_{\alpha\alpha} = -\frac{A^2}{R_\alpha}, \quad b_{\beta\beta} = -\frac{B^2}{R_\beta}. \qquad (1.20)_{1,2}$$

Further, by introducing the designation

$$b_{\alpha\beta} = \frac{AB}{R_{\alpha\beta}}, \qquad (1.20)_3$$

we rewrite equality (1.19)

$$\frac{1}{R} = \frac{\cos^2\gamma'}{R_\alpha} - \frac{2\sin\gamma'\cos\gamma'}{R_{\alpha\beta}} + \frac{\sin^2\gamma'}{R_\beta}. \qquad (1.21)$$

This shows, in particular, that the quantities $1/R_\alpha$ and $1/R_\beta$ are the normal curvatures of the surface in the directions of the coordinate lines.

Now, by using expressions (1.18) and (1.20) we can write

$$\frac{1}{R_\alpha} = -\frac{\partial^2 \vec{r}}{A\partial\alpha\, A\partial\alpha}\cdot\vec{n} = \frac{1}{A}\frac{\partial \vec{n}}{\partial\alpha}\cdot\vec{e}_\alpha,$$
$$\frac{1}{R_\beta} = -\frac{\partial^2 \vec{r}}{B\partial\beta\, B\partial\beta}\cdot\vec{n} = \frac{1}{B}\frac{\partial \vec{n}}{\partial\beta}\cdot\vec{e}_\beta, \qquad (1.22)$$
$$\frac{1}{R_{\alpha\beta}} = \frac{\partial^2 \vec{r}}{A\partial\alpha\, B\partial\beta}\cdot\vec{n} = \frac{\partial^2 \vec{r}}{B\partial\beta\, A\partial\alpha}\cdot\vec{n} = -\frac{1}{B}\frac{\partial \vec{n}}{\partial\beta}\cdot\vec{e}_\alpha = -\frac{1}{A}\frac{\partial \vec{n}}{\partial\alpha}\cdot\vec{e}_\beta.$$

Equation (1.21) determines the normal curvature of any curved line or the surface passing through the point in question. Let us determine the directions for which the normal curvature takes on an extreme value. For this purpose it is obviously necessary to find the extreme value of the function

$$f(\gamma') = \frac{\cos^2\gamma'}{R_\alpha} - \frac{2\sin\gamma'\cos\gamma'}{R_{\alpha\beta}} + \frac{\sin^2\gamma'}{R_\beta}.$$

By setting the first derivative of this function equal to zero, we obtain /16

$$\frac{df(\gamma')}{d\gamma'} = -2\sin\gamma'\cos\gamma'\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right) - 2(\cos^2\gamma' - \sin^2\gamma')\frac{1}{R_{\alpha\beta}} = 0, \qquad (1.23)$$

from which

$$\operatorname{tg} 2\gamma' = \frac{-\frac{2}{R_{\alpha\beta}}}{\frac{1}{R_\alpha} - \frac{1}{R_\beta}}. \qquad (1.24)$$

In the general case the last equation determines two mutually orthogonal directions ($d\alpha_1$, $d\alpha_2$). We shall show that for the established extreme directions

$$\frac{1}{R_{\alpha_1\alpha_2}} = 0.$$

Indeed, by replacing successively the differentiation with respect to α_1 and α_2 by differentiation with respect to α and β we find

$$\frac{1}{R_{\alpha_1\alpha_2}} = \frac{\partial^2 \vec{r}}{A_1\partial\alpha_1 A_2\partial\alpha_2}\cdot\vec{n} = \left[\frac{A d\alpha}{A_1 d\alpha_1}\frac{A d\alpha}{A_2 d\alpha_2}\frac{\partial^2 \vec{r}}{A\partial\alpha\, A\partial\alpha} + \frac{A d\alpha}{A_2 d\alpha_2}\frac{B d\beta}{A_1 d\alpha_1}\frac{\partial^2 \vec{r}}{B\partial\beta\, A\partial\alpha} + \right.$$

$$+\frac{Ad\alpha}{A_1 d\alpha_1}\frac{Bd\beta}{A_2 d\alpha_2}\frac{\partial^2\vec{r}}{\partial\alpha B\partial\beta}+\frac{Bd\beta}{A_1 d\alpha_1}\frac{Bd\beta}{A_2 d\alpha_2}\frac{\partial^2\vec{r}}{B\partial\beta B\partial\beta}+$$
$$+\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{Ad\alpha}{A_2 d\alpha_2}\right)\frac{1}{A}\frac{\partial\vec{r}}{\partial\alpha}+\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{Bd\beta}{A_2 d\alpha_2}\right)\frac{1}{B}\frac{\partial\vec{r}}{\partial\beta}\Bigg]\cdot\vec{n}.$$

In view of equality (1.12), the last two terms are equal to zero. By replacing, in the remaining terms, the ratios of the arcs with the sines and cosines of the angle (see (1.16)), we obtain the equality

$$\frac{1}{R_{\alpha_1\alpha_2}}=\sin\gamma'\cos\gamma'\left(\frac{1}{R_\alpha}-\frac{1}{R_\beta}\right)+(\cos^2\gamma'-\sin^2\gamma')\frac{1}{R_{\alpha\beta}}.$$

However, this expression is equal to zero in view of (1.23). Thus we have shown that the extreme directions of the quantity $\frac{1}{R_{\alpha_1\alpha_2}}$ are equal to zero. The directions which have been established are called the principal directions, while the quantities $R_1 = R_{\alpha_1}$ and $R_2 = R_{\alpha_2}$ are the respective principal radii of curvature. The line, whose tangents at each of its points coincide with the principal direction, is called the line of principal curvature, or more briefly, the line of curvature. In the subsequent chapters we shall consider almost exclusively the coordinate meshes formed by the curvature lines. In the course of this, while converting from an arbitrary orthogonal system of coordinates to the curvature lines, we shall replace the signs α, β by α_1, α_2 (above we have already carried out this replacement).

Expressed in terms of the curvature lines, relationship (1.21) assumes /17 a simpler form

$$\frac{1}{R}=\frac{\cos^2\gamma'}{R_1}+\frac{\sin^2\gamma'}{R_2}. \tag{1.25}$$

In the curvilinear system of coordinates the orthogonal unit vectors $(\vec{e}_\alpha, \vec{e}_\beta, \vec{n})$ cannot be considered as constant quantities. To obtain the various differential relationships, it is necessary to establish a rule for the differentiation of unit vectors. First, we have the obvious relationships (in view of the orthogonality and unit lengths of the orthogonal unit vectors)

$$\vec{e}_\alpha\cdot\vec{e}_\beta=\vec{e}_\alpha\cdot\vec{n}=\vec{e}_\beta\cdot\vec{n}=0,\quad \frac{\partial\vec{e}_\alpha}{\partial\alpha}\cdot\vec{e}_\alpha=\frac{1}{2}\frac{\partial\left(\vec{e}_\alpha^2\right)}{\partial\alpha}=\frac{1}{2}\frac{\partial(1)}{\partial\alpha}=0 \tag{1.26}$$

and the analogous expressions

$$\frac{\partial \vec{e}_\alpha}{\partial \beta} \cdot \vec{e}_\alpha = \frac{\partial \vec{e}_\beta}{\partial \alpha} \cdot \vec{e}_\beta = \frac{\partial \vec{e}_\beta}{\partial \beta} \cdot \vec{e}_\beta = \frac{\partial \vec{n}}{\partial \alpha} \cdot \vec{n} = \frac{\partial \vec{n}}{\partial \beta} \cdot \vec{n} = 0. \tag{1.27}$$

Further, by writing the obvious identity

$$\frac{\partial^2 \vec{r}}{\partial \beta \partial \alpha} = \frac{\partial^2 \vec{r}}{\partial \alpha \partial \beta}$$

and using equality (1.4) in the form

$$\frac{\partial A \vec{e}_\alpha}{\partial \beta} = \frac{\partial B \vec{e}_\beta}{\partial \alpha},$$

we obtain, by expanding the latter,

$$A \frac{\partial \vec{e}_\alpha}{\partial \beta} + \frac{\partial A}{\partial \beta} \vec{e}_\alpha = B \frac{\partial \vec{e}_\beta}{\partial \alpha} + \frac{\partial B}{\partial \alpha} \vec{e}_\beta.$$

By taking the scalar product of the last expression and $\vec{e}_\alpha$ and $\vec{e}_\beta$, we obtain the relationships

$$\frac{\partial \vec{e}_\beta}{\partial \alpha} \cdot \vec{e}_\alpha = \frac{1}{B} \frac{\partial A}{\partial \beta}, \quad \frac{\partial \vec{e}_\alpha}{\partial \beta} \cdot \vec{e}_\beta = \frac{1}{A} \frac{\partial B}{\partial \alpha}.$$

It follows from this that

$$\frac{\partial \vec{e}_\alpha}{\partial \alpha} \cdot \vec{e}_\beta = \frac{\partial (\vec{e}_\alpha \cdot \vec{e}_\beta)}{\partial \alpha} - \frac{\partial \vec{e}_\beta}{\partial \alpha} \cdot \vec{e}_\alpha = - \frac{\partial \vec{e}_\beta}{\partial \alpha} \cdot \vec{e}_\alpha = - \frac{1}{B} \frac{\partial A}{\partial \beta},$$

In a similar manner

$$\frac{\partial \vec{e}_\beta}{\partial \beta} \cdot \vec{e}_\alpha = - \frac{1}{A} \frac{\partial B}{\partial \alpha}.$$

Relationships (1.22) lead to the equalities

$$\frac{\partial \vec{n}}{\partial \alpha}\cdot\vec{e}_\alpha = \frac{A}{R_\alpha}, \quad \frac{\partial \vec{n}}{\partial \alpha}\cdot\vec{e}_\beta = -\frac{A}{R_{\alpha\beta}}, \quad \frac{\partial \vec{n}}{\partial \beta}\cdot\vec{e}_\alpha = -\frac{B}{R_{\alpha\beta}},$$
$$\frac{\partial \vec{n}}{\partial \beta}\cdot\vec{e}_\beta = \frac{B}{R_\beta}.$$

Thus we have obtained expressions for the projections of all the derivatives of the orthogonal unit vectors along both coordinates. Now it becomes easy to determine the derivatives themselves and to obtain a table of differentiation for the orthogonal unit vectors which makes it possible to differentiate the required vectors. Although we shall not make use of this table below, it will prove to be useful in the second part of the book where we shall consider systems of coordinates which have a more general form /18

$$\begin{aligned}
&\frac{1}{A}\frac{\partial \vec{e}_\alpha}{\partial \alpha} = -\frac{1}{AB}\frac{\partial A}{\partial \beta}\vec{e}_\beta - \frac{1}{R_\alpha}\vec{n}, && \frac{1}{B}\frac{\partial \vec{e}_\alpha}{\partial \beta} = \frac{1}{AB}\frac{\partial B}{\partial \alpha}\vec{e}_\beta + \frac{1}{R_{\alpha\beta}}\vec{n},\\
&\frac{1}{A}\frac{\partial \vec{e}_\beta}{\partial \alpha} = \frac{1}{AB}\frac{\partial A}{\partial \beta}\vec{e}_\alpha + \frac{1}{R_{\alpha\beta}}\vec{n}, && \frac{1}{B}\frac{\partial \vec{e}_\beta}{\partial \beta} = -\frac{1}{AB}\frac{\partial B}{\partial \alpha}\vec{e}_\alpha - \frac{1}{R_\beta}\vec{n},\\
&\frac{1}{A}\frac{\partial \vec{n}}{\partial \alpha} = \frac{1}{R_\alpha}\vec{e}_\alpha - \frac{1}{R_{\alpha\beta}}\vec{e}_\beta, && \frac{1}{B}\frac{\partial \vec{n}}{\partial \beta} = -\frac{1}{R_{\alpha\beta}}\vec{e}_\alpha + \frac{1}{R_\beta}\vec{e}_\beta.
\end{aligned} \tag{1.28}$$

Along the curvature lines the rules for differentiation have a simpler form ($\vec{e}_{\alpha_1} = \vec{e}_1$, $\vec{e}_{\alpha_2} = \vec{e}_2$)

$$\begin{aligned}
&\frac{1}{A_1}\frac{\partial \vec{e}_1}{\partial \alpha_1} = -\frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\vec{e}_2 - \frac{1}{R_1}\vec{n}, && \frac{1}{A_2}\frac{\partial \vec{e}_1}{\partial \alpha_2} = \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\vec{e}_2,\\
&\frac{1}{A_1}\frac{\partial \vec{e}_2}{\partial \alpha_1} = \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\vec{e}_1, && \frac{1}{A_2}\frac{\partial \vec{e}_2}{\partial \alpha_2} = -\frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\vec{e}_1 - \frac{1}{R_2}\vec{n},\\
&\frac{1}{A_1}\frac{\partial \vec{n}}{\partial \alpha_1} = \frac{1}{R_1}\vec{e}_1, && \frac{1}{A_2}\frac{\partial \vec{n}}{\partial \alpha_2} = \frac{1}{R_2}\vec{e}_2.
\end{aligned} \tag{1.29}$$

A comparison of the third and sixth relationships in (1.28) and (1.29) leads to an interesting property of the curvature lines: the derivative with respect to the unit vector of the normal to the surface along the arc of the curvature line coincides in direction with the tangent to the line. This property is known in the theory of surfaces as the Rodrigues theorem. It provides a simple geometric method of finding the curvature lines on a surface.

The quantities A_1, A_2, R_1, R_2 cannot be assigned as arbitrary functions of the surface point. They must satisfy equalities which are known as the Codazzi-Gauss relationships in the theory of surfaces. The latter may be obtained by utilizing the obvious identities

$$\frac{\partial^2 \vec{e}_\alpha}{\partial\alpha\partial\beta} = \frac{\partial^2 \vec{e}_\alpha}{\partial\beta\partial\alpha}, \quad \frac{\partial^2 \vec{e}_\beta}{\partial\alpha\partial\beta} = \frac{\partial^2 \vec{e}_\beta}{\partial\beta\partial\alpha}, \quad \frac{\partial^2 \vec{n}}{\partial\alpha\partial\beta} = \frac{\partial^2 \vec{n}}{\partial\beta\partial\alpha}. \tag{1.30}$$

For this purpose it is necessary to utilize twice the rule for differentiating the orthogonal unit vectors (1.29), eliminating from the last equality the derivatives with respect to unit vectors. If this is carried out, the results give us the Gauss relationship

$$\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right) = -\frac{A_1A_2}{R_1R_2} \tag{1.31}_1$$

and two Codazzi relationships

$$\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right) = \frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}, \quad \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right) = \frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}. \tag{1.31}_{2,3}$$

In courses on the theory of surfaces (for example, ref. 17), it is shown that the assignment of four quantities A_1, A_2, R_1, R_2, which satisfy the /19

Codazzi-Gauss relationships, determines the surface completely (of course, with an accuracy up to its position in space).

2. Lines on the Surface

Let us consider some line Γ on a surface which is referred to the curvature lines. We draw two orthogonal families of curves on the surface, so that one of them contains the line Γ. The remainder of these families may be completely arbitrary. We associate the orthogonal unit vectors $\{\vec{\nu}, \vec{t}, \vec{n}\}$ with the curves of the family (fig. 4). Because they are of unit length and are orthogonal, the following relationships are valid

$$\vec{\nu}^2 = \vec{t}^2 = \vec{n}^2 = 1, \quad \vec{t}\cdot\vec{n} = \vec{t}\cdot\vec{\nu} = \vec{\nu}\cdot\vec{n} = 0; \tag{2.1}$$

$$[\vec{\nu}, \vec{t}] = \vec{n}, \quad [\vec{t}, \vec{n}] = \vec{\nu}, \quad [\vec{n}, \vec{\nu}] = \vec{t}. \tag{2.2}$$

Let us represent the derivatives of the orthogonal unit vectors along Γ by means of expansions in terms of these orthogonal unit vectors

$$\frac{d\vec{t}}{ds_t} = \alpha_{11}\vec{t} + \alpha_{12}\vec{\nu} + \alpha_{13}\vec{n}, \tag{2.3}$$

$$\frac{d\vec{v}}{ds_t} = a_{21}\vec{t} + a_{22}\vec{v} + a_{23}\vec{n},$$
$$\frac{d\vec{n}}{ds_t} = a_{31}\vec{t} + a_{32}\vec{v} + a_{33}\vec{n}.$$

Relationships (2.1) place certain associations on the coefficients of the expansions. Thus

$$a_{11} = \vec{t} \cdot \frac{d\vec{t}}{ds_t} = \frac{1}{2}\frac{d(\vec{t})^2}{ds_t} = \frac{1}{2}\frac{d(1)}{ds_t} = 0.$$

Furthermore,

$$a_{12} = \vec{v} \cdot \frac{d\vec{t}}{ds_t} = \frac{d(\vec{t}\cdot\vec{v})}{ds_t} - \frac{d\vec{v}}{ds_t}\cdot\vec{t} = -\frac{d\vec{v}}{ds_t}\cdot\vec{t} = -a_{21}.$$

In exactly the same manner the following equalities are established

$$a_{22} = a_{33} = 0, \quad a_{13} = -a_{31}, \quad a_{23} = -a_{32}.$$

By taking into account the relationships which we have obtained, we can write (2.3) in the form /20

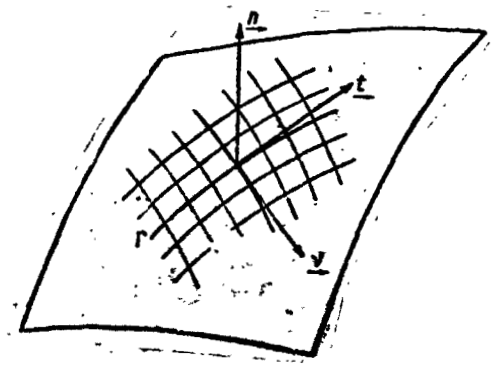

Figure 4

$$\begin{aligned}
&\frac{d\vec{t}}{ds_t}=c_t\vec{n}-\varkappa_t\vec{\nu}, && \sigma_t=\vec{n}\cdot\frac{d\vec{t}}{ds_t}=-\vec{t}\cdot\frac{d\vec{n}}{ds_t},\\
&\frac{d\vec{\nu}}{ds_t}=\varkappa_t\vec{t}-\tau_t\vec{n}, && \tau_t=\vec{\nu}\cdot\frac{d\vec{n}}{ds_t}=-\vec{n}\cdot\frac{d\vec{\nu}}{ds_t},\\
&\frac{d\vec{n}}{ds_t}=\tau_t\vec{\nu}-\sigma_t\vec{t}, && \varkappa_t=\vec{t}\cdot\frac{d\vec{\nu}}{ds_t}=-\vec{\nu}\cdot\frac{d\vec{t}}{ds_t}.
\end{aligned} \tag{2.4}$$

The quantity σ is called the normal curvature, $\varkappa$ is called the geodetic curvature, while τ is called the geodetic torsion (ref. 17).

In exactly the same manner, by differentiating along the normal to Γ we obtain

$$\begin{aligned}
&\frac{d\vec{t}}{ds_\nu}=\varkappa_\nu\vec{\nu}-\tau_\nu\vec{n}, && \sigma_\nu=\vec{n}\cdot\frac{d\vec{\nu}}{ds_\nu}=-\vec{\nu}\cdot\frac{d\vec{n}}{ds_\nu},\\
&\frac{d\vec{\nu}}{ds_\nu}=\sigma_\nu\vec{n}-\varkappa_\nu\vec{t}, && \tau_\nu=\vec{t}\cdot\frac{d\vec{n}}{ds_\nu}=-\vec{n}\cdot\frac{d\vec{t}}{ds_\nu},\\
&\frac{d\vec{n}}{ds_\nu}=\tau_\nu\vec{t}-\sigma_\nu\vec{\nu}, && \varkappa_\nu=\vec{\nu}\cdot\frac{d\vec{t}}{ds_\nu}=-\vec{t}\cdot\frac{d\vec{\nu}}{ds_\nu}.
\end{aligned} \tag{2.5}$$

We designate by γ the angle between the line α_1 and the normal to the curve Γ. Then it is easy to establish from figure 5 that

$$\begin{aligned}
\vec{\nu}&=\cos\gamma\,\vec{e}_1+\sin\gamma\,\vec{e}_2,\\
\vec{t}&=-\sin\gamma\,\vec{e}_1+\cos\gamma\,\vec{e}_2;
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\vec{e}_1&=\cos\gamma\,\vec{\nu}-\sin\gamma\,\vec{t},\\
\vec{e}_2&=\sin\gamma\,\vec{\nu}+\cos\gamma\,\vec{t};
\end{aligned} \tag{2.7}$$

$$\frac{d}{ds_t}=-\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}; \tag{2.8}$$

$$\frac{d}{ds_\nu}=\frac{\cos\gamma}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\sin\gamma}{A_2}\frac{\partial}{\partial\alpha_2}. \tag{2.9}$$

We replace in the left part of equalities (2.4) and (2.5) the vectors $\vec{t}$ and $\vec{\nu}$ by their expressions from (2.6) and then carry out differentiation by

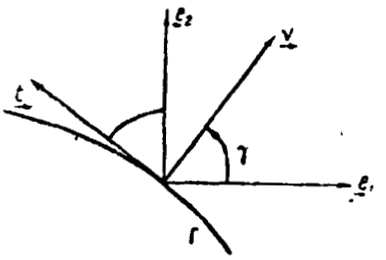

Figure 5

means of expressions (1.29) and (2.9). In the resulting equalities we replace the vectors $\vec{e}_1$, $\vec{e}_2$ by expressions (2.7) and, comparing the left and right sides, obtain

$$
\begin{aligned}
-\sigma_t &= \left(\frac{\sin^2\gamma}{R_1}+\frac{\cos^2\gamma}{R_2}\right)=\frac{1}{R_t},\\
-\sigma_\nu &= \left(\frac{\cos^2\gamma}{R_1}+\frac{\sin^2\gamma}{R_2}\right)=\frac{1}{R_\nu},\\
\tau_t &= \tau_\nu = -\sin\gamma\cos\gamma\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=-\frac{1}{R_{t\nu}},\\
\varkappa_t &= \frac{d\gamma}{ds_t}+\frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2},\\
\varkappa_\nu &= -\frac{d\gamma}{ds_\nu}-\frac{\sin\gamma}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\cos\gamma}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}.
\end{aligned}
\tag{2.10}
$$

We note that in accordance with (1.25) the quantities $1/R_t$ and $1/R_\nu$ /21

are the normal curvatures of the surface in the direction of the tangent and the normal to the surface Γ, respectively. In terms of the new designations, relationships (2.4) and (2.5) take on the following form

$$
\begin{aligned}
\frac{d\vec{t}}{ds_t} &= -\varkappa_t\vec{\nu}-\frac{1}{R_t}\vec{n},\\
\frac{d\vec{\nu}}{ds_t} &= \varkappa_t\vec{t}+\frac{1}{R_{t\nu}}\vec{n},\\
\frac{d\vec{n}}{ds_t} &= \frac{1}{R_t}\vec{t}-\frac{1}{R_{t\nu}}\vec{\nu};
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
\frac{d\vec{t}}{ds_\nu} &= \varkappa_\nu\vec{\nu}+\frac{1}{R_{t\nu}}\vec{n},\\
\frac{d\vec{\nu}}{ds_\nu} &= -\varkappa_\nu\vec{t}-\frac{1}{R_\nu}\vec{n},\\
\frac{d\vec{n}}{ds_\nu} &= -\frac{1}{R_{t\nu}}\vec{t}+\frac{1}{R_\nu}\vec{\nu}.
\end{aligned}
\tag{2.12}
$$

3. The Geometry of a Shell

As we have already stated above, the surface which is equidistant from the two faces is called the middle surface. We shall determine the position of an arbitrary point on the shell by its distance (along the perpendicular) from the middle surface (ζ) and by the position of the base of the perpendicular ($\vec{r}$). In this case the radius vector of the considered point ($\vec{R}$) and the radius vector of the base of the perpendicular ($\vec{r}$) are associated by a vector relationship

$$\vec{R}=\vec{r}+\zeta\vec{n}. \tag{3.1}$$

In the future we shall consider shells of constant thickness. For these shells the faces will have corresponding values $\zeta = \pm \frac{h}{2}$. We shall assume that the lateral surfaces (if the shell is not closed) are formed by the motion of the segment of the perpendicular along a contour which bounds the middle surface.

For a fixed value $\zeta = \zeta_0$ the end of the radius vector $\vec{R}$ describes a /22 so-called parallel surface. If, at the same time, the end $\vec{r}$ is displaced along the line α, the end of $\vec{R}$ on the parallel surface will trace a curve whose points are conveniently expressed by the same coordinates $\alpha(\beta = \beta_0)$. The lines β are introduced on the parallel surface in a similar fashion.

Differentiating expression (3.1) with respect to α and β we obtain, by remembering equality (1.28),

$$\begin{aligned}\frac{\partial\vec{R}}{\partial\alpha}&=\frac{\partial\vec{r}}{\partial\alpha}+\zeta\frac{\partial\vec{n}}{\partial\alpha}=A\left\{\left(1+\frac{\zeta}{R_\alpha}\right)\vec{e}_\alpha-\frac{\zeta}{R_{\alpha\beta}}\vec{e}_\beta\right\},\\ \frac{\partial\vec{R}}{\partial\beta}&=\frac{\partial\vec{r}}{\partial\beta}+\zeta\frac{\partial\vec{n}}{\partial\beta}=B\left\{\left(1+\frac{\zeta}{R_\beta}\right)\vec{e}_\beta-\frac{\zeta}{R_{\alpha\beta}}\vec{e}_\alpha\right\}.\end{aligned} \tag{3.2}$$

From this we have

$$\begin{aligned}A^\zeta&=\left|\frac{\partial\vec{R}}{\partial\alpha}\right|=A\sqrt{\left(1+\frac{\zeta}{R_\alpha}\right)^2+\frac{\zeta^2}{R_{\alpha\beta}^2}},\\ B^\zeta&=\left|\frac{\partial\vec{R}}{\partial\beta}\right|=B\sqrt{\left(1+\frac{\zeta}{R_\beta}\right)^2+\frac{\zeta^2}{R_{\alpha\beta}^2}}.\end{aligned}$$

We shall consider the shell to be thin, if the quantity h/R for the shell (R is any linear dimension which characterizes the middle surface) is small

compared to unity. Therefore, with an accuracy up to terms of order $(h/R)^2$, which are neglected compared with unity, the expressions for the Lamé coefficients of the parallel surface may be simplified

$$A^\zeta = A\left(1+\frac{\zeta}{R_\alpha}\right), \quad B^\zeta = B\left(1+\frac{\zeta}{R_\beta}\right). \tag{3.3}$$

Furthermore,

$$\vec{e}_\alpha^\zeta = \frac{1}{A^\zeta}\left|\frac{\partial \vec{R}}{\partial \alpha}\right| = \frac{1}{1+\zeta/R_\alpha}\left\{\left(1+\frac{\zeta}{R_\alpha}\right)\vec{e}_\alpha - \frac{\zeta}{R_{\alpha\beta}}\vec{e}_\beta\right\} \approx \vec{e}_\alpha - \frac{\zeta}{R_{\alpha\beta}}\vec{e}_\beta$$

and

$$\vec{e}_\beta^\zeta = \vec{e}_\beta - \frac{\zeta}{R_{\alpha\beta}}\vec{e}_\alpha.$$

By determining the cosine of the angle between the coordinate lines on the parallel surface we obtain

$$\cos\chi^\zeta = \vec{e}_\alpha^\zeta \cdot \vec{e}_\beta^\zeta = -\frac{2\zeta}{R_{\alpha\beta}}.$$

From this we can see that the system of coordinates on the parallel surface will be orthogonal only when the curvature lines are the coordinate lines on the middle surface. In this case

$$\vec{e}_1^\zeta = \vec{e}_1, \quad \vec{e}_2^\zeta = \vec{e}_2, \quad \vec{n}^\zeta = [\vec{e}_1^\zeta, \vec{e}_2^\zeta] = [\vec{e}_1, \vec{e}_2] = \vec{n}. \tag{3.4}$$

Furthermore, the orthogonal coordinate lines for the parallel surface are also the curvature lines. Indeed, it follows from (3.4) and (1.29) that /23

$$\frac{1}{A_1}\frac{\partial \vec{n}^\zeta}{\partial \alpha_1} = \frac{1}{A_1}\frac{\partial \vec{n}}{\partial \alpha_1} = \frac{1}{R_1}\vec{e}_1 = \frac{1}{R_1}\vec{e}_1^\zeta \tag{3.5}_1$$

and

$$\frac{1}{A_2}\frac{\partial \vec{n}^{\zeta}}{\partial \alpha_2} = \frac{1}{R_2}\vec{e}_2^{\zeta} \qquad (3.5)_2$$

Thus the conditions of the Rodrigues theorem are satisfied on the parallel surface. It follows from this that the coordinate curves coincide with the curvature lines. Let us write the expressions which we have just obtained in a slightly different form

$$\frac{1}{A_1\left(1+\frac{\zeta}{R_1}\right)}\frac{\partial \vec{n}^{\zeta}}{\partial \alpha_1} = \frac{1}{R_1\left(1+\frac{\zeta}{R_1}\right)}\vec{e}_1^{\zeta}, \quad \frac{1}{A_2\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial \vec{n}^{\zeta}}{\partial \alpha_2} = \frac{1}{R_2\left(1+\frac{\zeta}{R_2}\right)}\vec{e}_2^{\zeta}. \qquad (3.5')$$

Since, according to (3.3)

$$A_1^{\zeta} = A_1\left(1+\frac{\zeta}{R_1}\right), \quad A_2^{\zeta} = A_2\left(1+\frac{\zeta}{R_2}\right), \qquad (3.6)$$

a comparison of relationships (3.5') and (1.29) gives us

$$R_1^{\zeta} = R_1\left(1+\frac{\zeta}{R_1}\right), \quad R_2^{\zeta} = R_2\left(1+\frac{\zeta}{R_2}\right). \qquad (3.7)$$

Furthermore, by taking into account equalities (2.11) and (2.12) we obtain

$$\frac{d\vec{R}}{ds_t} = \frac{d\vec{r}}{ds_t} + \zeta\frac{d\vec{n}}{ds_t} = \left(1+\frac{\zeta}{R_t}\right)\vec{t} - \frac{\zeta}{R_{vt}}\vec{v},$$
$$\frac{d\vec{R}}{ds_v} = \frac{d\vec{r}}{ds_v} + \zeta\frac{d\vec{n}}{ds_v} = \left(1+\frac{\zeta}{R_v}\right)\vec{v} - \frac{\zeta}{R_{vt}}\vec{t},$$

from which it follows that

$$A_t^{\zeta} = \left|\frac{d\vec{R}}{ds_t}\right| = \left(1+\frac{\zeta}{R_t}\right), \quad A_v^{\zeta} = \left(1+\frac{\zeta}{R_v}\right); \qquad (3.8)$$

$$\vec{t}^{\zeta} = \vec{t} - \frac{\zeta}{R_{vt}}\vec{v}, \quad \vec{v}^{\zeta} = \vec{v} - \frac{\zeta}{R_{vt}}\vec{t}. \qquad (3.9)$$

CHAPTER II. THE THEORY OF DEFORMATION

1. Law Governing the Variations in the Strains Along the Thickness of the Shell

In the preceding section it was established that if the middle surface is referred to the curvature lines, α_1, α_2 and ζ determine an orthogonal /24

system of coordinates in the entire shell. Therefore, in deriving the basic relationships of the theory of shells we can utilize the expressions of the three-dimensional theory of elasticity given in reference 32. In this connection it is necessary to take into account the fact that in our case

$$\alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \zeta,$$
$$H_1 = A_1^\zeta = A_1\left(1 + \frac{\zeta}{R_1}\right), \quad H_2 = A_2^\zeta = A_2\left(1 + \frac{\zeta}{R_2}\right), \quad H_3 = 1. \tag{1.1}$$

We somewhat change the designations adopted in reference 32 and, taking into account (1.1), write the expressions for the components of the vector describing the rotation of the region close to the considered point of the shell

$$\delta_1 = \frac{1}{2}(\operatorname{rot}\vec{U}^\zeta)_1 = \frac{1}{2A_2\left(1+\frac{\zeta}{R_2}\right)}\left\{\frac{\partial U_\zeta^\zeta}{\partial \alpha_2} - \frac{\partial A_2\left(1+\frac{\zeta}{R_2}\right)U_2^\zeta}{\partial \zeta}\right\},$$
$$\delta_2 = \frac{1}{2}(\operatorname{rot}\vec{U}^\zeta)_2 = \frac{1}{2A_1\left(1+\frac{\zeta}{R_1}\right)}\left\{\frac{\partial A_1\left(1+\frac{\zeta}{R_1}\right)U_1^\zeta}{\partial \zeta} - \frac{\partial U_\zeta^\zeta}{\partial \alpha_1}\right\}, \tag{1.2}$$
$$\delta_3 = \frac{1}{2}(\operatorname{rot}\vec{U}^\zeta)_3 = \frac{1}{2A_1A_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)}\left\{\frac{\partial A_2\left(1+\frac{\zeta}{R_2}\right)U_2^\zeta}{\partial \alpha_1} - \frac{\partial A_1\left(1+\frac{\zeta}{R_1}\right)U_1^\zeta}{\partial \alpha_2}\right\}$$

and for the strain components

$$e_{1\zeta}=\frac{\partial U_1^\zeta}{\partial\zeta}+\frac{1}{A_1\left(1+\frac{\zeta}{R_1}\right)}\frac{\partial U_\zeta^\zeta}{\partial\alpha_1}-\frac{1}{R_1\left(1+\frac{\zeta}{R_1}\right)}U_1^\zeta,$$
$$e_{2\zeta}=\frac{\partial U_2^\zeta}{\partial\zeta}+\frac{1}{A_2\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial U_\zeta^\zeta}{\partial\alpha_2}-\frac{1}{R_2\left(1+\frac{\zeta}{R_2}\right)}U_2^\zeta, \qquad (1.3)$$
$$e_{\alpha\alpha}=\frac{\partial U_\zeta^\zeta}{\partial\zeta}.$$

of which the first two determine the variation, during the process of deformation, in the initial deformation of right angles between the fibers of the parallel surface and between the fibers which are normal to these surfaces. The latter also determines the relative elongation of the normal fibers. /25

We expand the components of the vector for the displacements of the point on the shell

$$\vec{U}^\zeta=U_1^\zeta\vec{e}_1+U_2^\zeta\vec{e}_2+U_\zeta^\zeta\vec{n} \qquad (1.4)$$

in power series of ζ

$$U_1^\zeta=(U_1^\zeta)_0+\left(\frac{\partial U_1^\zeta}{\partial\zeta}\right)_0\cdot\zeta+\ldots,$$
$$U_2^\zeta=(U_2^\zeta)_0+\left(\frac{\partial U_2^\zeta}{\partial\zeta}\right)_0\cdot\zeta+\ldots,$$
$$U_\zeta^\zeta=(U_\zeta^\zeta)_0+\left(\frac{\partial U_\zeta^\zeta}{\partial\zeta}\right)_0\cdot\zeta+\ldots$$

(the symbol "0" shows that the quantity is computed on the middle surface). Since the shell thickness is small, we can limit ourselves to the terms of the expansions which have been written above, writing them in the following form

$$U_1^\zeta=(U_1^\zeta)_0+\zeta\left[\underline{\frac{1}{2}(e_{1\zeta})_0}+(\delta_2)_0\right],$$
$$U_2^\zeta=(U_2^\zeta)_0+\zeta\left[\underline{\frac{1}{2}(e_{2\zeta})_0}-(\delta_1)_0\right], \qquad (1.5)$$
$$U_\zeta^\zeta=(U_\zeta^\zeta)_0+\underline{\zeta(e_{\alpha\alpha})_0}.$$

where, according to (1.2) and (1.3)

$$
\begin{aligned}
(e_{1\zeta})_0 &= \left(\frac{\partial U_1^\zeta}{\partial \zeta}\right)_0 + \frac{1}{A_1}\frac{\partial (U_\zeta^\zeta)_0}{\partial \alpha_1} - \frac{(U_1^\zeta)_0}{R_1},\\
(\delta_2)_0 &= \frac{1}{2}\left\{\left(\frac{\partial U_1^\zeta}{\partial \zeta}\right)_0 - \frac{1}{A_1}\frac{\partial (U_\zeta^\zeta)_0}{\partial \alpha_1} + \frac{(U_1^\zeta)_0}{R_1}\right\},\\
(e_{2\zeta})_0 &= \left(\frac{\partial U_2^\zeta}{\partial \zeta}\right)_0 + \frac{1}{A_2}\frac{\partial (U_\zeta^\zeta)_0}{\partial \alpha_2} - \frac{(U_2^\zeta)_0}{R_2},\\
(\delta_1)_0 &= \frac{1}{2}\left\{-\left(\frac{\partial U_2^\zeta}{\partial \zeta}\right)_0 + \frac{1}{A_2}\frac{\partial (U_\zeta^\zeta)_0}{\partial \alpha_2} - \frac{(U_2^\zeta)_0}{R_2}\right\},\\
(e_{\zeta\zeta})_0 &= \left(\frac{\partial U_\zeta^\zeta}{\partial \zeta}\right)_0.
\end{aligned}
\tag{1.6}
$$

The flexibility of a shell is associated with the thinness of its walls; therefore, generally speaking, the angles of rotation in a shell are substantially greater than the elongations and the shear strains. Comparing the expressions for the angles of rotation $(\delta_1)_0$, $(\delta_2)_0$ with the corresponding sheer strains $(e_{1\zeta})_0$, $(e_{2\zeta})_0$, we see that the same terms are contained in them, except with a different sign. It follows from this that the angles of rotation may be substantially large quantities compared with the shear strains only in the case when /26

$$
\begin{aligned}
\left(\frac{\partial U_1^\zeta}{\partial \zeta}\right)_0 + \frac{1}{A_1}\frac{\partial (U_\zeta^\zeta)_0}{\partial \alpha_1} - \frac{(U_1^\zeta)_0}{R_1} &\approx 0,\\
\left(\frac{\partial U_2^\zeta}{\partial \zeta}\right)_0 + \frac{1}{A_2}\frac{\partial (U_\zeta^\zeta)_0}{\partial \alpha_2} - \frac{(U_2^\zeta)_0}{R_2} &\approx 0.
\end{aligned}
\tag{1.7}
$$

The last equalities should be interpreted in the sense that the terms which are contained in them form small differences between close values.

Introducing the designations which are traditional for the theory of shells

$$
\begin{aligned}
(U_1^\zeta)_0 = u_1, \quad (U_2^\zeta)_0 = u_2, \quad (U_\zeta^\zeta)_0 = w,\\
\vartheta_1 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u_1}{R_1}, \quad \vartheta_2 = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{u_2}{R_2},
\end{aligned}
\tag{1.8}
$$

we may, in accordance with (1.7), consider approximately that

$$
\left(\frac{\partial U_1^\zeta}{\partial \zeta}\right)_0 = \vartheta_1, \quad \left(\frac{\partial U_2^\zeta}{\partial \zeta}\right)_0 = \vartheta_2. \tag{1.9}
$$

From this and from (1.6) it follows that

$$(\delta_1)_0 = -\vartheta_2, \quad (\delta_2)_0 = \vartheta_1.$$

If we further neglect, in expression (1.5), the underlined components of deformation compared with the angles of rotation and replace the latter with expressions from (1.9), we transform to the following law for the variation in displacements along the thickness of the shell

$$\begin{aligned} U_1^\zeta &= u_1 + \zeta\vartheta_1, \\ U_2^\zeta &= u_2 + \zeta\vartheta_2, \\ U_\zeta^\zeta &= w. \end{aligned} \tag{1.10}$$

Substituting the expressions obtained for the displacements into equality (1.3), we become convinced that for them

$$e_{1\zeta} = e_{2\zeta} = e_{\zeta\zeta} = 0. \tag{1.11}$$

In other words, in considering the geometric picture of the deformation of /27
a thin shell we may assume that the fiber perpendicular to the middle surface remains orthogonal with respect to it, before deformation and after deformation without changing its length in the process. This represents the first (geometric) Kirchhoff hypothesis.

Let us introduce one other designation

$$\Omega_n = (\delta_\zeta)_0. \tag{1.12}$$

According to equality $(1.2)_3$

$$\Omega_n = \frac{1}{2A_1A_2}\left(\frac{\partial A_2 u_2}{\partial \alpha_1} - \frac{\partial A_1 u_1}{\partial \alpha_2}\right). \tag{1.13}$$

Now expressions (1.9) and (1.12) make it possible for us to write the expression for the vector, which represents the rotation angles of the neighborhood of an arbitrary point on the middle surface, in the form

$$\vec{\Omega} = -\vartheta_2\vec{e}_1 + \vartheta_1\vec{e}_2 + \Omega_n\vec{n}. \tag{1.14}$$

In this case, since equality (1.11) is valid, the angles between the normal to the middle surface and the tangent to the coordinate lines may be considered to be invariable, ϑ_1 and ϑ_2 are simultaneously the angles of rotation both of the normal as well as of the respective tangent to the coordinate line around another tangent. The positive directions of the angles of rotation are shown in figure 6. The situation is somewhat different with respect to the angle of rotation Ω_n. Since the displacement between the coordinate lines α_1 and α_2 is not equal to zero, Ω_n characterizes only the average rotation of the entire neighborhood, associated with the considered point on the middle surface, around the normal.

By using the general equations of the theory of elasticity it can be shown that the rotation of tangents to the coordinate lines of the middle surface around the normal are, respectively, equal to

$$\omega_1 = \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_1, \quad \omega_2 = \frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}u_2 \tag{1.15}$$

(see fig. 6). It is easy to verify that

$$\Omega_n = \frac{\omega_1 - \omega_2}{2}. \tag{1.16}$$

As in the case of the general theory of elasticity, the quantity

$$\omega = \omega_1 + \omega_2 \tag{1.17}$$

determines the displacement between the coordinate lines of the middle surface.

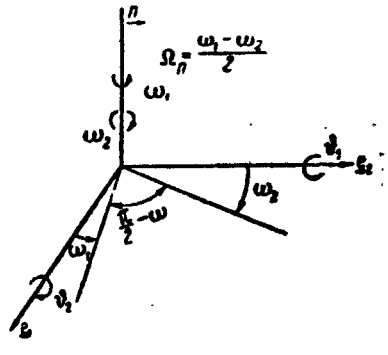

Figure 6

Now, by introducing the vector for the displacement of points on the mean surface /28

$$\vec{U}=u_1\vec{e}_1+u_2\vec{e}_2+w\vec{n} \tag{1.18}$$

and recalling expression (1.14), it is easy to write equality (1.10) in vector form

$$\vec{U}^{\zeta}=\vec{U}+\left[\vec{\Omega},\ \zeta\vec{n}\right]. \tag{1.19}$$

2. Deformation of s hell

The following expressions (ref. 32) exist for the basic strain components

$$\begin{aligned}
e_{11}&=\frac{1}{A_1\left(1+\frac{\zeta}{R_1}\right)}\frac{\partial U_1^{\zeta}}{\partial\alpha_1}+\frac{1}{A_1A_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial A_1\left(1+\frac{\zeta}{R_1}\right)}{\partial\alpha_2}U_2^{\zeta}+\\
&\quad+\frac{1}{R_1\left(1+\frac{\zeta}{R_1}\right)}U_{\zeta}^{\zeta},\\
e_{22}&=\frac{1}{A_2\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial U_2^{\zeta}}{\partial\alpha_2}+\frac{1}{A_1A_2\left(1+\frac{\zeta}{R_2}\right)\left(1+\frac{\zeta}{R_1}\right)}\frac{\partial A_2\left(1+\frac{\zeta}{R_2}\right)}{\partial\alpha_1}U_1^{\zeta}+\\
&\quad+\frac{1}{R_2\left(1+\frac{\zeta}{R_2}\right)}U_{\zeta}^{\zeta},\\
e_{12}&=\frac{1}{A_1\left(1+\frac{\zeta}{R_1}\right)}\frac{\partial U_2^{\zeta}}{\partial\alpha_1}-\frac{1}{A_1A_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial A_1\left(1+\frac{\zeta}{R_1}\right)}{\partial\alpha_2}U_1^{\zeta}+\\
&\quad+\frac{1}{A_2\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial U_1^{\zeta}}{\partial\alpha_2}-\frac{1}{A_1A_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)}\frac{\partial A_2\left(1+\frac{\zeta}{R_2}\right)}{\partial\alpha_1}U_2^{\zeta}.
\end{aligned} \tag{2.1}$$

We somewhat transform the terms which are contained in the above expressions. First of all, by using the Codazzi relationships (I, 1.31), we obtain

$$\frac{\partial A_1\left(1+\frac{\zeta}{R_1}\right)}{\partial\alpha_2}=\frac{\partial A_1}{\partial\alpha_2}+\zeta\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right)=\left(1+\frac{\zeta}{R_2}\right)\frac{\partial A_1}{\partial\alpha_2}$$

and

$$\frac{\partial A_2\left(1+\frac{\zeta}{R_2}\right)}{\partial \alpha_1}=\left(1+\frac{\zeta}{R_1}\right)\frac{\partial A_2}{\partial \alpha_1}. \tag{2.2}$$

By using the resulting equalities and also relationships (1.10), we obtain /29 the following expressions for the first two quantities

$$e_{11}=\frac{\varepsilon_1+\zeta\varkappa_1}{1+\frac{\zeta}{R_1}},\quad e_{22}=\frac{\varepsilon_2+\zeta\varkappa_2}{1+\frac{\zeta}{R_2}}, \tag{2.3}_{1,2}$$

where

$$\begin{aligned}\varepsilon_1&=\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u_2+\frac{w}{R_1},\quad \varkappa_1=\frac{1}{A_1}\frac{\partial \vartheta_1}{\partial \alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\vartheta_2,\\ \varepsilon_2&=\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u_1+\frac{w}{R_2},\quad \varkappa_2=\frac{1}{A_2}\frac{\partial \vartheta_2}{\partial \alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\vartheta_1.\end{aligned} \tag{2.4}$$

From the third expression we find

$$e_{12}=\frac{\omega_1+\zeta\tau_1}{1+\frac{\zeta}{R_1}}+\frac{\omega_2+\zeta\tau_2}{1+\frac{\zeta}{R_2}}. \tag{2.3}_3$$

Here ω_1 and ω_2 are the expressions obtained above (see (1.15)), while

$$\tau_1=\frac{1}{A_1}\frac{\partial \vartheta_2}{\partial \alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\vartheta_1,\quad \tau_2=\frac{1}{A_2}\frac{\partial \vartheta_1}{\partial \alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\vartheta_2. \tag{2.5}$$

If we use the Codazzi relationships, it is easy to show the validity of the following identity

$$\tau_1+\frac{\omega_2}{R_1}=\tau_2+\frac{\omega_1}{R_2}\equiv\tau. \tag{2.6}$$

Here, in addition to the function ω which has already been introduced, the total value of the right and left sides of identity (2.6) is designated by τ. The shear displacement of the parallel surface is written in the following form in terms of functions ω and τ

$$e_{12}=\frac{1}{\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)}\left\{\left(1-\frac{\zeta^2}{R_1R_2}\right)\omega+\right.$$
$$\left.+2\left[1+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\zeta}{2}\right]\zeta\tau\right\}. \qquad (2.7)$$

An examination of equalities (2.3) and (2.7) shows that the deformation of the shell is determined completely by the assignment of six quantities

$$\varepsilon_1=\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2+\frac{w}{R_1},$$
$$\varepsilon_2=\frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_1+\frac{w}{R_2},$$
$$\omega=\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right).$$
$$\varkappa_1=\frac{1}{A_1}\frac{\partial\vartheta_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\vartheta_2=$$
$$=-\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u_1}{R_1}\right)-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}-\frac{u_2}{R_2}\right), \qquad (2.8)$$
$$\varkappa_2=\frac{1}{A_2}\frac{\partial\vartheta_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\vartheta_1=$$
$$=-\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}-\frac{u_2}{R_2}\right)-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u_1}{R_1}\right),$$
$$\tau=\frac{1}{2}\left\{\frac{1}{A_1}\frac{\partial\vartheta_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\vartheta_1+\frac{1}{A_2}\frac{\partial\vartheta_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\vartheta_2+\right.$$
$$\left.+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_2\right)+\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_1\right)\right\}=$$
$$=-\frac{1}{A_1A_2}\left(\frac{\partial^2w}{\partial\alpha_1\partial\alpha_2}-\frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1}-\frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right)+$$
$$+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_1\right)+\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2\right).$$

By neglecting the quantities $\frac{\zeta}{R_1}$, $\frac{\zeta}{R_2}$ with respect to unity in equalities (2.3) and (2.7), we arrive at the following approximate expressions /30

$$e_{11}\approx\varepsilon_1+\zeta\varkappa_1,\quad e_{22}\approx\varepsilon_2+\zeta\varkappa_2,\quad e_{12}\approx\omega+\zeta\,2\tau.$$

From this the meaning of the six quantities which have been introduced becomes clear. The first three (ε_1; ε_2, ω) characterize the deformation which is uniform along the thickness of the shell and which is determined by the tension and shear associated with the middle surface; ($\varkappa_1$, $\varkappa_2$, τ) determine the deformation which varies in a linear fashion along the thickness and which is associated with the flexure and torsion of the middle surface. Therefore, below

we shall call the first three quantities the components of the tangential strain and the last three the components of the flexural strain. Let us derive certain relationships which will be useful in the future. First of all, by differentiating the displacement vector $\vec{U}$ with respect to α_1 and α_2, we obtain, by using

the rule for the differentiation of orthogonal unit vectors (I, 1.29), the equalities

$$\begin{aligned}\frac{1}{A_1}\frac{\partial \vec{U}}{\partial \alpha_1} &= \varepsilon_1\vec{e}_1 + \omega_1\vec{e}_2 - \vartheta_1\vec{n} = \varepsilon_1\vec{e}_1 + \frac{1}{2}\omega\vec{e}_2 + \left[\vec{\Omega}, \vec{e}_1\right],\\ \frac{1}{A_2}\frac{\partial \vec{U}}{\partial \alpha_2} &= \omega_1\vec{e}_1 + \varepsilon_2\vec{e}_2 - \vartheta_2\vec{n} = \varepsilon_2\vec{e}_2 + \frac{1}{2}\omega\vec{e}_1 + \left[\vec{\Omega}, \vec{e}_2\right].\end{aligned} \tag{2.9}$$

Carrying out the same operation with the vector for the angles of rotation $\vec{\Omega}$ we find

$$\begin{aligned}\frac{1}{A_1}\frac{\partial \vec{\Omega}}{\partial \alpha_1} &= -\tau_1\vec{e}_1 + \varkappa_1\vec{e}_2 + \zeta_1\vec{n} + \frac{\Omega_n}{R_1}\vec{e}_1,\\ \frac{1}{A_2}\frac{\partial \vec{\Omega}}{\partial \alpha_2} &= -\varkappa_2\vec{e}_1 + \tau_2\vec{e}_2 + \zeta_2\vec{n} + \frac{\Omega_n}{R_2}\vec{e}_2,\end{aligned} \tag{2.10}$$

where

$$\zeta_1 = \frac{1}{A_1}\frac{\partial \Omega_n}{\partial \alpha_1} + \frac{\vartheta_2}{R_2}, \quad \zeta_2 = \frac{1}{A_2}\frac{\partial \Omega_n}{\partial \alpha_2} - \frac{\vartheta_1}{R_1}. \tag{2.11}$$

Expressions (2.9) and (2.10) may be regarded as a system of equations for determining vectors $\vec{U}$ and $\vec{\Omega}$. It is natural to expect that the requirements for their compatibility place certain limitations on the values contained in the right side of the considered relationships. Thus, in order to determine the vector of displacements from (2.9), it is necessary that the following equality be satisfied

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$$\frac{\partial}{\partial \alpha_2}\left(\frac{\partial \vec{U}}{\partial \alpha_1}\right) = \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \vec{U}}{\partial \alpha_2}\right), \tag{2.12}$$

i.e.,

$$\begin{aligned}&\frac{\partial}{\partial \alpha_2}\left\{A_1\left(\varepsilon_1\vec{e}_1 + \frac{1}{2}\omega\vec{e}_2 + \left[\vec{\Omega}, \vec{e}_1\right]\right)\right\} =\\ &= \frac{\partial}{\partial \alpha_1}\left\{A_2\left(\varepsilon_2\vec{e}_2 + \frac{1}{2}\omega\vec{e}_1 + \left[\vec{\Omega}, \vec{e}_2\right]\right)\right\}.\end{aligned}$$

By expanding this expression we obtain the following, after simple transformations and after taking into account equalities (I, 1.29)

$$
\begin{aligned}
-\zeta_1 &= \frac{1}{A_1A_2}\left\{\frac{\partial A_1\varepsilon_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial}{\partial\alpha_1}\left(A_2^2\frac{\omega}{2}\right) - \frac{\partial A_1}{\partial\alpha_2}\varepsilon_2\right\}\\
\zeta_2 &= \frac{1}{A_1A_2}\left\{\frac{\partial A_2\varepsilon_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial}{\partial\alpha_2}\left(A_1^2\frac{\omega}{2}\right) - \frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right\},\\
\tau_1 &+ \frac{\omega_2}{R_1} - \left(\tau_2 + \frac{\omega_1}{R_2}\right) = 0.
\end{aligned}
\tag{2.13}
$$

The last equality is satisfied identically (see (2.6)). The first two give expressions for ζ_1, ζ_2 in term of the strain components. Below the values ζ_1, ζ_2 will designate expressions (2.13).

Conditions for the integration of system (2.10) have the form

$$
\frac{\partial}{\partial\alpha_2}\left(\frac{\partial\vec{\Omega}}{\partial s_1}\right) = \frac{\partial}{\partial\alpha_1}\left(\frac{\partial\vec{\Omega}}{\partial\alpha_2}\right)
\tag{2.14}
$$

or, according to (2.10),

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_2}A_1\left[\left(-\tau_1 + \frac{\Omega_n}{R_1}\right)\vec{e}_1 + \varkappa_1\vec{e}_2 + \zeta_1\vec{n}\right] =\\
&= \frac{\partial}{\partial\alpha_1}A_2\left[-\varkappa_2\vec{e}_1 + \left(\tau_2 + \frac{\Omega_n}{R_2}\right)\vec{e}_2 + \zeta_2\vec{n}\right].
\end{aligned}
\tag{2.15}
$$

If we carry out the differentiation and compare the factors in front of the same orthogonal unit vectors in the right and left parts, we obtain three scalar relationships. If, in these relationships we replace τ_1 and τ_2 by their expressions from (2.6), we arrive at three equations of compatibility

$$
\begin{aligned}
&\frac{\partial A_2\varkappa_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial A_1^2\tau}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\varkappa_1 - \frac{1}{R_1}\left(\frac{\partial A_2\varepsilon_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial A_1^2\frac{\omega}{2}}{\partial\alpha_1} - \frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right) +\\
&\qquad + \frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\frac{\omega}{2}\right) + \frac{1}{R_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\omega}{2} = 0,\\
&\frac{\partial A_1\varkappa_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial A_2^2\tau}{\partial\alpha_1} - \frac{\partial A_1}{\partial\alpha_2}\varkappa_2 - \frac{1}{R_2}\left(\frac{\partial A_1\varepsilon_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial A_2^2\frac{\omega}{2}}{\partial\alpha_1} - \frac{\partial A_1}{\partial\alpha_2}\varepsilon_2\right) +\\
&\qquad + \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\frac{\omega}{2}\right) + \frac{1}{R_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\omega}{2} = 0,\\
&\frac{\varkappa_2}{R_1} + \frac{\varkappa_1}{R_2} + \frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left(\frac{\partial A_2\varepsilon_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial A_1^2\frac{\omega}{2}}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right) +\right.\\
&\qquad \left. + \frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left(\frac{\partial A_1\varepsilon_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial A_2^2\frac{\omega}{2}}{\partial\alpha_1} - \frac{\partial A_1}{\partial\alpha_2}\varepsilon_2\right)\right\} = 0.
\end{aligned}
\tag{2.16}
$$

In the future we shall encounter systems of this type quite frequently. Therefore, in order to shorten our notations it is rational to introduce compact designations. Let us assume that the functions Φ_1, Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 are certain solutions of the following system of differential equations /32

$$\begin{aligned}
&\frac{\partial A_2\Phi_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\Phi_3}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\Phi_2+\frac{1}{R_1}\left(\frac{\partial A_2\Phi_4}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\Phi_6}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\Phi_5\right)+\\
&\qquad+\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\Phi_6\right)+\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}\Phi_6=0,\\
&\frac{\partial A_1\Phi_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\Phi_3}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\Phi_1+\frac{1}{R_2}\left(\frac{\partial A_1\Phi_5}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\Phi_6}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\Phi_4\right)+\\
&\qquad+\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\Phi_6\right)+\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}\Phi_6=0,\\
&\frac{\Phi_1}{R_1}+\frac{\Phi_2}{R_2}-\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left(\frac{\partial A_2\Phi_4}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\Phi_6}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\Phi_5\right)+\right.\\
&\qquad\left.+\frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left(\frac{\partial A_1\Phi_5}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\Phi_6}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\Phi_4\right)\right\}=0.
\end{aligned}\tag{2.17}$$

We shall describe system (2.17) in the following condensed form

$$\begin{aligned}
&L_1\{\Phi_1,\ \Phi_2,\ \Phi_3,\ \Phi_4,\ \Phi_5,\ \Phi_6\}=0,\\
&L_2\{\ldots\}=0,\ L_3\{\ldots\}=0.
\end{aligned}\tag{2.18}$$

Here, for example, $L_1\{\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6\}$ represents the differentiation operation performed on the functions Φ_1, Φ_2, ..., Φ_6 in the left part of the first equation in system (2.17), $L_2\{\ldots\}$ of the second and $L_3\{\ldots\}$ of the third.

In line with the introduced abbreviated designations, the equations of compatibility may be written in the following manner

$$\begin{aligned}
&L_1\left\{\varkappa_2,\ \varkappa_1-\tau,\ -\varepsilon_2,\ -\varepsilon_1,\ \frac{\omega}{2}\right\}=0,\\
&L_2\{\ldots\}=0,\ L_3\{\ldots\}=0.
\end{aligned}$$

According to the sense of the derivation, equations (2.16) are the equations of compatibility. The fulfillment of these equations is necessary and sufficient for the assigned six functions $\varepsilon_1(\alpha_1, \alpha)$, ..., $\tau(\alpha_1, \alpha_2)$ to /33

correspond to some definite vector of displacements. It is only in this case that the introduced quantities may be looked upon as strain components. We have established the fact that the conditions are necessary. The fact that the conditions are sufficient will be shown in the next section.

The conditions of compatibility are frequently called equations of continuity for the middle surface. Let us clarify this. Equations (2.16) may also be obtained in the following manner (ref. 3). The Codazzi-Gauss conditions are formulated for a deformed surface. These are transformed by means of the Codazzi-Gauss equation for the nondeformed surface. As a result we arrive at the same equations (2.16). As we have already stated above, the Gauss-Codazzi conditions provide for the existence of a middle surface with assigned Lamé coefficients and radii of curvature. From this it becomes clear that if equations (2.16) are satisfied, this provides for a continuous transition of the initial (nondeformed) surface into some deformed surface. This justifies the commonly adopted definition of equations (2.16). For the sake of brevity we shall call these equations the continuity equations.

3. Deformation of the Boundary Element. Determination of Displacement and Angular Rotation Vectors From the Assigned Strain Components

If we use coordinates fixed with respect to the line of the middle surface Γ (I, sec. 2), the vector of displacements may be written in the following form

$$\vec{U}=u_\nu\vec{\nu}+u_t\vec{t}+w\vec{n} \tag{3.1}$$

$$(u_\nu=\vec{\nu}\cdot\vec{U},\ u_t=\vec{t}\cdot\vec{U}). \tag{3.2}$$

By differentiating $\vec{U}$ along the arc we obtain, taking into account the rule for the differentiation of orthogonal unit vectors (I, 2.11),

$$\frac{d\vec{U}}{ds_t}=-\omega_{tt}\vec{\nu}+\varepsilon_{tt}\vec{t}-\vartheta_t\vec{n}, \tag{3.3}$$

where

$$\varepsilon_{tt}=\frac{du_t}{ds_t}+\varkappa_t u_\nu+\frac{w}{R_t},$$

$$-\omega_{tt}=\frac{du_\nu}{ds_t}-\varkappa_t u_t-\frac{w}{R_{\nu t}}, \tag{3.4}$$

$$\vartheta_t=-\frac{dw}{ds_t}+\frac{u_t}{R_t}-\frac{u_\nu}{R_{\nu t}}$$

$$\left(\varkappa_t=\frac{d\gamma}{ds_t}+\frac{\cos\gamma}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\sin\gamma}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2},\ \varepsilon_{tt}=\vec{t}\cdot\frac{d\vec{U}}{ds_t},\right.$$

$$\left.\omega_{tt}=-\vec{\nu}\frac{d\vec{U}}{ds_t},\ \vartheta_t=-\vec{n}\cdot\frac{d\vec{U}}{ds_t}\right). \tag{3.5}$$

By using an analogy for expression (1.14) we introduce the vector for the angles of rotation of the normal element associated with the line Γ, /34

$$\vec{\Omega}_t=-\vartheta_t\vec{\nu}+\vartheta_\nu\vec{t}+\omega_{tt}\vec{n}, \tag{3.6}$$

where

$$\vartheta_\nu=-\vec{n}\cdot\frac{d\vec{U}}{ds_\nu}=-\frac{dw}{ds_\nu}+\frac{u_\nu}{R_\nu}-\frac{u_t}{R_{\nu t}}. \tag{3.7}$$

The positive direction of the introduced angles of rotation is shown in figure 7. It is easy to check (by means of (3.3) and (3.6)) the validity of the following relationship

$$\frac{d\vec{U}}{ds_t}=\varepsilon_{tt}\vec{t}+[\vec{\Omega}_t,\vec{t}]. \tag{3.8}$$

Furthermore, by using equalities (3.2), (3.5), (3.7), (I, 2.11; 2.12) and (2.9), it is easy to obtain the following expression

$$u_\nu=\cos\gamma\,u_1+\sin\gamma\,u_2,\quad u_t=-\sin\gamma\,u_1+\cos\gamma\,u_2; \tag{3.9}$$

$$\vartheta_\nu=\cos\gamma\,\vartheta_1+\sin\gamma\,\vartheta_2,\quad \vartheta_t=-\sin\vartheta_1+\cos\gamma\,\vartheta_2;$$
$$\varepsilon_{tt}=\cos^2\gamma\,\varepsilon_2-\sin\gamma\cos\gamma\,\omega+\sin^2\gamma\,\varepsilon_1, \tag{3.10}$$

$$\omega_{tt}=\sin\gamma\cos\gamma\,(\varepsilon_1-\varepsilon_2)-\cos^2\gamma\,\omega_2+\sin^2\gamma\,\omega_1. \tag{3.11}$$

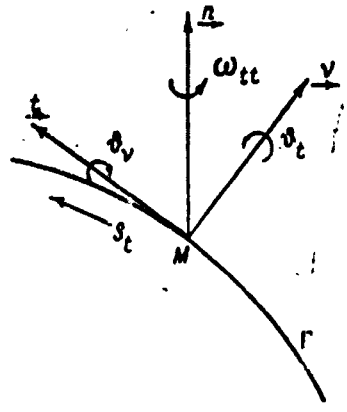

Figure 7

As we can see from (3.6) and figure 7, the vector $\vec{\Omega}_t$ characterizes the rotation of the normal element associated with the curve Γ in the vicinity of the considered point. At the same time the vector $\vec{\Omega}$, introduced earlier, describes the average rotation of the entire neighborhood of the point. The latter can be verified directly without making reference to the geometric meaning of the quantity $(1.2)_3$, borrowed by us from the theory of elasticity.

Indeed, by comparing expressions (3.6) and (1.14) we become convinced that the vectors $\vec{\Omega}_t$ and $\vec{\Omega}$ differ only in their normal components. Let us consider /35 the entire sets of values of γ from 0 to 2π. Then the following equality follows from $(3.11)_2$

$$(\omega_{tt})_{cp} = \frac{1}{2\pi}\int_0^{2\pi} \omega_{tt}\,d\gamma = \frac{\omega_1 - \omega_2}{2} = \Omega_n,$$

which confirms the meaning of the vector $\vec{\Omega}$. Comparing once again the normal components of both vectors, we arrive at the following relationship between them

$$\vec{\Omega}_t = \vec{\Omega} - \varepsilon_{t\nu}\vec{n}, \tag{3.12}$$

where the quantity

$$\varepsilon_{t\nu} = \sin\gamma\cos\gamma\,(\varepsilon_2 - \varepsilon_1) + (\cos^2\gamma - \sin^2\gamma)\frac{\omega}{2} \tag{3.13}$$

is equal to half of the displacement between the tangent and normal to the line Γ.

The vector describing the variation in the curvature of the normal element associated with curve Γ will be determined as the derivative of the vector for the angular rotations $\vec{\Omega}_t$ along the arc

$$\vec{\varkappa}_t = \frac{d\vec{\Omega}_t}{ds_t} = -\varkappa_{tt}\vec{\nu} + \varkappa_{t\nu}\vec{t} - \varkappa_{tn}\vec{n}. \tag{3.14}$$

This method of determining the vector which characterizes the variation in curvature is quite natural, because the vector introduced in this fashion determines the vector for the angles of rotation along Γ, i.e., the distortion corresponding to the normal cross section. In particular, if $\vec{\Omega}_t$ = const along Γ, then $\vec{\varkappa}_t = 0$ and the normal element rotates like a whole rigid element.

We obtained the expression for the components of vector $\vec{\varkappa}_t$ by taking into account the rule for the correct differentiation of the orthogonal unity vectors (I, 2.5)

$$\begin{aligned} \varkappa_{tt} &= \frac{d\vartheta_t}{ds_t} + \varkappa_t\vartheta_\nu + \frac{\omega_{tt}}{R_{\nu t}}, \\ \varkappa_{t\nu} &= \frac{d\vartheta_\nu}{ds_t} - \varkappa_t\vartheta_t + \frac{\omega_{tt}}{R_t}, \\ -\varkappa_{tn} &= \frac{d\omega_{tt}}{ds_t} - \frac{\vartheta_\nu}{R_t} - \frac{\vartheta_t}{R_{\nu t}}. \end{aligned} \tag{3.15}$$

Then, by utilizing relationships (I, 2.9) and (2.10), we find

$$\begin{aligned} \vec{\varkappa}_t = &-\sin\gamma\left[\left(-\tau_1 + \frac{\Omega_n}{R_1}\right)\vec{e}_1 + \varkappa_1\vec{e}_2 + \zeta_1\vec{n}\right] + \\ &+\cos\gamma\left[-\varkappa_2\vec{e}_1 + \left(\tau_2 + \frac{\Omega_n}{R_2}\right)\vec{e}_2 + \zeta_2\vec{n}\right] - \frac{d\varepsilon_{t\nu}\vec{n}}{ds_t}. \end{aligned} \tag{3.16}$$

Utilizing the same rules for the differentiation of orthogonal unit vectors and replacing the vectors $\vec{e}_1$ and $\vec{e}_2$ by expressions (I, 2.7) we obtain the following equalities /36

$$\begin{aligned} \varkappa_{tt} &= \cos^2\gamma\,\varkappa_2 + \sin^2\gamma\,\varkappa_1 - \sin\gamma\cos\gamma\left(\tau_1 + \tau_2 + \frac{\Omega_n}{R_2} - \frac{\Omega_n}{R_1}\right) - \frac{\varepsilon_{t\nu}}{R_{\nu t}}, \\ \varkappa_{t\nu} &= -\sin\gamma\cos\gamma(\varkappa_2 - \varkappa_1) + \cos^2\gamma\left(\tau_2 + \frac{\Omega_n}{R_2}\right) - \sin^2\gamma\left(\tau_1 - \frac{\Omega_n}{R_1}\right) - \frac{\varepsilon_{t\nu}}{R_t}, \\ \varkappa_{tn} &= \frac{d\varepsilon_{t\nu}}{ds_t} + \sin\gamma\,\zeta_1 - \cos\gamma\,\zeta_2. \end{aligned} \tag{3.17}$$

By using (2.6) and expanding the quantities $\varepsilon_{\tau\nu}$, ζ_1, ζ_2 (see (3.13), (2.13), (1.16)), we rewrite these equalities

$$\begin{aligned}
\varkappa_{tt} &= \cos^2\gamma\,\varkappa_2 - 2\sin\gamma\cos\gamma\,\tau + \sin^2\gamma\,\varkappa_1 + \sin\gamma\cos\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\omega}{2} - \\
&\quad - \frac{1}{R_{vt}}\left[\sin\gamma\cos\gamma\,(\varepsilon_2-\varepsilon_1) + (\cos^2\gamma-\sin^2\gamma)\frac{\omega}{2}\right], \\
\varkappa_{t\nu} &= \sin\gamma\cos\gamma\,(\varkappa_2-\varkappa_1) + (\cos^2\gamma-\sin^2\gamma)\,\tau + \left(\frac{\sin^2\gamma}{R_1}-\frac{\cos^2\gamma}{R_2}\right)\frac{\omega}{2} \\
&\quad - \frac{1}{R_t}\left[\sin\gamma\cos\gamma\,(\varepsilon_2-\varepsilon_1) + (\cos^2\gamma-\sin^2\gamma)\frac{\omega}{2}\right], \qquad (3.18) \\
\varkappa_{tn} &= \left(-\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}\right)\left[\sin\gamma\cos\gamma\,(\varepsilon_2-\varepsilon_1) + (\cos^2\gamma-\sin^2\gamma)\frac{\omega}{2}\right] - \\
&\quad - \frac{\cos\gamma}{A_1A_2}\left\{\frac{\partial A_2\varepsilon_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial}{\partial\alpha_2}\left(A_1^2\frac{\omega}{2}\right) - \frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right\} - \\
&\quad - \frac{\sin\gamma}{A_1A_2}\left\{\frac{\partial A_1\varepsilon_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial}{\partial\alpha_1}\left(A_2^2\frac{\omega}{2}\right) - \frac{\partial A_1}{\partial\alpha_2}\varepsilon_2\right\}, \\
\varepsilon_{tt} &= \cos^2\gamma\,\varepsilon_2 - \sin\gamma\cos\gamma\,\omega + \sin^2\gamma\,\varepsilon_1.
\end{aligned}$$

In this manner we have put together four quantities, which characterize the deformation of a normal element associated with line Γ (we note that the considered normal element may be a boundary element).

As we can see from (3.14), the first quantity characterizes the distortion of the normal element in its plane, while the third characterizes it from the plane. The second characterizes the torsion of the element. The last quantity, on the other hand, determines the tension of the element along its middle line.

We introduce the following designations

$$\begin{aligned}
g_1\{\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5,\Phi_6\} &= \cos^2\gamma\,\Phi_1 + 2\sin\gamma\cos\gamma\,\Phi_3 + \\
&+ \sin^2\gamma\,\Phi_2 + \sin\gamma\cos\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\Phi_6 - \frac{1}{R_{vt}}[\sin\gamma\cos\gamma\,(\Phi_5-\Phi_4) + \\
&+ (\cos^2\gamma-\sin^2\gamma)\,\Phi_6], \\
g_2\{\ldots\} &= \sin\gamma\cos\gamma\,(\Phi_2-\Phi_1) + (\cos^2\gamma-\sin^2\gamma)\,\Phi_3 + \\
&+ \left(\frac{\cos^2\gamma}{R_2}-\frac{\sin^2\gamma}{R_1}\right)\Phi_6 + \frac{1}{R_t}[\sin\gamma\cos\gamma\,(\Phi_5-\Phi_4) + (\cos^2\gamma-\sin^2\gamma)\,\Phi_6], \\
g_3\{\ldots\} &= \frac{\cos\gamma}{A_1A_2}\left(\frac{\partial A_2\Phi_4}{\partial\alpha_1} + \frac{1}{A_1}\frac{\partial A_1^2\Phi_6}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\Phi_5\right) + \qquad (3.19) \\
&+ \frac{\sin\gamma}{A_1A_2}\left(\frac{\partial A_1\Phi_5}{\partial\alpha_2} + \frac{1}{A_2}\frac{\partial A_2^2\Phi_6}{\partial\alpha_1} - \frac{\partial A_1}{\partial\alpha_2}\Phi_4\right) + \\
&+ \left(-\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}\right)\left[\sin\gamma\cos\gamma\,(\Phi_5-\Phi_4) + (\cos^2\gamma-\sin^2\gamma)\,\Phi_6\right], \\
g_4\{\ldots\} &= \cos^2\gamma\,\Phi_4 + 2\sin\gamma\cos\gamma\,\Phi_6 + \sin^2\gamma\,\Phi_5, \\
g_5^i\{\ldots\} &= \left\{\sin\gamma\cos\gamma\,(\Phi_5-\Phi_4) + (\cos^2\gamma-\sin^2\gamma)\,\Phi_6\right\}_{s=s_i}
\end{aligned}$$

By using these we can write the strain components of the normal element in the following manner /37

$$\varkappa_{tt}=g_1\left\{\varkappa_2,\ \varkappa_1,\ -\tau,\ -\varepsilon_2,\ -\varepsilon_1,\ \frac{\omega}{2}\right\},$$
$$-\varkappa_{t_0}=g_2\{\ldots\},\ \varkappa_{tn}=g_3\{\ldots\},\ -\varepsilon_{tt}=g_4\{\ldots\}. \tag{3.20}$$

Now we turn to the determination of vectors for displacements and angles of rotation. First of all, if the strain components ε_1, ε_2, ..., τ are assigned, then in view of (3.18) we may assume that the strain components of the normal element are known. This element is associated with some line Γ along which the angle $\gamma = \gamma(s)$ is assumed to be a known function.

Integrating expression (3.14) along Γ from some initial point s_0 to the running point, we obtain an expression for the vector of rotation angles

$$\vec{\Omega}_t=\vec{\Omega}_t^0+\int_{s_0}^{s_t}\vec{\varkappa}_t\,ds_t' \tag{3.21}$$

(the sign "0" here and below will be used with quantities which have been computed for $s_t = s_0$). In order to determine the vector $\vec{\Omega}$ it is necessary to make use of equalities (3.12)-(3.13).

Proceeding with the analysis we find, from (3.8),

$$\vec{U}=\vec{U}^0+\int_{s_0}^{s_t}\left\{\varepsilon_{tt}\vec{t}'+[\vec{\Omega}_t,\ \vec{t}']\right\}ds_t'. \tag{3.22}$$

Let $\vec{r}_0$ and $\vec{r}$ be the radius vectors, respectively, of the initial and running point on the curve. Then the following obvious relationship is valid

$$\vec{t}=\frac{d(\vec{r}-\vec{r}_0)}{ds_t}. \tag{3.23}$$

By means of this relationship, taking into account (3.21), we obtain /38

$$\int_{s_0}^{s_t}[\vec{\Omega}_t, \vec{t}']\,ds_t' = \int_{s_0}^{s_t}\left[\vec{\Omega}_t, \frac{d(\vec{r}'-\vec{r}_0)}{ds_t'}\right]ds_t' =$$

$$= [\vec{\Omega}_t, (\vec{r}-\vec{r}_0)] - \int_{s_0}^{s_t}[\vec{x}_t, \vec{r}'-\vec{r}_0]\,ds_t'.$$

By substituting expression (3.21) for $\bar{\Omega}_t$, we replace the last term in (3.22) with the integral obtained in this manner. As a result we obtain an expression for the vector of displacements in terms of the strain components

$$\vec{U} = \vec{U}^0 + [\vec{\Omega}_t^0, (\vec{r}-\vec{r}_0)] + \left[\int_{s_0}^{s_t}\vec{x}_t u_t' ds_t', \vec{r}-\vec{r}_0\right] + \int_{s_0}^{s_t}\{[\vec{r}'-\vec{r}_0, \vec{x}_t] + \varepsilon_{tt}\vec{t}'\}\,ds_t' \qquad (3.24)$$

In view of the obvious identity

$$\frac{d}{ds_t}\left\{\left[\int_{s_0}^{s_t}\vec{x}_t ds_t', \vec{r}'-\vec{r}_0\right] - \int_{s_0}^{s_t}[\vec{x}_t, \vec{r}'-\vec{r}_0]\,ds_t'\right\} = \left[\int_{s_0}^{s_t}\vec{x}_t ds_t', \vec{t}\right]$$

we also have a slightly different expression

$$\vec{U} = \vec{U}^0 + [\vec{\Omega}_t^0, \vec{r}-\vec{r}_0] + \int_{s_0}^{s_t}\left\{\left[\int_{s_0}^{s_t'}\vec{x}_t ds_t', \vec{t}'\right] + \varepsilon_{tt}\vec{t}'\right\}ds_t'. \qquad (3.24')$$

The first two terms in the above expression describe the displacement of the shell as a whole rigid element.

Relationships (3.24), (3.24') and (3.21) give us direct expression for the vectors of displacement and angles of rotation in terms of the known strain components. These expressions are of value only in the case when the values of the vectors of displacement and angles of rotation do not depend on the form of the curve which connects the initial point and the running point of the middle surface (under conditions, of course, that the curves joining these points do not envelop some inner contour). Let us show that the expressions which have been introduced have this property. To be specific, we consider expression (3.24). The independence of the integration path is obviously achieved, because the following integrals are equal to zero along any /39

closed contour Γ_0 which bounds part of the middle surface S_0, which does not contain internal contours

$$J_1=\oint_{\Gamma_0}\vec{\varkappa}_t ds_t,$$
$$J_2=\oint_{\Gamma_0}\left\{\left[\vec{r}-\vec{r}_0,\ \vec{\varkappa}_t\right]+\varepsilon_{tt}\vec{t}\right\}ds_t. \tag{3.25}$$

First of all, since the strain components are continuous and single-valued, we obtain the following expression, in view of equalities (3.14) and (3.12),

$$J_1=\oint_{\Gamma_0}\vec{\varkappa}_t ds_t=\oint_{\Gamma_0}\left[\frac{d\vec{\Omega}}{ds_t}-\frac{d\varepsilon_{tn}\vec{n}}{ds_t}\right]ds_t=\oint_{\Gamma_0}\frac{d\vec{\Omega}}{ds_t}ds_t= \tag{3.26}$$
$$=\oint_{\Gamma_0}\left[-\frac{\sin\gamma}{A_1}\frac{\partial\vec{\Omega}}{\partial\alpha_1}+\frac{\cos\gamma}{A_2}\frac{\partial\vec{\Omega}}{\partial\alpha_2}\right]ds_t.$$

We shall now use the Ostrogradskiy equations (ref. 40, Part II). Since, in our notation (fig. 3)

$$\frac{A_1 d\alpha_1}{ds_t}=-\sin\gamma,\ \frac{A_2 d\alpha_2}{ds_t}=\cos\gamma, \tag{3.27}$$

the latter have the form

$$\oint_{\Gamma_0}\frac{\cos\gamma}{A_2}\Phi ds_t=\oint_{\Gamma_0}\Phi d\alpha_2=\iint_{S_0}\frac{\partial\Phi}{\partial\alpha_1}d\alpha_1 d\alpha_2,$$
$$\oint_{\Gamma_0}\frac{\sin\gamma}{A_1}\Phi ds_t=-\oint_{\Gamma_0}\Phi d\alpha_1=\iint_{S_0}\frac{\partial\Phi}{\partial\alpha_2}d\alpha_1 d\alpha_2 \tag{3.28}$$

Here Γ_0 is the contour which bounds part of the middle surface S_0. In this case the contour is traced in such a way that the region remains to the left. Applying the Ostrogradskiy equations to integral (3.26) we obtain

$$J_1=\iint_{S_0}\left\{-\frac{\partial^2\vec{\Omega}}{\partial\alpha_1\partial\alpha_2}+\frac{\partial^2\vec{\Omega}}{\partial\alpha_2\partial\alpha_1}\right\}d\alpha_1 d\alpha_2=0,$$

because, according to the continuity equations (2.14) the expression under the integral sign is equal to zero.

In a completely analogous fashion

$$J_2^{(1)}=\oint_{\Gamma_0}\left[\vec{r}-\vec{r}_0,\ \vec{\varkappa}_t\right]ds_t=\oint_{\Gamma_0}\left[\vec{r}\quad\vec{r}_0,\ \frac{d\vec{\Omega}}{ds_t}-\underline{\frac{d\varepsilon_t\vec{n}}{ds_t}}\right]ds_t.$$

Integrating the underlined term in parts and transforming the contour integral into an area integral, we obtain

$$J_2^{(1)}=\iint_{S_0}\left[\vec{r}-\vec{r}_0,\ -\frac{\partial^2\vec{\Omega}}{\partial\alpha_1\partial\alpha_2}+\frac{\partial^2\vec{\Omega}}{\partial\alpha_2\partial\alpha_1}\right]d\alpha_1d\alpha_2+$$

$$+\iint_{S_0}\left\{\left[A_2\vec{e}_2,\ -\frac{\partial\vec{\Omega}}{\partial\alpha_1}\right]+\left[A_1\vec{e}_1,\ \frac{\partial\vec{\Omega}}{\partial\alpha_2}\right]\right\}d\alpha_1d\alpha_2+\oint_{\Gamma_0}\varepsilon_t\vec{\nu}\,ds_t.$$

The first integral which enters into this expression is equal to zero, in view of the same equations of continuity, and may be dropped. Combining the contour integral with the last term in (3.25), we obtain the following expression, after simple transformations, /40

$$J_2=\iint_{S_0}\left\{\left[-\tau_1-\frac{1}{A_1A_2}\left(\frac{\partial A_1\varepsilon_1}{\partial\alpha_2}-\frac{1}{A_2}\frac{\partial}{\partial\alpha_1}\left(A_2^2\frac{\omega}{2}\right)-\frac{\partial A_1}{\partial\alpha_2}\varepsilon_2\right)\right]\vec{e}_1-\right.$$
$$-\left[\tau_2-\frac{1}{A_1A_2}\left(\frac{\partial A_2\varepsilon_2}{\partial\alpha_1}-\frac{1}{A_1}\frac{\partial}{\partial\alpha_2}\left(A_1^2\frac{\omega}{2}\right)-\frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right)\right]\vec{e}_2-$$
$$\left.-\left[\tau_1+\frac{\omega_2}{R_1}-\left(\tau_2+\frac{\omega_1}{R_2}\right)\right]\vec{n}\right\}A_1A_2d\alpha_1d\alpha_2.$$

However, the last expression is equal to zero in view of equality (2.13). Thus we have shown that the integral in question is equal to zero along any closed path, and consequently the path of integration may be selected in an arbitrary manner.

We note that in the case when the mean surface represents a multiconnected region, even if the strain components are single-valued, the displacements may not be single-valued. The question concerning the multivalued displacements will be considered in more detail in Part II.

CHAPTER III. THE THEORY OF STRESSES

1. Forces and Moments

In line with the idea of reducing a three-dimensional problem of elasticity to a two-dimensional problem, we replace the stresses acting on a normal element of length ds_t by the principal vector and the principal moment /41

$$\vec{T}_\nu ds_t = \int_{-h/2}^{h/2} \vec{\sigma}_\nu ds_t^\zeta d\zeta, \quad \vec{M}_\nu ds_t = \int_{-h/2}^{h/2} [\zeta \vec{n}, \vec{\sigma}_\nu] ds_t^\zeta d\zeta. \tag{1.1}$$

The components of the principal vector and of the principal moment computed per unit length of the mean line of the normal element will be called the forces and moments, respectively. Since

$$ds_t^\zeta = (1 + \zeta/R_t)\, ds_t,$$

the expressions for the force vectors and moment vectors are written in the form

$$\vec{T}_\nu = \int_{-h/2}^{h/2} \vec{\sigma}_\nu (1 + \zeta/R_t)\, d\zeta, \quad \vec{M}_\nu = \int_{-h/2}^{h/2} [\vec{n}, \vec{\sigma}_\nu] (1 + \zeta/R_t)\, \zeta d\zeta. \tag{1.2}$$

Resolving these vectors along the orthogonal unit vectors which are fixed with respect to the normal element, we obtain (fig. 8)

$$\begin{aligned} \vec{T}_\nu &= T_{\nu\nu}\vec{\nu} + T_{\nu t}\vec{t} + T_{\nu n}\vec{n}, \\ \vec{M}_\nu &= M_{\nu t}\vec{t} - M_{\nu\nu}\vec{\nu}. \end{aligned} \tag{1.3}$$

We shall call the quantity $T_{\nu\nu}$ the normal force, $T_{\nu t}$ the shear force, $T_{\nu n}$ the cutting force, $M_{\nu\nu}$ the bending moment, $M_{\nu t}$ the torsional moment.

Considering in sequence the cross sections which pass through the coordinates of the line we have /42

$$\begin{aligned}
\vec{T}_1 &= \int_{-h/2}^{h/2} \vec{\sigma}_1 (1 + \zeta/R_2)\, d\zeta = T_1\vec{e}_1 + T_{12}\vec{e}_2 + T_{1n}\vec{n}\,,\\
\vec{M}_1 &= \int_{-h/2}^{h/2} [\vec{n}, \vec{\sigma}_1] (1 + \zeta/R_2)\,\zeta d\zeta = M_1\vec{e}_2 - M_{12}\vec{e}_1,\\
\vec{T}_2 &= \int_{-h/2}^{h/2} \vec{\sigma}_2 (1 + \zeta/R_1)\, d\zeta = T_{21}\vec{e}_1 + T_2\vec{e}_2 + T_{2n}\vec{n},\\
\vec{M}_2 &= \int_{-h/2}^{h/2} [\vec{n}, \vec{\sigma}_2] (1 + \zeta/R_1)\,\zeta d\zeta = M_{21}\vec{e}_2 - M_2\vec{e}_1.
\end{aligned} \tag{1.4}$$

By comparing the components of the above vector expressions along the coordinate axes, we obtain the following expressions for the forces and the moments

$$\begin{aligned}
T_1 &= \int_{-h/2}^{h/2} \sigma_{11} (1 + \zeta/R_2)\, d\zeta,\\
T_2 &= \int_{-h/2}^{h/2} \sigma_{22} (1 + \zeta/R_1)\, d\zeta;\\
T_{12} &= \int_{-h/2}^{h/2} \sigma_{12} (1 + \zeta/R_2)\, d\zeta, \qquad T_{21} = \int_{-h/2}^{h/2} \sigma_{21} (1 + \zeta/R_1)\, d\zeta;\\
T_{1n} &= \int_{-h/2}^{h/2} \sigma_{1\zeta} (1 + \zeta/R_2)\, d\zeta, \qquad T_{2n} = \int_{-h/2}^{h/2} \sigma_{2\zeta} (1 + \zeta/R_1)\, d\zeta;\\
M_1 &= \int_{-h/2}^{h/2} \sigma_{11} (1 + \zeta/R_2)\,\zeta d\zeta, \qquad M_2 = \int_{-h/2}^{h/2} \sigma_{22} (1 + \zeta/R_1)\,\zeta d\zeta;\\
M_{12} &= \int_{-h/2}^{h/2} \sigma_{12} (1 + \zeta/R_2)\,\zeta d\zeta, \qquad M_{21} = \int_{-h/2}^{h/2} \sigma_{21} (1 + \zeta/R_1)\,\zeta d\zeta.
\end{aligned} \tag{1.5}$$

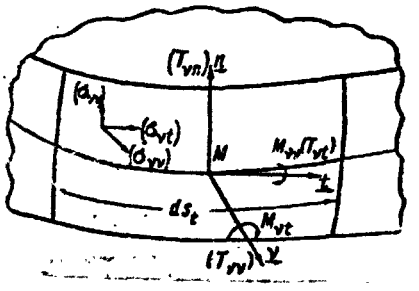

Figure 8

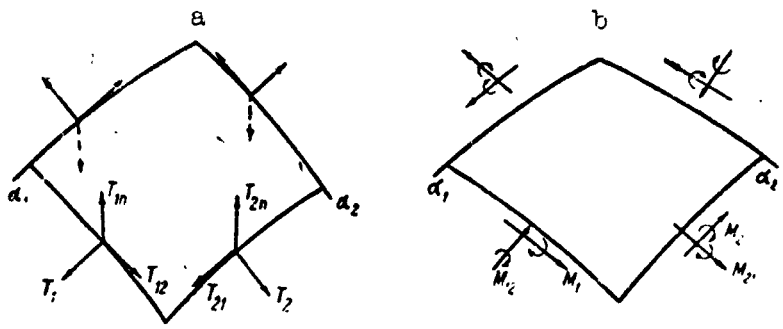

Figure 9

The last expressions make it possible for us to determine the positive directions for the forces and moments, which are in agreement with those accepted in the theory of elasticity for the positive directions of stress components. These directions are shown in figure 9a and b. We note that even though $\sigma_{12} = \sigma_{21}$, as it follows from (1.5), generally speaking, $T_{12} \neq T_{21}$, $M_{12} \neq M_{21}$. Thus, in the theory of shells the parity law for shear forces and torsional moments does not exist.

In Chapter II, when we considered the deformation picture we neglected /43 the effect of shear strains $e_{1\zeta}$, $e_{2\zeta}$, corresponding to the stresses $\sigma_{1\zeta}$, $\sigma_{2\zeta}$. It would seem, that as a result of this, we would have to neglect the cutting forces T_{1n} and T_{2n}. However, this cannot be done because the latter play a substantial role in the equilibrium conditions of the shell.

Thus in the theory of thin shells the cutting forces are not associated directly with stresses, but rather are determined from the equilibrium conditions of the shell.

2. Equilibrium Equations. Static Boundary Conditions

In the derivation of the equilibrium equations we shall make use of the principle of possible displacements. Let us assume that the shell, which is subjected to external forces and which is considered as a three-dimensional body, is in a state of equilibrium, and that the displacements which occur in it are determined by the displacement vector $\vec{U}^{\zeta}$. Let us subject the points of the shell to certain additional displacements (variations) $\delta\vec{U}^{\zeta}$. The variations $\delta\vec{U}^{\zeta}$, in addition to the natural requirement of continuity and the specific number of derivatives, will also be required to be geometrically permissible, i.e., on those regions of the body's surface where the displacements or some component

of the displacement vector are assigned, the corresponding components of variation must be equal to zero. In other words, the variations of displacements must be compatible with the geometric constraints imposed on the shell.

According to the theory of possible displacements, the work of the internal forces expended to produce displacement variations is equal to the work of external forces performed in the course of the same variations. In terms of the notation used by us the principle of possible displacements has the following form /44

$$\begin{aligned}
&\iint_S \left\{ \int_{-h/2}^{h/2} (\sigma_{11}\delta e_{11} + \sigma_{22}\delta e_{22} + \sigma_{12}\delta e_{12} + \sigma_{1\zeta}\delta e_{1\zeta} + \sigma_{2\zeta}\delta e_{2\zeta} + \right. \\
&\quad \left. + \sigma_{\zeta\zeta}\delta e_{\zeta\zeta})(1+\zeta/R_1)(1+\zeta/R_2)\right\} A_1A_2 d\alpha_1 d\alpha_2 = \\
&\qquad = \int_g \left\{ \int_{-h/2}^{h/2} \vec{\sigma}_\nu \delta\vec{U}^\zeta (1+\zeta/R_t)\, d\zeta \right\} ds_t + \\
&+ \iint_S \left\{ \int_{-h/2}^{h/2} \vec{F}\delta\vec{U}^\zeta (1+\zeta/R_1)(1+\zeta/R_2)\, d\zeta \right\} A_1A_2 d\alpha_1 d\alpha_2 + \\
&+ \iint_S \left\{ \vec{\sigma}_\zeta^{h/2} \cdot \delta\vec{U}^{h/2} (1+h/2R_1)(1+h/2R_2) \right\} A_1A_2 d\alpha_1 d\alpha_2 + \\
&+ \iint_S \left\{ \vec{\sigma}_\zeta^{-h/2} \delta\vec{U}^{-h/2} (1-h/2R_1)(1-h/2R_2) \right\} A_1A_2 d\alpha_1 d\alpha_2.
\end{aligned} \tag{2.1}$$

Here g designates the boundary of the region containing the middle surface. The integral in the left part represents the work performed by internal forces. The first integral in the right part represents the work performed by the surface forces applied to the lateral surface, while the second integral represents the work of bulk forces. Finally, the third and fourth integrals represent the work of surface forces, which are applied to the upper and lower faces of the shell.

Since we have adopted the kinematic Kirchhoff hypothesis, the displacement vector is equal to

$$\begin{aligned}
\vec{U}^\zeta &= \vec{U} + [\vec{\Omega}, \zeta\vec{n}] \quad \text{(in region)} \\
\vec{U}^\zeta &= \vec{U} + [\vec{\Omega}_t, \zeta\vec{n}] \quad \text{(on contour)}
\end{aligned} \tag{II, 1.19'}$$

Substituting these expressions into the right side of relationship (2.1), we obtain the following expression for the work performed by external forces (designating it by δL)

$$\delta L = \int_g \left\{ \delta\vec{U} \cdot \int_{-h/2}^{h/2} \vec{\sigma}_\nu (1+\zeta/R_t)\, d\zeta + \delta\vec{\Omega}_t \cdot \int_{-h/2}^{h/2} [\vec{n}, \vec{\sigma}_\nu] (1+\zeta/R_t)\zeta\, d\zeta \right\} ds_t + \tag{2.2}$$

$$+\iint_S\left\{\delta\vec{U}\cdot\int_{-h/2}^{h/2}\vec{F}(1+\zeta/R_1)(1+\zeta/R_2)\,d\zeta+\right.$$
$$\left.+\delta\vec{\Omega}\cdot\int_{-h/2}^{h/2}\left[\vec{n},\vec{F}\right](1+\zeta/R_1)(1+\zeta/R_2)\,\zeta d\zeta\right\}A_1A_2d\alpha_1d\alpha_2+$$
$$+\iint_S\left\{\delta\vec{U}\cdot\vec{\sigma}_\zeta^{h/2}+\frac{h}{2}\delta\vec{\Omega}\cdot\left[\vec{n},\vec{\sigma}_\zeta^{h/2}\right]\right\}(1+h/2R_1)(1+h/2R_2)\times$$
$$\times A_1A_2d\alpha_1d\alpha_2+\iint_S\left\{\delta\vec{U}\cdot\vec{\sigma}_{-\zeta}^{-h/2}-\frac{h}{2}\delta\vec{\Omega}\cdot\left[\vec{n},\vec{\sigma}_{-\zeta}^{-h/2}\right]\right\}\times$$
$$\times(1-h/2R_1)(1-h/2R_2)A_1A_2d\alpha_1d\alpha_2.$$

By using designations (1.2) we can reduce it to a compact form /45

$$\delta L=\int_g\left\{\vec{T}_v\cdot\delta\vec{U}+\vec{M}_v\cdot\delta\vec{\Omega}_t\right\}ds_t+$$
$$+\iint_S\left\{\vec{p}\cdot\delta\vec{U}+\vec{m}\cdot\delta\vec{\Omega}\right\}A_1A_2d\alpha_1d\alpha_2, \tag{2.3}$$

where

$$\vec{p}=\int_{-h/2}^{h/2}\vec{F}(1+\zeta/R_1)(1+\zeta/R_2)\,d\zeta+\vec{\sigma}_\zeta^{h/2}(1+h/2R_1)(1+h/2R_2)+$$
$$+\vec{\sigma}_{-\zeta}^{-h/2}(1-h/2R_1)(1-h/2R_2),$$
$$\vec{m}=\int_{-h/2}^{h/2}\left[\vec{n},\vec{F}\right](1+\zeta/R_1)(1+\zeta/R_2)\,\zeta d\zeta+ \tag{2.4}$$
$$+\frac{h}{2}\left[\vec{n},\vec{\sigma}_\zeta^{h/2}\right](1+h/2R_1)(1+h/2R_2)-$$
$$-\frac{h}{2}\left[\vec{n},\vec{\sigma}_{-\zeta}^{-h/2}\right](1-h/2R_1)(1-h/2R_2)$$

are the principal vector and the principal moment of external forces reduced to the middle surface and referred to a unit area of the middle surface.

Utilizing expressions (1.3) (II, 3.6, 3.1), we write expression (2.3) in expanded form

$$\delta L=\int_g\left\{T_v\delta u_v+T_{vt}\delta u_t+T_{vn}\delta w+M_v\delta\vartheta_v+M_{vt}\delta\vartheta_t\right\}ds_t+$$
$$+\iint_S\left\{p_1\delta u_1+p_2\delta u_2+p_n\delta w+m_1\delta\vartheta_1+m_2\delta\vartheta_2\right\}A_1A_2d\alpha_1d\alpha_2. \tag{2.5}$$

The advantage of the relationship obtained in this manner is the simplici of its structure: each term under the integral sign of the expression represents a product of the generalized force and the corresponding variation of the generalized displacement. The disadvantage of this expression is the fact that not all of the variations of the generalized displacements are independent. It is easy to eliminate this difficulty.

First of all, according to (II, 3.4)$_3$

$$\int_s M_{vt}\delta\vartheta_t ds_t = \int_s M_{vt}\delta\left(-\frac{dw}{ds_t}+\frac{u_t}{R_t}-\frac{u_v}{R_{vt}}\right)ds_t = \int_s\left\{-\frac{M_{vt}}{R_{vt}}\delta u_v+\frac{M_{vt}}{R_t}\delta u_t+\frac{dM_{vt}}{ds_t}\delta w\right\}ds_t+\sum_i Q_n^i\delta w_i, \tag{2.6}$$

where

$$Q_n^i = \{M_{vt}^+ - M_{vt}^-\}_i \tag{2.7}$$

is the discontinuity in the torsional moment at the i-th point of its dis- /46
continuity which has become apparent during differentiation by parts. Strictly speaking, the term outside the integral sign should have been written in the form

$$\sum_i\{(M_{vt}\delta w)^+ - (M_{vt}\delta w)^-\}.$$

However, since the quantity w is continuous along the contour (its discontinuity would have meant the discontinuity of the middle surface at the point $s = s_1$), the form of notation (2.6) is valid. A comparison of the term outside the integral sign with the remaining terms of equality (2.5) makes it possible to identify the quantities Q_n^i with the normal concentrated forces, which are

applied at the points of discontinuity of the torsional moment.

Furthermore,

$$\iint_S\{m_1\delta\vartheta_1+m_2\delta\vartheta_2\}A_1A_2d\alpha_1d\alpha_2 = \iint_S\left\{m_1\delta\left(-\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}+\frac{u_1}{R_1}\right)+m_2\delta\left(-\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}+\frac{u_2}{R_2}\right)\right\}A_1A_2d\alpha_1d\alpha_2 =$$
$$= -\iint_S\left\{A_2m_1\frac{\partial\delta w}{\partial\alpha_1}+A_1m_2\frac{\partial\delta w}{\partial\alpha_2}\right\}d\alpha_1d\alpha_2+\iint_S\left\{\frac{m_1}{R_1}\delta u_1+\frac{m_2}{R_2}\delta u_2\right\}A_1A_2d\alpha_1d\alpha_2.$$

Integrating the first integral in parts and utilizing the Ostrogradskiy equations (II, 3.28)

$$\iint_S \frac{\partial \Phi}{\partial \alpha_1} d\alpha_1 d\alpha_2 = \oint_\Gamma \frac{\cos \gamma}{A_2} \Phi ds_t, \quad \iint_S \frac{\partial \Phi}{\partial \alpha_2} d\alpha_1 d\alpha_2 = \oint_\Gamma \frac{\sin \gamma}{A_1} \Phi ds_t, \tag{2.8}$$

we obtain

$$\begin{aligned} \iint_S (m_1 \delta \vartheta_1 + m_2 \delta \vartheta_2) A_1 A_2 d\alpha_1 d\alpha_2 = -\int_\Gamma m_\nu \delta w ds_t + \\ + \iint_S \left\{ \frac{m_1}{R_1} \delta u_1 + \frac{m_2}{R_2} \delta u_2 + \frac{1}{A_1 A_2} \left(\frac{\partial A_2 m_1}{\partial \alpha_1} + \frac{\partial A_1 m_2}{\partial \alpha_2} \right) \delta w \right\} \times \\ \times A_1 A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{2.9}$$

where

$$m_\nu = \cos \gamma \, m_1 + \sin \gamma \, m_2. \tag{2.10}$$

Substituting expressions (2.9) and (2.6) into equality (2.5), we write the variation in the work performed by the external forces in the form

$$\begin{aligned} \delta L = \int_\Gamma (Q_\nu \delta u_\nu + Q_t \delta u_t + Q_n \delta w + M_\nu \delta \vartheta_\nu) ds_t + \\ + \sum_i Q_n^i \delta w_i + \iint_S (q_1 \delta u_1 + q_2 \delta u_2 + q_n \delta w) A_1 A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{2.11}$$

Here

$$\begin{aligned} q_1 = p_1 + \frac{m_1}{R_1}, \quad q_2 = p_2 + \frac{m_2}{R_2}, \\ q_n = p_n + \frac{1}{A_1 A_2} \left(\frac{\partial A_2 m_1}{\partial \alpha_1} + \frac{\partial A_1 m_2}{\partial \alpha_2} \right), \end{aligned} \tag{2.12}$$

$$Q_\nu = T_\nu - \frac{M_{\nu t}}{R_{\nu t}}, \quad Q_t = T_{\nu t} + \frac{M_{\nu t}}{R_t};$$

$$Q_{\nu n} = T_{\nu n} + \frac{dM_{\nu t}}{ds_t} - m_\nu. \tag{2.13}$$

We shall call the latter quantities the generalized boundary forces. /47

Now let us turn to the transformation of the integral contained in the latter part of equality (2.1). As we have already pointed out, it consists of

the work performed by the external forces for the variations of the displacements.

It was shown in Chapter II that the adopted kinematic Kirchhoff hypothesis leads to the equalities

$$e_{1\zeta} = e_{2\zeta} = e_{\zeta\zeta} = 0. \tag{2.14}$$

Therefore the expression for the work performed by the internal forces (we designate them by δA) may be taken in a simpler form

$$\delta A = \iint_S \left\{ \int_{-h/2}^{h/2} (\sigma_{11}\delta e_{11} + \sigma_{12}\delta e_{12} + \sigma_{22}\delta e_{22})(1 + \zeta/R_1)(1 + \zeta/R_2)\, d\zeta \right\} \times \\ \times A_1 A_2 d\alpha_1 d\alpha_2. \tag{2.15}$$

If, in this expression we replace e_{11}, e_{12}, e_{22} by expressions from (II, 2.3) and utilize equality (1.5), we obtain

$$\delta A = \iint_S \{ T_1\delta\varepsilon_1 + T_2\delta\varepsilon_2 + \underline{T_{12}\delta\omega_1 + T_{21}\delta\omega_2} + M_1\delta\varkappa_1 + \\ + M_2\delta\varkappa_2 + \underline{M_{12}\delta\tau_1 + M_{21}\delta\tau_2} \} A_1 A_2 d\alpha_1 d\alpha_2. \tag{2.16}$$

Let us transform the terms which are underlined. If in these terms we replace τ_1 and τ_2 by their expressions in terms of τ and ω (see (II, 2.6)), we find

$$T_{12}\delta\omega_1 + T_{21}\delta\omega_2 + M_{12}\delta\tau_1 + M_{21}\delta\tau_2 = \\ = (M_{12} + M_{21})\delta\tau + \left(T_{12} - \frac{M_{21}}{R_2}\right)\delta\omega_1 + \left(T_{21} - \frac{M_{12}}{R_1}\right)\delta\omega_2. \tag{2.17}$$

The following identity is valid

$$\left(T_{12} - \frac{M_{21}}{R_2}\right) - \left(T_{21} - \frac{M_{12}}{R_1}\right) = 0. \tag{2.18}$$

It is easy to convince ourselves that this identity is satisfied, due to the symmetry of the stress components. Indeed, according to (1.5)

$$T_{12}-\frac{M_{21}}{R_2}-T_{21}+\frac{M_{12}}{R_1}=\int\limits_{-h/2}^{h/2}(1+\zeta/R_1)(1+\zeta/R_2)(\sigma_{12}-\sigma_{21})\,d\zeta=0.$$

Identity (2.18) makes it possible for us to introduce the symmetric V. V. Novozhilov force. /48

$$S=T_{12}-\frac{M_{21}}{R_2}=T_{21}-\frac{M_{12}}{R_1}. \tag{2.19}$$

As the second symmetric quantity V. V. Novozhilov adopted the half sum of the torsional moments

$$H=\frac{M_{12}+M_{21}}{2}. \tag{2.20}$$

Now equality (2.17) makes it possible to express the work performed by the internal forces in terms of the introduced symmetric functions

$$\delta A=\iint\limits_S\{T_1\delta\varepsilon_1+T_2\delta\varepsilon_2+S\delta\omega+M_1\delta\varkappa_1+M_2\delta\varkappa_2+2H\delta\tau\}\times \times A_1A_2\,d\alpha_1\,d\alpha_2, \tag{2.21}$$

from which, in particular, we can see that quantities S and 2H are generalized forces, if we take the deformation components ω, τ introduced earlier as the generalized displacements.

Substituting into (2.21) the expressions for the variation in the deformation components

$$\begin{aligned}
\delta\varepsilon_1&=\frac{1}{A_1}\frac{\partial\delta u_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta u_2+\frac{\delta w}{R_1},\\
\delta\varepsilon_2&=\frac{1}{A_2}\frac{\partial\delta u_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta u_1+\frac{\delta w}{R_2},\\
\delta\omega&=\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\delta u_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\delta u_1}{A_1}\right),\\
\delta\varkappa_1&=\frac{1}{A_1}\frac{\partial\delta\vartheta_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta\vartheta_2,\\
\delta\varkappa_2&=\frac{1}{A_2}\frac{\partial\delta\vartheta_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta\vartheta_1.
\end{aligned} \tag{2.22}$$

$$\delta(2\tau)=\frac{1}{A_1}\frac{\partial\delta\vartheta_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta\vartheta_1+\frac{1}{A_2}\frac{\partial\delta\vartheta_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta\vartheta_2+$$
$$+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial\delta u_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta u_2\right)+\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial\delta u_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta u_1\right),$$

we obtain, after integration by parts and transformations by means of the Ostrogradskiy equations (2.8)

$$\begin{aligned}\delta A=\iint_S\Big\{&-\left[\frac{\partial A_2T_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2S}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_2}T_2+\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}H\right)+\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}H\right]\delta u_1-\\&-\left[\frac{\partial A_1T_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2S}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}T_1+\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}H\right)+\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}H\right]\delta u_2+\\&+A_1A_2\left(\frac{T_1}{R_1}+\frac{T_2}{R_2}\right)\delta w-\left[\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{\partial A_1H}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}H-\frac{\partial A_2}{\partial\alpha_1}M_2\right]\delta\vartheta_1-\\&-\left[\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{\partial A_2H}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}H-\frac{\partial A_1}{\partial\alpha_2}M_1\right]\delta\vartheta_2\Big\}d\alpha_1d\alpha_2+\\&+\int_s\Big\{\Big(\cos\gamma\,T_1+\sin\gamma\,S+\frac{1}{R_1}\sin\gamma\,H\Big)\delta u_1+\\&+\Big(\sin\gamma\,T_2+\cos\gamma\,S+\frac{1}{R_2}\cos\gamma\,H\Big)\delta u_2+(\cos\gamma\,M_1+\sin\gamma\,H)\,\delta\vartheta_1+\\&+(\sin\gamma\,M_2+\cos\gamma\,H)\,\delta\vartheta_2\Big\}\,ds_t.\end{aligned}\tag{2.23}$$

We replace the variations $\delta\vartheta_1$, $\delta\vartheta_2$ in the area integral by their expressions /49

$$\delta\vartheta_1=-\frac{1}{A_1}\frac{\partial\delta w}{\partial\alpha_1}+\frac{\delta u_1}{R_1},\quad \delta\vartheta_2=-\frac{1}{A_2}\frac{\partial\delta w}{\partial\alpha_2}+\frac{\delta u_2}{R_2}.\tag{2.24}$$

and in the contour integrals by the expressions (see II, 3.9, 3.10)

$$\delta\vartheta_1=\cos\gamma\,\delta\vartheta_n-\sin\gamma\,\delta\vartheta_t,\quad \delta\vartheta_2=\sin\gamma\,\delta\vartheta_n+\cos\gamma\,\delta\vartheta_t;\tag{2.25}$$
$$\delta u_1=\cos\gamma\,\delta u_n-\sin\gamma\,\delta u_t,\quad \delta u_2=\sin\gamma\,\delta u_n+\cos\gamma\,\delta u_t.\tag{2.26}$$

Furthermore, by transforming the area integral in the same manner and utilizing designations (II, 2.17) it becomes possible for us to write (2.23) in the form

$$\begin{aligned}\delta A=\iint_S\{&-L_1\{T_1,T_2,S,M_1,M_2,H\}\,\delta u_1-L_2\{\ldots\}\,\delta u_2+\\&+A_1A_2L_3\{\ldots\}\,\delta w\}\,d\alpha_1d\alpha_2+\int_s\Big\{\Big[\cos^2\gamma\,T_1+2\sin\gamma\cos\gamma\,S+\\&+\sin^2\gamma\,T_2+\sin\gamma\cos\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)H\Big]\delta u_n+\end{aligned}\tag{2.27}$$

$$+\left[\sin\gamma\cos\gamma(T_2-T_1)+(\cos^2\gamma-\sin^2\gamma)S+\left(\frac{\cos^2\gamma}{R_2}-\frac{\sin^2\gamma}{R_1}\right)H\right]\delta u_t+$$
$$+\left[\frac{\cos\gamma}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2H}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}M_2\right)+\right.$$
$$\left.+\frac{\sin\gamma}{A_1A_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2H}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}M_1\right)\right]\delta w+[\cos^2\gamma M_1+$$
$$+2\sin\gamma\cos\gamma H+\sin^2\gamma M_2]\,\delta\vartheta_1+[\sin\gamma\cos\gamma(M_2-M_1)+$$
$$+(\cos^2\gamma-\sin^2\gamma)H]\,\delta\vartheta_t\}\,ds_t.$$

Let us equate the expression for the work performed by the internal forces to the work of the external forces (2.11). In the region S the variations δu_1, δu_2, δw are arbitrary. Therefore, by equating the factors before them we obtain three equilibrium equations

$$L_1\{T_1, T_2, S, M_1, M_2, H\}=-A_1A_2q_1,$$
$$L_2\{\ldots\}=-A_1A_2q_2,\quad L_3\{\ldots\}=q_n. \tag{2.28}$$

which are written as follows in the expanded form

$$\frac{\partial A_2T_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2S}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}T_2+\frac{1}{R_1}\left(\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2H}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}M_2\right)+$$
$$+\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}H\right)+\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}H=-A_1A_2q_1,$$
$$\frac{\partial A_1T_2}{\partial\alpha_1}+\frac{1}{A_2}\frac{\partial A_2^2S}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}T_1+\frac{1}{R_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2H}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}M_1\right)+ \tag{2.29}$$
$$+\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}H\right)+\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}H=-A_1A_2q_2,$$
$$\frac{T_1}{R_1}+\frac{T_2}{R_2}-\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left(\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2H}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}M_2\right)+\right.$$
$$\left.+\frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2H}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}M_1\right)\right\}=q_n,$$

where

$$q_1=p_1+\frac{m_1}{R_1}\approx p_1,\quad q_2=p_2+\frac{m_2}{R_2}\approx p_2,$$
$$q_n=p_n+\frac{1}{A_1A_2}\left(\frac{\partial A_2m_1}{\partial\alpha_1}+\frac{\partial A_1m_2}{\partial\alpha_2}\right)\approx p_n,$$

since in the majority of the problems the quantities m_i are of order hp, /50

so that by identifying q_i and p_i we drop quantities of order h/R compared with unity.

Let us now consider the contour integral. In this integral the variation $\delta\vartheta_t$ cannot be considered independent, because according to (II, 3.4)$_3$

$$\delta\vartheta_t = -\frac{d\delta w}{ds_t} + \frac{\delta u_t}{R_t} - \frac{\delta u_\nu}{R_{\nu t}} \tag{2.30}$$

is a known quantity, as long as δu_ν, δu_t, δw are assigned on the contour. Therefore, if we replace $\delta\vartheta_t$ in the contour integral with the last expression and integrate by parts, it becomes possible (see (II, 3.19)) to express it in the form

$$\int_s \{g_1\{T_1, T_2, S, M_1, M_2, H\}\delta u_\nu + g_2\{\ldots\}\delta u_t + g_3\{\ldots\}\delta w + g_4\{\ldots\}\delta\vartheta_\nu\}\, ds_t + \sum_i g_5^i\{\ldots\}\delta w_i. \tag{2.31}$$

In practice, versions of boundary conditions are encountered in which either the generalized displacements u_ν u_t, w, ϑ_ν, w_i or the corresponding generalized $Q_{\nu\nu}$, $Q_{\nu t}$, $Q_{\nu n}$, $M_{\nu\nu}$, Q_n^i forces are assigned along the contour, so that from each pair of values some one value is assigned. Boundary conditions of this type will be called conventional in the future. In the case of conventional boundary conditions, all generalized forces in the contour integral may be assumed to be assigned values, while the variations of the generalized displacements may be considered arbitrary. Indeed, if the generalized displacement rather than the generalized force is assigned in some pair, the corresponding variation of this must be equal to zero and the corresponding term in the expression under the integral sign is in fact absent. /51

Let us assume that the boundary conditions are conventional. Then, by comparing the factors in front of the displacement variations, we obtain the static boundary conditions

$$\begin{aligned} Q_{\nu\nu} &= g_1\{T_1, T_2, S, M_1, M_2, H\} = Q_{\nu\nu}^0, \\ Q_{\nu t} &= g_2\{\ldots\} = Q_{\nu t}^0, \\ Q_{\nu n} &= g_3\{\ldots\} = Q_{\nu n}^0, \\ M_{\nu\nu} &= g_4\{\ldots\} = M_{\nu\nu}^0, \\ Q_n^i &= g_5^i\{\ldots\} = Q_n^{i0}. \end{aligned} \tag{2.32}$$

Let us write the left sides in expanded form

$$
\begin{aligned}
Q_\nu &= \cos^2\gamma\, T_1 + 2\sin\gamma\cos\gamma\, S + \sin^2\gamma\, T_2 + \sin\gamma\cos\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)H - \\
&- \sin\gamma\cos\gamma\left(\frac{1}{R_1}-\frac{1}{R_2}\right)[\sin\gamma\cos\gamma\,(M_2-M_1) + \cos^2\gamma - \sin^2\gamma)\,H], \\
Q_\alpha &= \sin\gamma\cos\gamma\,(T_2-T_1) + (\cos^2\gamma-\sin^2\gamma)\,S + \left(\frac{\cos^2\gamma}{R_2}-\frac{\sin^2\gamma}{R_1}\right)H + \\
&+ \left(\frac{\sin^2\gamma}{R_1}+\frac{\cos^2\gamma}{R_2}\right)[\sin\gamma\cos\gamma\,(M_2-M_1) + (\cos^2\gamma-\sin^2\gamma)\,H], \\
Q_{\nu n} &= \frac{\cos\gamma}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2H}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}M_2\right) + \\
&+ \frac{\sin\gamma}{A_1A_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2H}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}M_1\right) + \\
&+ \left(-\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}\right)[\sin\gamma\cos\gamma(M_2-M_1)+(\cos^2\gamma-\sin^2\gamma)H] + m_\nu \\
&\qquad (m_\nu = \cos\gamma\, m_1 + \sin\gamma\, m_2), \\
M_\nu &= \cos^2\gamma\, M_1 + 2\sin\gamma\cos\gamma\, H + \sin^2\gamma\, M_2, \\
Q_n' &= [\sin\gamma\cos\gamma\,(M_2-M_1) + (\cos^2\gamma-\sin^2\gamma)\,H]_{s=s_i}.
\end{aligned}
\tag{2.33}
$$

From the relationships obtained above we can see, first of all, that both the equations of equilibrium and the static boundary conditions are formulated in terms of the symmetric forces and moments T_1, T_2, S, M_1, N_2, H introduced by V. V. Novozhilov. The quantities T_1, T_2, T_{1n}, T_{2n}, M_{12}, M_{21} could be necessary for determining the stresses $\sigma_{1\zeta}$, $\sigma_{2\zeta}$, σ_{12}. However, the stresses $\sigma_{1\zeta}$, $\sigma_{2\zeta}$ are small and we are never interested in their values. As far as σ_{12} is concerned, it is easily determined from S and H. Indeed, according to (1.5)

$$
S = T_{12} - \frac{M_{21}}{R_2} = \int_{-h/2}^{h/2}\sigma_{12}\left(1-\frac{\zeta^2}{R_1R_2}\right)d\zeta,
$$

$$
H = \frac{M_{12}+M_{21}}{2} = \int_{-h/2}^{h/2}\sigma_{12}\left[1+\frac{\zeta}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]\zeta\,d\zeta.
$$

Neglecting the underlined terms which are small compared to unity, we obtain/52

$$
S = \int_{-h/2}^{h/2}\sigma_{12}\,d\zeta; \quad H = \int_{-h/2}^{h/2}\sigma_{12}\zeta\,d\zeta. \tag{2.34}
$$

Let us assume the simplest proposition concerning the linear distribution of the principal stresses along the thickness of the shell

$$\sigma_{12} = \sigma_{12}^0 + \frac{\zeta}{h/2}\sigma_{12}^1; \quad \sigma_1 = \sigma_1^0 + \frac{\zeta}{h/2}\sigma_1^1; \quad \sigma_2 = \sigma_2^0 + \frac{\zeta}{h/2}\sigma_2^1.$$

By substituting the first of these expressions into (2.34) we obtain

$$\sigma_{12}^0 = \frac{S}{h}; \quad \sigma_{12}^1 = \frac{6H}{h^2},$$

from which

$$\sigma_{12} = \frac{S}{h} + \frac{\zeta}{h/2}\frac{6H}{h^2}. \qquad (2.35)_3$$

Two other stresses are determined in exactly the same manner

$$\sigma_1 = \frac{T_1}{h} + \frac{\zeta}{h/2}\frac{6M_1}{h^2}; \quad \sigma_2 = \frac{T_2}{h} + \frac{\zeta}{h/2}\frac{6M_2}{h^2}. \qquad (2.35)_{1,2}$$

The terms which are constant over the thickness of the shell

$$\sigma_{T_1} = \frac{T_1}{h}, \quad \sigma_{T_2} = \frac{T_2}{h}, \quad \sigma_S = \frac{S}{h} \qquad (2.36)$$

are called tangential (sometimes chain) stresses. In the same way the following terms which varied in a linear manner

$$\sigma_{M_1} = \frac{6M_1}{h^2}, \quad \sigma_{M_2} = \frac{6M_2}{h^2}, \quad \sigma_H = \frac{6H}{h^2} \qquad (2.37)$$

are called flexural stresses.

CHAPTER IV. COMPLEX TRANSFORMATION

1. Stress Functions

By comparing the homogeneous equilibrium equations (III, 2.29) and the continuity equations (II, 2.16), we see that they are similar and that they can be obtained from each other by a simple replacement of quantities T_1, T_2, S, M_1, M_2, H, respectively, with the quantities $\varkappa_2$, $\varkappa_1$, $-\tau$, $-\epsilon_2$, $-\epsilon_1$, $\frac{\omega}{2}$. This situation points to the way which can be used to introduce stress functions, i.e., functions which satisfy the equations of equilibrium identically. Let us construct the following expressions

$$\begin{aligned}
T_1 &= T_1^* + Ehc\bar{\varkappa}_2(\bar{u}_1, \bar{u}_2, \bar{w}),\\
T_2 &= T_2^* + Ehc\bar{\varkappa}_1(\bar{u}_1, \bar{u}_2, \bar{w}),\\
S &= S^* - Ehc\bar{\tau}(\bar{u}_1, \bar{u}_2, \bar{w}),\\
M_1 &= M_1^* - Ehc\bar{\epsilon}_2(\bar{u}_1, \bar{u}_2, \bar{w}),\\
M_2 &= M_2^* - Ehc\bar{\epsilon}_1(\bar{u}_1, \bar{u}_2, \bar{w}),\\
H &= H^* + Ehc\frac{1}{2}\bar{\omega}(\bar{u}_1, \bar{u}_2, \bar{w})
\end{aligned} \tag{1.1}$$

(E is Young's modulus, $c = \frac{h}{\sqrt{12(1-\nu^2)}}$, ν is the Poisson ratio). Here $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, ..., $\bar{\tau}$ are quantities which are analogous to the strain components but which are constructed not from the displacements, but from certain functions $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$, which need not satisfy any special conditions other than the natural requirements concerning continuity and the existence of the required number of derivatives. We shall require that the system of functions T_1^*, T_2^*, S^*, M_1^*, M_2^*, H^* yield certain solutions for the nonhomogeneous system of equilibrium equations. In all other respects they are arbitrary. In particular, it is not necessary at all that any displacements be associated with these functions. In other words, these functions should not be identified with forces and with moments which occur in the shell under the action of a given surface load. We shall call them the static system of functions.

We shall show that the expressions in the right sides of relationships (1.1) give a general solution of the system of equilibrium equations, and this

justifies the designations of the left parts. Before we can prove the above propositions, it is necessary to convince ourselves that the following are true:

(a) the constructed expressions for any values $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$ yield a solution of the system of equilibrium equations;

(b) for any solution of the system of equilibrium equations functions $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$ can be found such that with their aid the solution in question may be represented in the form (1.1).

To prove both propositions we transform the left side of the first equilibrium equation by means of expression (1.1). Substituting the latter into the left side of the equation (III, 2.29) we obtain

$$\begin{aligned} &L_1\{T_1, T_2, S, M_1, M_2, H\} + A_1A_2q_1 = \\ &= Ehc\, L_1\left\{\bar{\varkappa}_2, \bar{\varkappa}_1, -\bar{\tau}, -\bar{\varepsilon}_2, -\bar{\varepsilon}_1, \frac{\bar{\omega}}{2}\right\} + \\ &+ L_1\{T_1^*, T_2^*, S^*, M_1^*, M_2^*, H^*\} + A_1A_2q_1. \end{aligned} \tag{1.2}$$

We start with the first proposition. The functions $\bar{u}_1(\alpha_1, \alpha_2)$, $\bar{u}_2(\alpha_1, \alpha_2)$, $\bar{w}(\alpha_1, \alpha_2)$ may be considered as certain displacements of points on the middle surface. Then the strain components $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, ..., $\bar{\tau}$, which are based on them, must satisfy the equations of continuity and particularly the first one. Therefore, the term in the second line of equality (1.2), which represents the left side of the first continuity equation, must be equal to zero. The term in the third line must also be equal to zero, because the functions of the static system satisfy the nonhomogeneous equilibrium equations. In this case the left part is also equal to zero. This in turn means that the expressions which have been derived satisfy the first equilibrium equation. In exactly the same way it can be established that the remaining equations satisfy the established solution. Proposition "a" has been proven.

Now let T_1, T_2, ..., H be the assigned solution of the equilibrium equations. It is required to prove that the system of equations

$$\bar{\varkappa}_2 = \frac{T_1 - T_1^*}{Ehc}, \quad \bar{\varkappa}_1 = \frac{T_2 - T_2^*}{Ehc}, \quad \ldots, \quad \frac{1}{2}\bar{\omega} = \frac{H - H^*}{Ehc} \tag{1.3}$$

is consistent in that it can be used to determine $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$. A sufficient condition for the existence of a solution to expression (1.3), as we know from chapter II, is that they satisfy the equations of continuity. We shall show

that this condition is satisfied. Turning again to relationship (1.2) we see that now its left side and the last line of the right side are equal to zero. Then the second line is also equal to zero, i.e., the first continuity equation is satisfied. In the same way we can prove that the remaining two are satisfied. Thus, we have proven the condition for the solubility of system (1.3) and consequently we have proven proposition "b". /55

It is appropriate to call the quantities $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$ the stress functions. They generalize in a natural way the Lur'ye Gol'denveyzer functions (refs. 10 and 22) for the case of a nonhomogeneous problem.

Let us substitute expression (1.1) into equalities (III, 2.33). If we take into account relationships (II, 3.18) it is easy to see that

$$\begin{aligned} Q_{\nu} &= Q_{\nu}^{*} + Ehc\bar{\varkappa}_{tt}(\bar{u}_1, \bar{u}_2, \bar{w}), \\ Q_{\alpha} &= Q_{\alpha}^{*} - Ehc\bar{\varkappa}_{t\alpha}, \\ Q_{\nu n} &= Q_{\nu n}^{*} + Ehc\bar{\varkappa}_{tn}, \\ M_{\nu} &= M_{\nu}^{*} - Ehc\bar{\varepsilon}_{tt}, \\ Q_n^i &= Q_n^{i*} - Ehc\bar{\varkappa}_n^i. \end{aligned} \tag{1.4}$$

Here

$$\varkappa_n^i = \left\{\sin\gamma\cos\gamma(\varepsilon_1 - \varepsilon_2) + \cdots \cos^2\gamma - \sin^2\gamma)\,\frac{\omega}{2}\right\}_{s=s_i} \tag{1.5}$$

The asterisks show the generalized boundary conditions constructed in accordance with the functions of the static system, while the bars above the symbols designate the strain components of the normal element computed by using $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, ..., $\bar{\tau}$.

If we introduce the vector for generalized boundary conditions

$$\vec{Q}_{\nu} = Q_{\nu\nu}\vec{\nu} + Q_{\nu t}\vec{t} + Q_{\nu n}\vec{n}, \tag{1.6}$$

then the first three equalities of (1.4) may be replaced by one vector

$$\vec{Q}_{\nu} = \vec{Q}_{\nu}^{*} - Ehc\vec{\varkappa}_t. \tag{1.7}$$

2. Principal Vector and Principal Moment of Forces and Moments Applied to the Arc of the Curve. Determination of Stress Functions

Let us consider the region $\overset{\frown}{s_0 s_t}$ of some contour Γ (fig. 10). The principal vector of forces applied to the arc in question is equal to

$$\vec{F} = \int_{s_0}^{s_t} \vec{Q}_\nu ds_t' = \int_{s_0}^{s_t} \vec{Q}_\nu^* ds_t' - Ehc \int_{s_0}^{s_t} \vec{\vec{\varkappa}}_t ds_t'. \tag{2.1}$$

By means of an equality analogous to (II, 3.14),

$$\vec{\vec{\varkappa}}_t = \frac{d\vec{\Omega}}{ds_t} \tag{2.2}$$

we obtain

$$\vec{F} = \vec{F}^* - Ehc\left\{\vec{\Omega}_t - \vec{\Omega}_t^0\right\}. \tag{2.3}$$

Here $\vec{F}^* = \int_{s_0}^{s_t} \vec{Q}_\nu^* ds_t'$ is the principal vector which corresponds to the selected system of static functions. Furthermore, the principal moment, as we can easily see from figure 10, equals /56

$$\begin{aligned}\vec{B}_{(0)} = \int_{s_0}^{s_t} M_{\nu\nu} \vec{t}\, ds_t' + \int_{s_0}^{s_t} \left[\vec{r}, \vec{Q}_\nu\right] ds_t' = \int_{s_0}^{s_t} M_{\nu\nu}^* \vec{t}\, ds_t' + \\ + \int_{s_0}^{s_t} \left[\vec{r}, \vec{Q}_\nu^*\right] ds_t' - Ehc\left\{\int_{s_0}^{s_t} \varepsilon_{tt} \vec{t}\, ds_t' + \int_{s_0}^{s_t} \left[\vec{r}, \vec{\vec{\varkappa}}_t\right] ds_t'\right\}.\end{aligned} \tag{2.4}$$

We replace $\vec{\vec{\varkappa}}_t$ by expression (2.2) and integrate the last integral by parts. At the same time, taking into account equalities (II, 3.22, 3.23), we obtain

$$\begin{aligned}\vec{B}_{(0)} = \int_{s_0}^{s_t} M_{\nu\nu}^* \vec{t}\, ds_t' + \int_{s_0}^{s_t} \left[\vec{r}, \vec{Q}_\nu^*\right] ds_t' - \\ - Ehc\left\{\left[\vec{r}, \vec{\Omega}_t\right] - \left[\vec{r}^0, \vec{\Omega}_t^0\right] + \vec{U} - \vec{U}^0\right\}.\end{aligned} \tag{2.5}$$

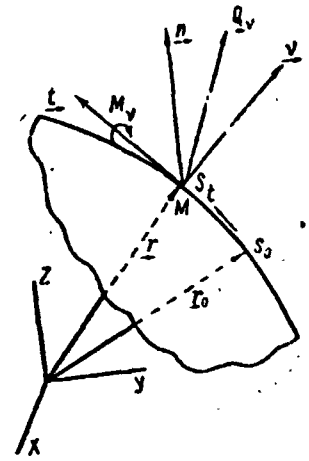

Figure 10

In the calculation of $\vec{F}$ and $\vec{B}_{(0)}$ the reference point was taken as the origin of the coordinates. If, however, we reduce the principal vector and the principal moment to the running point of the considered arc (s_t), then $\vec{F}$ will retain its value, while the position of $\vec{B}_{(0)}$ will be occupied by

$$\vec{B} = \vec{B}_{(0)} - [\vec{r}, \vec{F}].$$

Computing $\vec{B}$ by means of (2.3) and (2.5) we obtain

$$\vec{B} = \vec{B}^* - Ehc\{\vec{U} - \vec{U}^0 + [\vec{r} - \vec{r}^0, \vec{\Omega}_t^0]\} \tag{2.6}$$

where $\vec{B}^* = \int_{s_0}^{s_t} M_{\nu\nu}^* \vec{t}\, ds_t + \int_{s_0}^{s_t} [\vec{r}' - \vec{r}, \vec{Q}_*^*]\, ds_t$ is the principal moment of the static system functions.

Solving equalities (2.3), (2.6) with respect to the vectors of the stress functions and the "stress angles," we find /57

$$\vec{\Omega}_t = \vec{\Omega}_t^0 - \frac{\vec{F} - \vec{F}^*}{Ehc},$$
$$\vec{U} = \vec{U}^0 + [\vec{\Omega}_t^0, \vec{r} - \vec{r}^0] - \frac{\vec{B} - \vec{B}^*}{Ehc}. \tag{2.7}$$

These relationships can be used to formulate the static boundary conditions in terms of the stress functions and the "stress angles." In this

connection it is sufficient to apply them to the contours which bound the middle surface. In case of a singly connected surface we may write $\vec{U}^0 = \vec{\Omega}_t^0 = 0$.

Indeed, as we can see from (2.7), this is equivalent to the variation in the vector of stress functions by a term of the "rigid displacement" type

$$\vec{U} = \vec{U}^0 + \left[\vec{\Omega}_t^0, \vec{r} - \vec{r}^0\right],$$

which does not affect the shell's state of stress. In the case of an n-connected region consisting of n pairs of vector constants, only one pair can be fixed arbitrarily. The remaining ones are determined from the conditions of periodicity governing the displacement of the points on the shell. These conditions (a unique generalization of the Bredt theorem in the Saint-Venant problem concerning the torsion of a rod) may be obtained by using expressions (II, 3.14, 3.18), (4.10), (1.1) to replace $\varkappa_{tt}$, $\varkappa_{t\nu}$, $\varkappa_{tn}$, ϵ_{tt} by their expressions in terms of $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, ..., $\bar{\tau}$ under conditions of periodicity. There is no purpose in writing them down in explicit form, because in the case of multiconnected regions the formulation of boundary conditions in the form considered above is hardly expedient. It is difficult not to agree with this, if we remember that, compared with the problem of bar torsion, the number of constants increases from one to six (two of which are vector quantities).

We shall use expressions (2.7) to clarify the requirements which must be satisfied by the functions of the static system and which are important to us. Let Γ be some closed contour which, in particular, may be one of the internal contours of a multiconnected region S. We shall use the symbol $[\Phi]_\Gamma$ to designate the increment in the quantity Φ when the closed contour Γ is traced. Then, from equality (2.7) we find

$$\left[\vec{\Omega}_t\right]_\Gamma = -\frac{1}{Ehc}\left\{\vec{F}_\Gamma - \vec{l}_\Gamma\right\}, \qquad \left[\vec{U}\right]_\Gamma = -\frac{1}{Ehc}\left\{\vec{B}_\Gamma - \vec{B}_\Gamma^*\right\}. \tag{2.8}$$

Here $\vec{F}_\Gamma$ is the principal vector, while $\vec{B}_\Gamma$ is the principal moment of forces /58 and moments applied to the closed contour Γ.

Thus, during the tracing of the closed contour the stress functions may undergo a certain increment. In other words, the stress functions, generally speaking, are multivalued functions. We can eliminate this shortcoming by

agreeing to select the functions of the static system in such a way that along any closed contour they will also yield a principal vector and a principal moment, together with the unknown solution. Then $\vec{F}_\Gamma = \vec{F}^*_\Gamma$, $\vec{B}_\Gamma = \vec{B}^*_\Gamma$, and it follows from (2.8) that the stress functions are single-valued.

Let us consider two types of contours: let us assume that contour I does not envelop the internal contours (fig. 11). Then the following equality follows from the conditions of equilibrium for that part of the region which is contained inside the contour I

$$\vec{F}_\Gamma = \vec{B}_\Gamma = 0.$$

Assuming that the surface load is absent, we shall consider contour II which envelops the internal contour Γ_i.

From the equilibrium conditions for the portion of the region contained between contours II and Γ_i it follows that $\vec{F}_{II} = \vec{F}_{\Gamma_i}$, $\vec{B}_{II} = -\vec{B}_{\Gamma_i}$.

Thus the values of the principal vector and of the principal moment do not depend on the form of the contour, but only on the number of internal contours which it envelops. The forces and moments on the internal contours are usually known, so that quantities $\vec{F}_{\Gamma_i}$, $\vec{B}_{\Gamma_i}$ may be determined before the problem is solved. This substantially simplifies the construction of the static system which has the properties indicated above. In the second part of this book we shall present examples showing how to construct the functions of the static system effectively.

3. Static-Geometric and Complex Analogies

If we examine carefully the relationships which we have obtained, it is easy to note an analogy between the static and the geometric relationships and

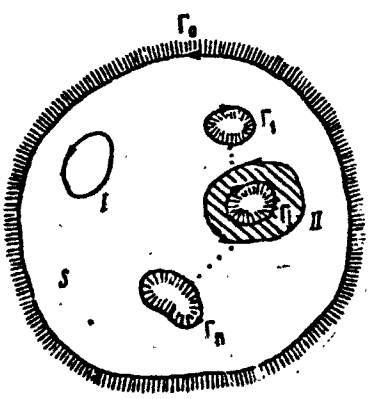

Figure 11

quantities. Indeed, if we make the stress functions u_1, u_2, w correspond to the displacements $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$, we can see that relationships (1.1) make a definite strain component correspond to each force and moment. On the other hand, relationships (1.4) establish a correspondence between the boundary static /59 and the geometric quantities. Finally, by comparing the homogeneous equilibrium equations (III, 2.9) with the continuity equations (II, 2.16) we see that they are of the same type.

The observed correspondence is usually called the static-geometric analogy. It can be formulated in the following manner: each static relationship (static quantity) may be correlated with a corresponding geometric relationship or quantity.

The concept of static-geometric analogy was introduced by A. L. Gol'denveyzer and was widely used by him in the simplification and transformation of the relationships from the theory of shells (ref. 13). We shall go somewhat further and introduce the concept of the complex alogy, according to which every static and geometric relationship and quantity has some corresponding complex relationship or quantity.

Thus we shall use the term complex displacements to designate the following complex combinations of displacements and stress functions

$$\tilde{u}_1 = u_1 + i\bar{u}_1, \quad \tilde{u}_2 = u_2 + i\bar{u}_2, \quad \tilde{w} = w + i\bar{w}. \tag{3.1}$$

Complex forces and moments are introduced by means of the following relationships

$$\begin{aligned} \tilde{T}_1 &= T_1 - iEhc\varkappa_2, & \tilde{M}_1 &= M_1 + iEhc\varepsilon_2, \\ \tilde{T}_2 &= T_2 - iEhc\varkappa_1, & \tilde{M}_2 &= M_2 + iEhc\varepsilon_1, \\ \tilde{S} &= S + iEhc\tau, & \tilde{H} &= H - iEhc\frac{\omega}{2}. \end{aligned} \tag{3.2}$$

By using relationships (1.1) they can also be represented in the form

$$\begin{aligned} \tilde{T}_1 &= T_1^* - iEhc\tilde{\varkappa}_2\left(\tilde{u}_1, \tilde{u}_2, \tilde{w}\right), & \tilde{M}_1 &= M_1 + iEhc\tilde{\varepsilon}_2, \\ \tilde{T}_2 &= T_2^* - iEhc\tilde{\varkappa}_1, & \tilde{M}_2 &= M_2 + iEhc\tilde{\varepsilon}_1, \\ \tilde{S} &= S^* + iEhc\tilde{\tau}, & \tilde{H} &= H - iEhc\frac{\tilde{\omega}}{2}. \end{aligned} \tag{3.3}$$

Here the quantities $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, ..., $\tilde{\tau}$ represent the strain components formulated from the complex displacements. Below we shall call them the components of complex deformation.

If we multiply the continuity equations (II, 2.16) by -iEhc and add them to the corresponding equilibrium equations, we obtain, by using equality (3.2), equations in terms of complex forces and moments

$$\begin{gathered} L_1\{\tilde{T}_1,\ \tilde{T}_2,\ \tilde{S},\ \tilde{M}_1,\ \tilde{M}_2,\ \tilde{H}\} = -A_1A_2q_1, \\ L_2\{\ldots\} = -A_1A_2q_2,\quad L_3\{\ldots\} = q_n. \end{gathered} \tag{3.4}$$

We further introduce complex boundary quantities, determining them by means /60 of the following equalities

$$\begin{aligned} \tilde{Q}_{vv} &= Q_{vv} - iEhc\varkappa_{tt} = Q^*_{vv} - iEhc\tilde{\varkappa}_{tt}, \\ \tilde{Q}_{vt} &= Q_{vt} + iEhc\varkappa_{tv} = Q^*_{vt} + iEhc\tilde{\varkappa}_{tv}, \\ \tilde{Q}_{vn} &= Q_{vn} - iEhc\varkappa_{tn} = Q^*_{vn} - iEhc\tilde{\varkappa}_{tn}, \\ \tilde{M}_{vv} &= M_{vv} + iEhc\varepsilon_{tt} = M^*_{vv} + iEhc\tilde{\varepsilon}_{tt}, \\ \tilde{Q}^i_n &= Q^i_n + iEhc\varkappa^i_n = Q^{i*}_n + iEhc\tilde{\varkappa}^i_n \end{aligned} \tag{3.5}$$

(in deriving these relationships we utilized equalities (1.4)). By introducing the vector for the complex boundary forces

$$\vec{\tilde{Q}}_v = \tilde{Q}_{vv}\vec{v} + \tilde{Q}_{vt}\vec{t} + \tilde{Q}_{vn}\vec{n}, \tag{3.6}$$

we can write the first three equalities of (3.5) in the form of a vector relationship

$$\vec{\tilde{Q}}_v = \vec{Q}_v + iEhc\vec{\varkappa}_t = \vec{Q}^*_v + iEhc\vec{\tilde{\varkappa}}_t. \tag{3.7}$$

Finally we multiply the first and third equality in (II, 3.18) by -iEhc; the second, fourth and also (1.5) by iEhc and add these to the corresponding expressions from (III, 2.33). As a result, taking into account the equality (II, 3.19), we obtain

$$\widetilde{Q}_{\nu\nu}=g_1\{\widetilde{T}_1,\ \widetilde{T}_2,\ \widetilde{S},\ \widetilde{M}_1,\ \widetilde{M}_2,\ \widetilde{H}\},\quad \widetilde{Q}_{\nu t}=g_2\{\ldots\},$$
$$\widetilde{Q}_{\nu n}=g_3\{\ldots\},\quad \widetilde{M}_{\nu\nu}=g_4\{\ldots\},\quad \widetilde{Q}'_n=g'_5\{\ldots\}. \tag{3.8}$$

All of the relationships obtained until now have been of a static-geometric nature. We have not taken into consideration the physical properties of the shell material at any time, so that everything we have stated is applicable to any continuous medium. In order to associate the static quantities (forces--moments) with the geometric ones (strain components), it is necessary to assign a physical law which characterizes the resistance of the shell material to deformation. Since we are considering only elastic materials, this law will be the generalized Hooke's law of the theory of elasticity.

4. Relationships Between Forces--Moments and Strain Components. Potential Strain Energy of the Middle Surface

In accordance with the general relationships of the theory of elasticity (ref. 32) the potential energy of a shell which is considered as a three-dimensional body may be written in the form

$$V=\frac{1}{2}\iint_S\left\{\int_{-h/2}^{h/2}[\sigma_{11}e_{11}+\sigma_{22}e_{22}+\sigma_{12}e_{12}+\sigma_{1\zeta}e_{1\zeta}+\sigma_{2\zeta}e_{2\zeta}+\right.$$
$$\left.+\sigma_{\zeta\zeta}e_{\zeta\zeta}](1+\zeta/R_1)(1+\zeta/R_2)\,d\zeta\right\}A_1A_2d\alpha_1d\alpha_2. \tag{4.1}$$

As in section 2 of Chapter III, the last three terms under the integral sign must be dropped. In addition, within the framework of adopted accuracy, the product $(1+\zeta/R_1)(1+\zeta/R_2)$ should be replaced by unity. Then the expression for the potential energy is substantially simplified /61

$$V=\frac{1}{2}\iint_S\left\{\int_{-h/2}^{h/2}[\sigma_{11}e_{11}+\sigma_{22}e_{22}+\sigma_{12}e_{12}]\,d\zeta\right\}A_1A_2d\alpha_1d\alpha_2. \tag{4.2}$$

For an elastic isotropic body the relationships of the generalized Hooke's law are valid

$$e_{11}=\frac{1}{E}[\sigma_{11}-\nu(\sigma_{22}+\sigma_{\zeta\zeta})],$$
$$e_{22}=\frac{1}{E}[\sigma_{22}-\nu(\sigma_{11}+\sigma_{\zeta\zeta})],\quad e_{12}=\frac{2(1+\nu)}{E}\sigma_{12}. \tag{4.3}$$

In accordance with the second (static) Kirchhoff hypothesis, which was formulated in the Introduction, we can neglect the stress $\sigma_{\zeta\zeta}$ compared with σ_{11} and σ_{22} in the above relationship.

The essence of Kirchhoff's static hypothesis can be illustrated most simply by using an example well known from the elementary course in physics, involving a cable secured at both ends with a weight hanging at its center. We can see from figure 12 that the force which occurs in the cable is equal to

$$T=\frac{1}{4}\left(\frac{L}{w_0}\right)P.$$

Thus, when the deflection is small, the transverse force generates a longitudinal tension force which exceeds it by many times. A similar situation takes place in thin shells.

Neglecting quantity $\sigma_{\zeta\zeta}$ in relationships (4.3) and solving them with respect to the stress components, we obtain

$$\sigma_{11}=\frac{E}{1-\nu^2}(e_{11}+\nu e_{22}),\quad \sigma_{22}=\frac{E}{1-\nu^2}(e_{22}+\nu e_{11}),$$
$$\sigma_{12}=\frac{E}{2(1+\nu)}e_{12}. \tag{4.4}$$

Within the framework of the adopted accuracy the strain components contained/62 here should be taken in a simplified form (see II, 2.3)

$$e_{11}=\varepsilon_1+\zeta\varkappa_1,\quad e_{22}=\varepsilon_2+\zeta\varkappa_2,\quad e_{12}=\omega+\zeta 2\tau. \tag{4.5}$$

If these expressions are substituted into the simplified relationships (4.2) and integration carried out over ζ, we obtain the following expression for the potential energy of the shell

$$V=\frac{Eh}{2(1-\nu^2)}\iint_S\left\{\varepsilon_1^2+2\nu\varepsilon_1\varepsilon_2+\varepsilon_2^2+2(1-\nu)\left(\frac{\omega}{2}\right)^2\right\}A_1A_2d\alpha_1d\alpha_2+$$
$$+\frac{Eh^3}{24(1-\nu^2)}\iint_S\{\varkappa_1^2+2\nu\varkappa_1\varkappa_2+\varkappa_2^2+2(1-\nu)\tau^2\}A_1A_2d\alpha_1d\alpha_2. \tag{4.6}$$

A more detailed analysis carried out by V. V. Novozhilov (ref. 31, p. 41) leads to the same relationship. The first part of the expression is the potential energy due to tension and shear, while the second part is due to flexure and torsion. We note that the expressions under the integral sign may be reduced to the form

$$\left[\frac{1+\nu}{2}(\varepsilon_1+\varepsilon_2)^2+\frac{1-\nu}{2}(\varepsilon_1-\varepsilon_2)^2+2(1-\nu)\left(\frac{\omega}{2}\right)^2\right],$$
$$\left[\frac{1+\nu}{2}(\varkappa_1+\varkappa_2)^2+\frac{1-\nu}{2}(\varkappa_1-\varkappa_2)^2+2(1-\nu)\tau^2\right].$$

Since $\nu < 0.5$, it follows that the potential strain energy of the shell is not negative. It becomes equal to zero only when the following equalities are satisfied simultaneously

$$\varepsilon_1=\varepsilon_2=\omega=\varkappa_1=\varkappa_2=\tau=0. \tag{4.7}$$

The displacements which satisfy these equalities are not accompanied by the deformation of the middle surface, and consequently are the displacements of the shell as a rigid whole. This obvious property of the displacements is also observed from equalities (II, 3.24), according to which the vector of displacements takes on the following form when conditions (4.7) are satisfied

$$\vec{U}=\vec{U}^0+\left[\vec{\Omega}^0,\ (\vec{r}-\vec{r}_0)\right].$$

In this case the vector $\vec{U}^0$ produces a linear displacement of points on the middle surface, while the vector $\vec{\Omega}^0$ produces its rotation around the point M ($\vec{r}_0$) by an angle $\vec{\Omega}^0$.

We now vary expression (4.6) with respect to the strain components

$$\delta V=\frac{Eh}{(1-\nu^2)}\iint_S\left\{(\varepsilon_1+\nu\varepsilon_2)\,\delta\varepsilon_1+(\varepsilon_2+\nu\varepsilon_1)\,\delta\varepsilon_2+(1-\nu)\frac{\omega}{2}\,\delta\omega\right\}\times$$
$$\times A_1A_2d\alpha_1d\alpha_2+$$
$$+\frac{Eh^3}{12(1-\nu^2)}\iint_S\left\{(\varkappa_1+\nu\varkappa_2)\,\delta\varkappa_1+(\varkappa_2+\nu\varkappa_1)\,\delta\varkappa_2+2(1-\nu)\tau d\tau\right\}\times \tag{4.8}$$
$$\times A_1A_2d\alpha_1d\alpha_2$$

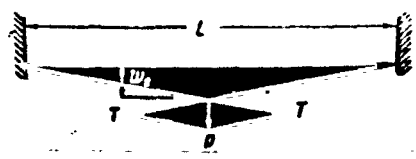

Figure 12

and return to the expression for the work performed by the internal forces (III, 2.21). Since the shell is elastic, the quantity δA may be identified with the variation of the potential strain energy δV. Then, by comparing its factors in front of the variations in the strain components with the analogous factors in (4.8) we obtain relationships which associate the forces--moments with the strain components along the middle surface

/63

$$\begin{aligned} T_1 &= \frac{Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2), & M_1 &= \frac{Eh^3}{12(1-\nu^2)}(\varkappa_1 + \nu\varkappa_2), \\ T_2 &= \frac{Eh}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1), & M_2 &= \frac{Eh^3}{12(1-\nu^2)}(\varkappa_2 + \nu\varkappa_1), \\ S &= \frac{Eh}{2(1+\nu)}\omega, & H &= \frac{Eh^3}{12(1+\nu)}\tau. \end{aligned} \tag{4.9}$$

These can be solved easily for the strain components

$$\begin{aligned} \varepsilon_1 &= \frac{1}{Eh}(T_1 - \nu T_2), & \varkappa_1 &= \frac{12}{Eh^3}(M_1 - \nu M_2), \\ \varepsilon_2 &= \frac{1}{Eh}(T_2 - \nu T_1), & \varkappa_2 &= \frac{12}{Eh^3}(M_2 - \nu M_1), \\ \omega &= \frac{2(1+\nu)}{Eh}S, & \tau &= \frac{12(1+\nu)}{Eh^3}H. \end{aligned} \tag{4.10}$$

For the sake of brevity we shall call these relationships Hooke's law. Hooke's law in this form was obtained by V. V. Novozhilov and L. I. Balabukh independently of each other and by different methods. Above we have reproduced the somewhat simpler derivation made by V. V. Novozhilov.

We note that by using expression (4.10) the potential strain energy may be expressed in terms of the forces and moments in the following manner

$$\begin{aligned} \bar{V} = \frac{1}{2Eh}\iint_S \{T_1^2 - 2\nu T_1 T_2 + T_2^2 + 2(1+\nu)S^2\} A_1 A_2 d\alpha_1 d\alpha_2 + \\ + \frac{6}{Eh^3}\iint_S \{M_1^2 - 2\nu M_1 M_2 + M_2^2 + 2(1+\nu)H^2\} A_1 A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{4.11}$$

Relationship (4.6) may be written in the form

$$V = \iint_S W A_1 A_2 d\alpha_1 d\alpha_2, \tag{4.12}$$

where the quantity

$$W=\frac{Eh}{2(1-\nu^2)}\left\{\varepsilon_1^2+2\nu\varepsilon_1\varepsilon_2+\varepsilon_2^2+2(1-\nu)\left(\frac{\omega}{2}\right)^2\right\}+ \\ +\frac{Eh^3}{24(1-\nu^2)}\{\varkappa_1^2+2\nu\varkappa_1\varkappa_2+\varkappa_2^2 2(1-\nu)\tau^2\} \tag{4.13}$$

may be pertinently called the specific strain energy. By using Hooke's law (4.10) it is easy to verify that /64

$$T_1=\frac{\partial W}{\partial\varepsilon_1},\quad T_2=\frac{\partial W}{\partial\varepsilon_2},\quad S=\frac{\partial W}{\partial\omega}, \\ M_1=\frac{\partial W}{\partial\varkappa_1},\quad M_2=\frac{\partial W}{\partial\varkappa_2},\quad 2H=\frac{\partial W}{\partial\tau}. \tag{4.14}$$

Thus the function W may be considered as the elastic potential of forces and moments.

In exactly the same way we introduce another form for the specific strain energy

$$\overline{W}=\frac{1}{2Eh}\{T_1^2-2\nu T_1T_2+T_2^2+2(1+\nu)S^2\}+ \\ +\frac{6}{Eh^3}\{M_1^2-2\nu M_1M_2+M_2^2+2(1+\nu)H^2\}. \tag{4.15}$$

and obtain

$$\overline{V}=\iint_S\overline{W}A_1A_2da_1da_2: \tag{4.16}$$

$$\varepsilon_1=\frac{\partial\overline{W}}{\partial T_1},\quad \varepsilon_2=\frac{\partial\overline{W}}{\partial T_2},\quad \omega=\frac{\partial\overline{W}}{\partial S}, \\ \varkappa_1=\frac{\partial\overline{W}}{\partial M_1},\quad \varkappa_2=\frac{\partial\overline{W}}{\partial M_2},\quad \tau=\frac{\partial\overline{W}}{\partial 2H}. \tag{4.17}$$

5. Complex Equations

The complex forces and moments introduced in section 3 are not independent quantities. Indeed, it follows from the first three relationships (3.2) that

$$T_1=\operatorname{Re}\widetilde{T}_1,\quad \varkappa_1=-\frac{1}{Ehc}\operatorname{Im}\widetilde{T}_2,$$
$$T_2=\operatorname{Re}\widetilde{T}_2,\quad \varkappa_2=-\frac{1}{Ehc}\operatorname{Im}\widetilde{T}_1, \tag{5.1}$$
$$S=\operatorname{Re}\widetilde{S},\quad \tau=\frac{1}{Ehc}\operatorname{Im}\widetilde{S}.$$

By using these expressions and Hooke's law $\left(c^2=\frac{h^2}{12(1-\nu^2)}\right)$, we obtain

$$M_1=-c\operatorname{Im}\left\{\widetilde{T}_2+\nu\widetilde{T}_1\right\},\quad \varepsilon_1=\frac{1}{Eh}\operatorname{Re}\left\{\widetilde{T}_1-\nu\widetilde{T}_2\right\},$$
$$M_2=-c\operatorname{Im}\left\{\widetilde{T}_1+\nu\widetilde{T}_2\right\},\quad \varepsilon_2=\frac{1}{Eh}\operatorname{Re}\left\{\widetilde{T}_2-\nu\widetilde{T}_1\right\}, \tag{5.2}$$
$$H=c(1-\nu)\operatorname{Im}\widetilde{S},\qquad \frac{\omega}{2}=\frac{1+\nu}{Eh}\operatorname{Re}\widetilde{S}.$$

Substituting these equalities into the last three relationships (3.2) we find /65

$$\widetilde{M}_1=ic\left\{\widetilde{T}_2-\nu\overline{\widetilde{T}}_1\right\},\quad \widetilde{M}_2=ic\left\{\widetilde{T}_1-\nu\overline{\widetilde{T}}_2\right\},\quad \widetilde{H}=-ic\left\{\widetilde{S}+\nu\overline{\widetilde{S}}\right\} \tag{5.3}$$

(Here and below, the bar over the tilde designates the conjugate sign for the respective complex quantity.)

By solving expression (3.3) for the components of complex strain and replacing the complex moments in the resulting equalities by means of expression (5.3), we obtain

$$\tilde{\varkappa}_2=\frac{i}{Ehc}\left(\widetilde{T}_1-T_1^*\right),\quad \tilde{\varepsilon}_2=\frac{1}{Eh}\left(\widetilde{T}_2-\nu\overline{\widetilde{T}}_1\right)+\frac{i}{Ehc}M_1^*,$$
$$\tilde{\varkappa}_1=\frac{i}{Ehc}\left(\widetilde{T}_2-T_2^*\right),\quad \tilde{\varepsilon}_1=\frac{1}{Eh}\left(\widetilde{T}_1-\nu\overline{\widetilde{T}}_2\right)+\frac{i}{Ehc}M_2^*, \tag{5.4}$$
$$\tilde{\tau}=-\frac{i}{Ehc}\left(\widetilde{S}-S^*\right),\quad \frac{\tilde{\omega}}{2}=\frac{1}{Eh}\left(\widetilde{S}+\nu\overline{\widetilde{S}}\right)-\frac{i}{Ehc}H^*.$$

Separating the real parts in these relationships, we become convinced that they are the relationships of Hooke's law written in a slightly different form. In the same manner the imaginary parts yield equalities (1.1), introducing the stress functions. Therefore, it is natural to call the equalities which associate the components of complex strain with the complex forces, the complex Hooke's law.

Solving equality (5.4) for the complex forces we find that

$$\begin{aligned}
&\widetilde{T}_1 = T_1^* - iEhc\widetilde{\varkappa}_2, \quad \widetilde{T}_1 = \frac{Eh}{1-\nu^2}\left(\widetilde{\varepsilon}_1 + \nu\widetilde{\overline{\varepsilon}}_2\right) - \frac{i}{c}\frac{M_2^* - \nu M_1^*}{1-\nu^2},\\
&\widetilde{T}_2 = T_2^* - iEhc\widetilde{\varkappa}_1, \quad \widetilde{T}_2 = \frac{Eh}{1-\nu^2}\left(\widetilde{\varepsilon}_2 + \nu\widetilde{\overline{\varepsilon}}_1\right) - \frac{i}{c}\frac{M_1^* - \nu M_2^*}{1-\nu^2},\\
&\widetilde{S} = S^* + iEhc\widetilde{\tau}, \quad \widetilde{S} = \frac{Eh}{1-\nu^2}\left(\frac{\widetilde{\omega}}{2} - \nu\frac{\widetilde{\overline{\omega}}}{2}\right) + \frac{i}{c}\frac{H^*}{1-\nu}.
\end{aligned} \tag{5.5}$$

Due to the very manner in which they were constructed, systems (5.4) and (5.5) must be consistent. If the complex forces are excluded from the last one, we arrive at a system of equations in terms of complex displacements

$$\begin{aligned}
&\widetilde{\varepsilon}_1 + ic\left(\widetilde{\varkappa}_2 + \nu\widetilde{\overline{\varkappa}}_1\right) = \frac{1}{Eh}\left[T_1^* - \nu T_2^* + \frac{i}{c}M_2^*\right],\\
&\widetilde{\varepsilon}_2 + ic\left(\widetilde{\varkappa}_1 + \nu\widetilde{\overline{\varkappa}}_2\right) = \frac{1}{Eh}\left[T_2^* - \nu T_1^* + \frac{i}{c}M_1^*\right],\\
&\frac{\widetilde{\omega}}{2} - ic\left(\widetilde{\tau} - \nu\widetilde{\overline{\tau}}\right) = \frac{1}{Eh}\left[(1+\nu)S^* - \frac{i}{c}H^*\right].
\end{aligned} \tag{5.6}$$

This system contains the unknowns only as complex displacements and their derivatives and may serve as the basis for the theory of shells. Since the real parts of the complex displacements are geometric quantities (displacements), while the imaginary parts are static quantities (stress functions), the method based on the utilization of system (5.6) should be referred to the so-called mixed methods of the theory of elasticity. /66

By replacing the complex moments in equations (3.4) with their expressions given by (5.3) we arrive at a system of equations containing the complex forces

$$\begin{aligned}
&L_1\left\{\widetilde{T}_1, \widetilde{T}_2, \widetilde{S}, ic\left(\widetilde{T}_2 - \nu\widetilde{\overline{T}}_1\right), ic\left(\widetilde{T}_1 - \nu\widetilde{\overline{T}}_2\right), -ic\left(\widetilde{S} + \nu\widetilde{\overline{S}}\right)\right\} = \\
&\qquad = -A_1A_2q_1,\\
&L_2\{\ldots\} = -A_1A_2q_2, \quad L_3\{\ldots\} = q_n.
\end{aligned} \tag{5.7}$$

We write this system in the form

$$\begin{aligned}
&\frac{\partial A_2\widetilde{T}_1}{\partial\alpha_1} + \frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_2 + \frac{ic}{R_1}\left\{\frac{\partial A_2\widetilde{T}_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_1 - \right.\\
&\left. - \nu\left(\frac{\partial A_2\widetilde{T}_1}{\partial\alpha_1} + \frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2} - \frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_2\right)\right\} - ic\left\{\frac{\partial}{\partial\alpha_2}\left[\frac{A_1}{R_1}\left(\widetilde{S} + \nu\widetilde{\overline{S}}\right)\right] + \right.
\end{aligned} \tag{5.8}$$

$$+\frac{\partial A_1}{\partial \alpha_2}\frac{1}{R_2}\left(\widetilde{S}+\nu\overline{\widetilde{S}}\right)\Big\}=-A_1A_2q_1,$$

$$\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_2}+\frac{1}{A_1}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_1+\frac{ic}{R_2}\left\{\frac{\partial A_1\widetilde{T}_1}{\partial\alpha_2}-\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\right.$$

$$-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_2-\nu\overline{\left(\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_1\right)}\Big\}-ic\left\{\frac{\partial}{\partial\alpha_1}\left[\frac{A_2}{R_2}\left(\widetilde{S}+\nu\overline{\widetilde{S}}\right)\right]+\right.$$

$$\left.+\frac{\partial A_2}{\partial\alpha_1}\frac{1}{R_1}\left(\widetilde{S}+\nu\overline{\widetilde{S}}\right)\right\}=-A_1A_2q_2,$$

$$\frac{\widetilde{T}_1}{R_1}+\frac{\widetilde{T}_2}{R_2}-\frac{ic}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left[\frac{\partial A_2\widetilde{T}_2}{\partial\alpha_1}-\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_1-\right.\right.$$

$$-\nu\overline{\left(\frac{\partial A_1\widetilde{T}_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_2\right)}\Big]+\frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left[\frac{\partial A_1\widetilde{T}_1}{\partial\alpha_2}-\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\right.$$

$$\left.\left.-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_2-\nu\overline{\left(\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_1\right)}\right]\right\}=q_n.$$

The system of equations containing the complex forces is a synthesis of the equilibrium equations and continuity equations and may also serve as the basis for the theory of shells. /67

We conclude this section by transforming the expressions for the complex boundary quantities (3.8), using equalities (5.3), and then writing the expressions in the form

$$\begin{aligned}\widetilde{Q}_\nu&=g_1\left\{\widetilde{T}_1,\ \widetilde{T}_2,\ \widetilde{S},\ ic\left(\widetilde{T}_2-\nu\overline{\widetilde{T}}_1\right),\ ic\left(\widetilde{T}_1-\nu\overline{\widetilde{T}}_2\right),\right.\\&\qquad\left.-ic\left(\widetilde{S}+\nu\overline{\widetilde{S}}\right)\right\},\\ \widetilde{Q}_\tau&=g_2\{\ldots\},\quad \widetilde{Q}_\nu=g_3\{\ldots\},\quad \widetilde{M}_\nu=g_4\{\ldots\},\quad \widetilde{Q}_n'=g_5'\{\ldots\}.\end{aligned}\tag{5.9}$$

Expanding them, we obtain

$$\begin{aligned}\widetilde{Q}_\nu&=\cos^3\gamma\,\widetilde{T}_1+2\sin\gamma\cos\gamma\,\widetilde{S}+\sin^2\gamma\,\widetilde{T}_2-\\&-ic\left\{\sin\gamma\cos\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\widetilde{S}+\frac{1}{R_\tau}\left(\sin\gamma\cos\gamma\left(\widetilde{T}_1-\widetilde{T}_2\right)-\right.\right.\\&\qquad\left.-\left(\cos^2\gamma-\sin^2\gamma\right)\widetilde{S}\right)-\\&-\nu\overline{\left\{\sin\gamma\cos\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\widetilde{S}+\frac{1}{R_\tau}\left(\sin\gamma\cos\gamma\left(\widetilde{T}_1-\widetilde{T}_2\right)+\right.\right.}\\&\qquad\left.\left.\left.+\left(\cos^2\gamma-\sin^2\gamma\right)\widetilde{S}\right)\right\}\right\},\\ \widetilde{Q}_\tau&=\sin\gamma\cos\gamma\left(\widetilde{T}_2-\widetilde{T}_1\right)+\left(\cos^2\gamma-\sin^2\gamma\right)\widetilde{S}+\\&+ic\left\{-\left(\frac{\cos^2\gamma}{R_2}-\frac{\sin^2\gamma}{R_1}\right)\widetilde{S}+\frac{1}{R_\tau}\left(\sin\gamma\cos\gamma\left(\widetilde{T}_1-\widetilde{T}_2\right)-\right.\right.\end{aligned}\tag{5.10}$$

$$-(\cos^2\gamma-\sin^2\gamma)\widetilde{S}\Big)-$$
$$-\overline{\nu\left(\frac{\cos^2\gamma}{R_2}-\frac{\sin^2\gamma}{R_1}\right)\widetilde{S}+\frac{1}{R_1}\left(\sin\gamma\cos\gamma\left(\widetilde{T}_1-\widetilde{T}_2\right)+\right.}$$
$$\left.\overline{+(\cos^2\gamma-\sin^2\gamma)\widetilde{S}}\right)\Big]\Big\},$$

$$\widetilde{Q}_{\nu n}=ic\left\{\frac{\cos\gamma}{A_1A_2}\left[\frac{\partial A_2\widetilde{T}_2}{\partial\alpha_1}-\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_1-\nu\left(\overline{\frac{\partial A_2\widetilde{T}_1}{\partial\alpha_2}}+\right.\right.\right.$$
$$\left.\left.\left.+\overline{\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_2}\right)\right]+\frac{\sin\gamma}{A_1A_2}\left[\frac{\partial A_1\widetilde{T}_1}{\partial\alpha_2}-\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\right.$$
$$-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_2-\overline{\nu\left(\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_1}+\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_1\right)}\Big]+$$
$$+\frac{1}{1s}\left[\sin\gamma\cos\gamma\left(\widetilde{T}_1-\widetilde{T}_2\right)-(\cos^2\gamma-\sin^2\gamma)\widetilde{S}-\right.$$
$$\left.-\overline{\nu\left(\sin\gamma\cos\gamma\left(\widetilde{T}_2-\widetilde{T}_1\right)+(\cos^2\gamma-\sin^2\gamma)\widetilde{S}\right)}\right]\Big\}+$$
$$+(\cos\gamma\,m_1+\sin\gamma\,m_2),$$

$$\widetilde{M}_n=ic\left\{\cos^2\gamma\,\widetilde{T}_2-2\sin\gamma\cos\gamma\,\widetilde{S}+\sin^2\gamma\,\widetilde{T}_1-\right.$$
$$\left.-\overline{\nu\left(\cos^2\gamma\,\widetilde{T}_1+2\sin\gamma\cos\gamma\,\widetilde{S}+\sin^2\gamma\,\widetilde{T}_2\right)}\right\},$$

$$\widetilde{Q}_n^1=ic\left\{\sin\gamma\cos\gamma\left(\widetilde{T}_2-\widetilde{T}_1\right)-(\cos^2\gamma-\sin^2\gamma)\widetilde{S}-\right.$$
$$\left.-\overline{\nu\left(\sin\gamma\cos\gamma\left(\widetilde{T}_2-\widetilde{T}_1\right)+(\cos^2\gamma-\sin^2\gamma)\widetilde{S}\right)}\right\}_{s=s_i}.$$

6. Simplification of the Complex Relationships

The relationships which have been obtained in the preceding section /68 may be considered accurate to the same degree as the "real" relationships of the theory of shells. Their shortcoming is the presence of the conjugation operation. However, it can be shown that terms entering these relationships with the conjugate sign, generally speaking, are small and may be dropped either directly or by making a certain change in the introduced complex quantities.

We begin by simplifying the equations which contain the complex forces (5.8). First of all, it is clear that in the first two the underlined terms may be dropped compared with similar terms which do not contain small factors of the form $\frac{c}{R}=\frac{h}{R}\frac{1}{\sqrt{12(1-\nu^2)}}$. We introduce the designations

$$\widetilde{l}_1=\frac{\partial A_2\widetilde{T}_2}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_2,$$
$$\widetilde{l}_2=\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_1. \qquad (6.1)$$

It is easy to establish the validity of the following identities

$$\begin{aligned}\frac{\partial A_2\tilde{T}_2}{\partial\alpha_1}-\frac{1}{A_1}\frac{\partial A_1^2\tilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\tilde{T}_1&=A_2\frac{\partial\tilde{T}}{\partial\alpha_1}-\tilde{l}_1,\\ \frac{\partial A_1\tilde{T}_1}{\partial\alpha_2}-\frac{1}{A_2}\frac{\partial A_2^2\tilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\tilde{T}_2&=A_1\frac{\partial\tilde{T}}{\partial\alpha_2}-\tilde{l}_2,\end{aligned}\tag{6.2}$$

where $\tilde{T}=\tilde{T}_1+\tilde{T}_2$ is the principal complex function introduced by V. V. Novozhilov.

By taking into account these last relationships, the first two equations can be written in the form

$$\begin{aligned}\tilde{l}_1+\frac{ic}{R_1}\left(A_2\frac{\partial\tilde{T}}{\partial\alpha_1}-\underline{\tilde{l}_1-\nu\bar{\tilde{l}}_1}\right)&=-A_1A_2q_1,\\ \tilde{l}_2+\frac{ic}{R_2}\left(A_1\frac{\partial\tilde{T}}{\partial\alpha_2}-\underline{\tilde{l}_2-\nu\bar{\tilde{l}}_2}\right)&=-A_1A_2q_2.\end{aligned}$$

It is obvious that the underlined terms are small. By dropping them we obtain a simplified form for the first two equations /69

$$\begin{aligned}\tilde{l}_1+\frac{ic}{R_1}A_2\frac{\partial\tilde{T}}{\partial\alpha_1}&=-A_1A_2q_1,\\ \tilde{l}_2+\frac{ic}{R_2}A_1\frac{\partial\tilde{T}}{\partial\alpha_2}&=-A_1A_2q_2.\end{aligned}\tag{6.3}$$

We substitute expressions (6.2) into the third equation having first replaced $\tilde{l}_1$ and $\tilde{l}_2$ with quantities given by (6.3). As a result we have

$$\begin{aligned}\frac{\tilde{T}_1}{R_1}+\frac{\tilde{T}_2}{R_2}-\frac{ic}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\left[\frac{A_2}{A_1}\frac{\partial\tilde{T}}{\partial\alpha_1}+\underline{\frac{ic}{R_1}\frac{A_2}{A_1}\left(\frac{\partial\tilde{T}}{\partial\alpha_1}-\nu\frac{\partial\bar{\tilde{T}}}{\partial\alpha_1}\right)}+(1+\nu)A_2q_1\right]\right.\\ \left.+\frac{\partial}{\partial\alpha_2}\left[\frac{A_1}{A_2}\frac{\partial\tilde{T}}{\partial\alpha_2}+\underline{\frac{ic}{R_2}\frac{A_1}{A_2}\left(\frac{\partial\tilde{T}}{\partial\alpha_2}-\nu\frac{\partial\bar{\tilde{T}}}{\partial\alpha_2}\right)}+(1+\nu)A_1q_2\right]\right\}=q_n.\end{aligned}$$

Dropping the small (underlined) terms, we add the simplified equation obtained in this manner to the first two

$$\frac{\partial A_2\tilde{T}_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\tilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\tilde{T}_2+ic\frac{A_2}{R_1}\frac{\partial\tilde{T}}{\partial\alpha_1}=-A_1A_2q_3.\tag{6.4}$$

$$\frac{\partial A_1\tilde{T}_2}{\partial x_2}+\frac{1}{A_2}\frac{\partial A_2^2\tilde{S}}{\partial x_1}-\frac{\partial A_1}{\partial x_2}\tilde{T}_1+ic\frac{A_1}{R_2}\frac{\partial\tilde{T}}{\partial x_2}=-A_1A_2q_2,$$

$$\frac{\tilde{T}_1}{R_1}+\frac{\tilde{T}_2}{R_2}-ic\Delta\{\tilde{T}\}=q_n'.$$

Here $q_n'=q_n+i(1+\nu)\frac{c}{A_1A_2}\left\{\frac{\partial A_2q_1}{\partial x_1}+\frac{\partial A_1q_2}{\partial x_2}\right\}\approx q_n$, if q_1, q_2 are smooth functions which do not increase appreciably during differentiation. The operator used by us

$$\Delta=\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial x_1}\frac{A_2}{A_1}\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\frac{A_1}{A_2}\frac{\partial}{\partial x_2}\right\} \tag{6.5}$$

is the two-dimensional Laplace operator written in terms of orthogonal curvilinear coordinates. Equations (6.4) were first obtained by V. V. Novozhilov in a slightly different manner (ref. 31).

We now turn to the expressions for the complex boundary quantities. First of all we can see from (5.10) that, with conventional accuracy, the first two of these expressions may be taken in the form

$$\begin{aligned}\tilde{Q}_{vv}&=\cos^2\gamma\,\tilde{T}_1+2\sin\gamma\cos\gamma\,\tilde{S}+\sin^2\gamma\,\tilde{T}_2,\\ \tilde{Q}_{vt}&=\sin\gamma\cos\gamma\left(\tilde{T}_2-\tilde{T}_1\right)+(\cos^2\gamma-\sin^2\gamma)\,\tilde{S}.\end{aligned} \tag{6.6}$$

Furthermore, by utilizing equalities (6.2), (6.3) and the second equality of those just obtained, we find /70

$$\begin{aligned}\tilde{Q}_{vn}=ic\Bigg\{&\frac{\cos\gamma}{A_1A_2}\left[A_2\frac{\partial\tilde{T}}{\partial\alpha_1}+\underline{\frac{ic}{R_1}A_2\left(\frac{\partial\tilde{T}}{\partial x_1}-\nu\frac{\partial\bar{\tilde{T}}}{\partial x_1}\right)}+A_1A_2(1+\nu)q_1\right]+\\&+\frac{\sin\gamma}{A_1A_2}\left[A_1\frac{\partial\tilde{T}}{\partial x_2}+\frac{ic}{R_2}A_1\left(\frac{\partial\tilde{T}}{\partial x_2}-\nu\frac{\partial\bar{\tilde{T}}}{\partial x_2}\right)+A_1A_2(1+\nu)q_2\right]-\\&-\frac{d\left(\tilde{Q}_{vt}+\nu\bar{\tilde{Q}}_{vt}\right)}{ds_t}\Bigg\}+(\cos\gamma\,m_1+\sin\gamma\,m_2).\end{aligned}$$

Dropping the small underlined terms and remembering that $\frac{d}{ds_v}=\frac{\cos\gamma}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\sin\gamma}{A_2}\frac{\partial}{\partial\alpha_2}$, we obtain

$$\begin{aligned}\tilde{Q}_{vn}=ic\Bigg\{&\frac{d\tilde{T}}{ds_v}-\frac{d\left(\tilde{Q}_{vt}+\nu\bar{\tilde{Q}}_{vt}\right)}{ds_t}+(1+\nu)(\cos\gamma\,q_1+\sin\gamma\,q_2)\Bigg\}+\\&+(\cos\gamma\,m_1+\sin\gamma\,m_2).\end{aligned}$$

In view of the analogy with the preceding case, the last two quantities may be reduced to the form

$$\widetilde{M}_{\nu\nu} = ic\left\{\widetilde{T} - \left(\widetilde{Q}_{\nu\nu} + \nu\overline{\overline{\widetilde{Q}}}_{\nu\nu}\right)\right\},$$
$$\widetilde{Q}'_{n} = ic\left\{-\widetilde{Q}_{\nu t} - \nu\overline{\overline{\widetilde{Q}}}_{\nu t}\right\}_{s=s_t}.$$

We now collect the simplified expressions which have been obtained for the boundary complex quantities

$$\begin{gathered}
\widetilde{Q}_{\nu\nu} = \cos^2\gamma\,\widetilde{T}_1 + 2\sin\gamma\cos\gamma\,\widetilde{S} + \sin^2\gamma\,\widetilde{T}_2,\\
\widetilde{Q}_{\nu t} = \sin\gamma\cos\gamma\left(\widetilde{T}_2 - \widetilde{T}_1\right) + (\cos^2\gamma - \sin^2\gamma)\,\widetilde{S},\\
\widetilde{Q}_{\nu n} + ic\left\{\frac{d\left(\widetilde{Q}_{\nu t} + \nu\overline{\overline{\widetilde{Q}}}_{\nu t}\right)}{ds_t} - (1+\nu)\,q_\nu\right\} - m_\nu = ic\frac{d\widetilde{T}}{ds_\nu},\\
\widetilde{M}_{\nu\nu} + ic\left(\widetilde{Q}_{\nu\nu} + \nu\overline{\overline{\widetilde{Q}}}_{\nu\nu}\right) = ic\widetilde{T},\\
\widetilde{Q}'_{n} + ic\left\{\widetilde{Q}_{\nu t} + \nu\overline{\overline{\widetilde{Q}}}_{\nu t}\right\}_{s=s_t} = 0
\end{gathered} \tag{6.7}$$

$$\left(\begin{array}{l}
q_\nu = \cos\gamma\,q_1 + \sin\gamma\,q_2, \quad \dfrac{d}{ds_\nu} = \dfrac{\cos\gamma}{A_1}\dfrac{\partial}{\partial x_1} + \dfrac{\sin\gamma}{A_2}\dfrac{\partial}{\partial x_2},\\
m_\nu = \cos\gamma\,m_1 + \sin\gamma\,m_2, \quad \dfrac{d}{ds_t} = -\dfrac{\sin\gamma}{A_1}\dfrac{\partial}{\partial x_1} + \dfrac{\cos\gamma}{A_2}\dfrac{\partial}{\partial x_2}
\end{array}\right).$$

These relationships associate the boundary complex quantities with the complex forces. In this connection the latter are contained without the conjugation signs.

A general corollary can be obtained from the first two relationships. /71 For this purpose we introduce an element which is orthogonal with respect to the one considered, and we determine on this element the normal complex force which can be expressed by the following simplified expression (using a procedure analogous to the one carried out above)

$$\widetilde{Q}_{tt} = \sin^2\gamma\,\widetilde{T}_1 - 2\sin\gamma\cos\gamma\,\widetilde{S} + \cos^2\gamma\,\widetilde{T}_2. \tag{6.8}$$

A tangential complex force coincides with the one introduced earlier.

By adding the first two expressions from (6.7) to (6.8) we obtain

$$\begin{aligned}\widetilde{Q}_{vv} &= \cos^2\gamma\,\widetilde{T}_1 + 2\sin\gamma\cos\gamma\,\widetilde{S} + \sin^2\gamma\,\widetilde{T}_2,\\ \widetilde{Q}_{vt} &= \sin\gamma\cos\gamma\,(\widetilde{T}_2 - \widetilde{T}_1) + (\cos^2\gamma - \sin^2\gamma)\widetilde{S},\\ \widetilde{Q}_{tt} &= \sin^2\gamma\,\widetilde{T}_1 - 2\sin\gamma\cos\gamma\,\widetilde{S} + \sin^2\gamma\,\widetilde{T}_2.\end{aligned} \tag{6.9}$$

It follows from the structure of these relaticnships that if we remain within the framework of accuracy common to the theory of shells, we can identify the tangential complex boundary forces with the components of a tensor, which has been assigned by means of its components $\widetilde{T}_1$, $\widetilde{S}$, $\widetilde{T}_2$ along the lines of curvature.

The invariant nature of the basic V. V. Novozhilov complex function follows,

$$\widetilde{Q}_{vv} + \widetilde{Q}_{tt} = \widetilde{T}_1 + \widetilde{T}_2 = \widetilde{T}. \tag{6.10}$$

Separating the real and imaginary parts in relationships (6.9) and (6.10), we can formulate the same conclusions with respect to

(a) static quantities

$$Q_{vv},\ Q_{vt},\ Q_{tt},\ T_1,\ S,\ T_2,$$

(b) deformation quantities (geometric).

$$\varkappa_{tt},\ \varkappa_{tv},\ \varkappa_{vv},\ \varkappa_2,\ \tau,\ \varkappa_1.$$

As far as the simplification of the system of equations containing complex displacement (5.6) is concerned, we must change the relationships for the complex Hooke's law (5.4). We note that the latter, strictly speaking, are consistent only while the complex forces are determined as the exact solutions of equations (5.8). However, as we shall see in the future, except for a few simplest cases, it is not possible to solve the system of equations (5.8) in this form. Therefore, a transition to system (6.4) substantially simplifies the problem of obtaining a solution which is useful for practical applications. Furthermore, even a simplified system is in need of further simplification and as a rule its solution is approximated.

Thus the system of equations (5.4) is only approximately consistent. Therefore, there are no weighted bases for preferring it to some other system which is consistent with the same degree of accuracy. In the two subsequent sections we shall consider other, more convenient versions of the complex Hooke's law. /72

7. The V. V. Novozhilov Version

We take the complex Hooke's law in the following form

$$\begin{aligned}
\widetilde{\varkappa}_2 &= \frac{1}{Ehc}\left\{\widetilde{T}_1 - T_1^*\right\}, \quad \widetilde{\varepsilon}_2 = \frac{1}{Eh}\left\{\widetilde{T}_2 - \nu\widetilde{T}_1 + \frac{i}{c}M_1^*\right\}, \\
\widetilde{\varkappa}_1 &= \frac{1}{Ehc}\left\{\widetilde{T}_2 - T_2^*\right\}, \quad \widetilde{\varepsilon}_1 = \frac{1}{Eh}\left\{\widetilde{T}_1 - \nu\widetilde{T}_2 + \frac{i}{c}M_2^*\right\}, \\
\widetilde{\tau} &= -\frac{1}{Ehc}\left\{\widetilde{S} - S^*\right\}, \quad \frac{\widetilde{\omega}}{2} = \frac{1}{Eh}\left\{(1+\nu)\widetilde{S} - \frac{i}{c}H^*\right\}.
\end{aligned} \tag{7.1}$$

This system, which is simplified compared with the exact system, does not contain conjugation operations. In this case, as we can easily see, (by separating the real parts) Hooke's law retains its form. The removal of the conjugation operations has been reflected on the imaginary parts, where the relationships which introduce the stress functions are slightly distorted. Hooke's law in the form (7.1) was proposed by V. V. Novozhilov. In proposing this form he assumed that $M_1^* = M_2^* = H^* = 0$, i.e., he selected the membrane solution as the static system. We shall discuss this further below.

Solving relationships (7.1) for the complex forces, we obtain

$$\begin{aligned}
\widetilde{T}_1 &= T_1^* - iEhc\widetilde{\varkappa}_2, \quad \widetilde{T}_1 = \frac{Eh}{1-\nu^2}\left(\widetilde{\varepsilon}_1 + \nu\widetilde{\varepsilon}_2\right) - \frac{i}{c}\frac{M_2^* + \nu M_1^*}{1-\nu^2}, \\
\widetilde{T}_2 &= T_2^* - iEhc\widetilde{\varkappa}_1, \quad \widetilde{T}_2 = \frac{Eh}{1-\nu_2}\left(\widetilde{\varepsilon}_2 + \nu\widetilde{\varepsilon}_1\right) - \frac{i}{c}\frac{M_1^* + \nu M_2^*}{1-\nu^2}, \\
\widetilde{S} &= S^* + iEhc\widetilde{\tau}, \quad \widetilde{S} = \frac{Eh}{2(1+\nu)}\widetilde{\omega} + \frac{i}{c}\frac{H^*}{1+\nu}.
\end{aligned} \tag{7.2}$$

Eliminating from these equations the complex forces, we obtain equations containing the complex displacements in the form proposed by V. V. Novozhilov

$$\begin{aligned}
\widetilde{\varepsilon}_1 + ic\left(\widetilde{\varkappa}_2 - \nu\widetilde{\varkappa}_1\right) &= \frac{1}{Eh}\left[T_1^* - \nu T_2^* + \frac{i}{c}M_2^*\right], \\
\widetilde{\varepsilon}_2 + ic\left(\widetilde{\varkappa}_1 - \nu\widetilde{\varkappa}_2\right) &= \frac{1}{Eh}\left[T_2^* - \nu T_1^* + \frac{i}{c}M_1^*\right], \\
\frac{\widetilde{\omega}}{2} - ic\,(1+\nu)\,\widetilde{\tau} &= \frac{1}{Eh}\left[(1+\nu)S^* - \frac{i}{c}H^*\right].
\end{aligned} \tag{7.3}$$

Comparing equalities (7.2) and (3.3) we become convinced that the V. V. Novozhilov version is associated with the following expressions for the complex moments

$$\widetilde{M}_1 = ic\left\{\widetilde{T}_2 - \nu\widetilde{T}_1\right\},\ \widetilde{M}_2 = ic\left\{\widetilde{T}_1 - \nu\widetilde{T}_2\right\},\ \widetilde{H} = -ic(1+\nu)\widetilde{S}. \tag{7.4}$$

Therefore the system of equations containing the complex forces takes the form /73

$$\begin{gathered} L_1\left\{\widetilde{T}_1,\ \widetilde{T}_2,\ \widetilde{S},\ ic\left(\widetilde{T}_2 - \nu\widetilde{T}_1\right),\ ic\left(\widetilde{T}_1 - \nu\widetilde{T}_2\right),\ - \right. \\ \left. - ic(1+\nu)\widetilde{S}\right\} = -A_1A_2q_1, \\ L_2\{\ldots\} = -A_1A_2q_2,\quad L_3\{\ldots\} = q_n. \end{gathered} \tag{7.5}$$

This system varies from the exact system only in the absence of conjugate signs for terms containing the factor ν. However, as we have clarified in the preceding section, when transition is made from the exact system to the simplified one, all such terms are dropped because they are very small. Therefore we can immediately state that the simplified system for (7.5) will still be the system of equations containing the complex forces (6.4). As far as the boundary complex quantities are concerned, by following the entire reasoning of the preceding section, we obtain for them the following simplified expressions

$$\begin{gathered} \widetilde{Q}_{\nu} = \cos^2\gamma\,\widetilde{T}_1 + 2\sin\gamma\cos\gamma\,\widetilde{S} + \sin^2\gamma\,\widetilde{T}_2, \\ \widetilde{Q}_{\tau} = \sin\gamma\cos\gamma\left(\widetilde{T}_2 - \widetilde{T}_1\right) + (\cos^2\gamma - \sin^2\gamma)\widetilde{S}, \\ Q_{\nu n} + i(1+\nu)\,c\frac{d\widetilde{Q}_{\nu\tau}}{ds_\tau} - m_\nu - i(1+\nu)\,cq_\nu = ic\frac{d\widetilde{T}}{ds_\tau}, \\ \widetilde{M}_\nu + i(1+\nu)\,c\widetilde{Q}_\nu = ic\widetilde{T}, \\ \widetilde{Q}'_n + i(1+\nu)\,c\left\{\widetilde{Q}_{\nu\tau}\right\}_{-s_\tau} = 0; \end{gathered} \tag{7.6}$$

$$\begin{gathered} q_\nu = \cos\gamma\,q_1 + \sin\gamma\,q_2,\quad \frac{d}{ds_\tau} = -\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}, \\ m_\nu = \cos\gamma\,m_1 + \sin\gamma\,m_2,\quad \frac{d}{ds_\nu} = \frac{\cos\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \frac{\sin\gamma}{A_2}\frac{\partial}{\partial\alpha_2}. \end{gathered} \tag{7.7}$$

Comparing them with expressions (6.7) we see that they differ only in their real parts. Thus there is some distortion in the expressions for the generalized boundary cutting force, for the boundary moment and for the normal concentrated force. Below we shall show that these distortions do not lead to appreciable errors in most cases.

In regard to the equations expressed in terms of the complex displacements (7.3), we note the following: although they do not contain the conjugation operation, their structure is still rather complex. The small terms

contained in them make it difficult to carry out transformations and thus decrease the practical value of this system. This is precisely the reason why, although the method of complex forces has received wide application, the method of equations expressed in terms of complex displacements is almost never used. In the next section a form of the complex Hooke's law is proposed, in which the equations containing the complex displacements are free from this shortcoming.

8. One Other Version

Let us take the complex law in a form which is further simplified /74

$$\begin{aligned}
&\tilde{\varkappa}_2 = \frac{i}{Ehc}\left(\tilde{T}_1 - T_1^*\right), \quad \tilde{\varepsilon}_2 = \frac{1}{Eh}\left(\tilde{T} + \frac{i}{c}M_1^*\right),\\
&\tilde{\varkappa}_1 = \frac{i}{Ehc}\left(\tilde{T}_2 - T_2^*\right), \quad \tilde{\varepsilon}_1 = \frac{1}{Eh}\left(\tilde{T} + \frac{i}{c}M_2^*\right),\\
&\tilde{\tau} = -\frac{i}{Ehc}\left(\tilde{S} - S^*\right), \quad \frac{\tilde{\omega}}{2} = -\frac{i}{Ehc}H^*.
\end{aligned} \tag{8.1}$$

Solving the first three of these relationships for the complex forces, we obtain the familiar expressions

$$\tilde{T}_1 = T_1^* - iEhc\tilde{\varkappa}_2, \quad \tilde{T}_2 = T_2^* - iEhc\tilde{\varkappa}_1, \quad \tilde{S} = S^* + iEhc\tilde{\tau}. \tag{8.2}$$

If the complex forces are eliminated from system (8.1), we obtain a new version of equations in terms of the complex displacements

$$\begin{aligned}
&\tilde{\varepsilon}_1 + ic\left(\tilde{\varkappa}_2 + \tilde{\varkappa}\right) = \frac{1}{Eh}\left(T^* + \frac{i}{c}M_2^*\right),\\
&\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 = \frac{i}{Ehc}\left(M_2^* - M_1^*\right), \quad (T^* = T_1^* + T_2^*),\\
&\tilde{\omega} = -\frac{i}{Ehc}2H^*
\end{aligned} \tag{8.3}$$

which can also be written in a slightly different form

$$\begin{aligned}
\tilde{\varepsilon}_2 + ic\left(\tilde{\varkappa}_1 + \tilde{\varkappa}_2\right) &= \frac{1}{Eh}\left(T^* + \frac{i}{c}M_1^*\right),\\
\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 &= \frac{i}{Ehc}\left(M_2^* - M_1^*\right),\\
\tilde{\omega} &= -\frac{i}{Ehc}2H^*.
\end{aligned} \tag{8.4}$$

The system which has been obtained is substantially simpler than the preceding ones and is quite competitive with the simplified system of equations

in terms of the complex forces (6.4). It will be widely used in the second part of this monograph, both for considering theoretical problems and in the solution of specific problems.

System (8.1) must be consistent. Let us see what limitation this requirement places on the functions $\widetilde{T}_1$, $\widetilde{T}_2$, $\widetilde{S}$. For this purpose we formulate the expression

$$L_1\left\{\widetilde{\varkappa}_2,\ \widetilde{\varkappa}_1,\ -\widetilde{\tau},\ -\widetilde{\varepsilon}_2,\ -\widetilde{\varepsilon}_1,\ \frac{\widetilde{\omega}}{2}\right\}=$$
$$=-\frac{i}{Ehc}\left[L_1\left\{T_1^*,\ T_2^*,\ S^*,\ M_1^*,\ M_2^*,\ H^*\right\}+A_1A_2q_1\right]+$$
$$+\frac{i}{Ehc}\left[L_1\left\{\widetilde{T}_1,\ \widetilde{T}_2,\ \widetilde{S},\ ic\widetilde{T},\ ic\widetilde{T},\ 0\right\}+A_1A_2q_1\right].$$

In view of the requirement for the consistency of the system, the left side of this equality becomes equal to zero. The static system makes the first term of the right side equal to zero. Then the first equation for the quantities $\widetilde{T}_1$, $\widetilde{T}_2$, $\widetilde{S}$ follows from the equality /75

$$L_1\left\{\widetilde{T}_1,\ \widetilde{T}_2,\ \widetilde{S},\ ic\widetilde{T},\ ic\widetilde{T},\ 0\right\}=-A_1A_2q_1. \qquad (8.5)_1$$

In the same manner we obtain the two remaining equations

$$L_2\{\ldots\}=-A_1A_2q_2,\quad L_3\{\ldots\}=q_n. \qquad (8.5)_{2,3}$$

If we expand these equations we find that they coincide with the equations of system (6.4), with the only irsignificant difference that in these the position of the quantity

$$q_n'=q_n+i(1+\nu)\frac{c}{A_1A_2}\left\{\frac{\partial A_2q_1}{\partial x_1}+\frac{\partial A_1q_2}{\partial x_2}\right\}\approx q_n$$

is occupied by q_n.

Thus the system which we have obtained in terms of the complex displacements (8.3), which is a corollary of the proposed form of the complex Hooke's law, may be considered equivalent to the system of equations containing the V. V. Novozhilov complex forces (6.4). Equivalence should be understood in the following way. Let $\widetilde{T}_1$, $\widetilde{T}_2$, $\widetilde{S}$ be some solution of system (6.4). We substitute them into equalities (8.1) and use them to determine the complex displacements $\widetilde{u}_1$, $\widetilde{u}_2$, $\widetilde{w}$. Then the established values of the complex displacements

make the equations in terms of the complex displacements (8.3) identical. And vice versa, let $\mathfrak{U}_1$, $\mathfrak{U}_2$, $\mathfrak{W}$ be some solution of the system of equations in terms of the complex displacements. We use them to compute expressions (8.2). We can maintain that the constructed $\widetilde{T}_1$, $\widetilde{T}_2$, $\widetilde{S}$ satisfy the system of equations in terms of the complex forces.

The first proposition is obvious. The second can be proved quite simply. Indeed, let $\widetilde{u}_1$, $\widetilde{u}_2$, $\widetilde{w}$ be some solution of the system of equations (8.3). We use these to construct the following expressions

$$\begin{aligned}
&\widetilde{\varkappa}_2+\frac{i}{Ehc}T_1^*=\widetilde{\Psi}_1, \quad \widetilde{\varepsilon}_2-\frac{i}{Ehc}M_1^*=\widetilde{\Psi}_4,\\
&\widetilde{\varkappa}_1+\frac{i}{Ehc}T_2^*=\widetilde{\Psi}_2, \quad \widetilde{\varepsilon}_1-\frac{i}{Ehc}M_2^*=\widetilde{\Psi}_5,\\
&\widetilde{\tau}-\frac{i}{Ehc}S^*=\widetilde{\Psi}_3, \quad \frac{\widetilde{\omega}}{2}+\frac{i}{Ehc}H^*=\widetilde{\Psi}_6.
\end{aligned}$$

Substituting these quantities into system (8.3), we obtain

$$\widetilde{\Psi}_6=0,\ \widetilde{\Psi}_5=\widetilde{\Psi}_4=-ic\left(\widetilde{\Psi}_1+\widetilde{\Psi}_2\right).$$

Thus the preceding equality may be written in the following manner

$$\begin{aligned}
&\widetilde{\varkappa}_2+\frac{i}{Ehc}T_1^*=\widetilde{\Psi}_1, \quad \widetilde{\varepsilon}_2-\frac{i}{Ehc}M_1^*=-ic\left\{\widetilde{\Psi}_1+\widetilde{\Psi}_2\right\},\\
&\widetilde{\varkappa}_1+\frac{i}{Ehc}T_2^*=\widetilde{\Psi}_2, \quad \widetilde{\varepsilon}_1-\frac{i}{Ehc}M_2^*=-ic\left\{\widetilde{\Psi}_1+\widetilde{\Psi}_2\right\},\\
&\widetilde{\tau}-\frac{i}{Ehc}S^*=\widetilde{\Psi}_3, \quad \frac{\widetilde{\omega}}{2}+\frac{i}{Ehc}H^*=0.
\end{aligned} \tag{8.6}$$

Substituting the complex strain components determined from this equation into the continuity equations we find, in a conventional manner, that the quantities /76

$$-iEhc\widetilde{\Psi}_1,\ -iEhc\widetilde{\Psi}_2,\ -iEhc\widetilde{\Psi}_3$$

satisfy the system of equations expressed in terms of the complex forces (6.4). Introducing the conventional designations

$$\widetilde{T}_1=-iEhc\widetilde{\Psi}_1,\ \widetilde{T}_2=-iEhc\widetilde{\Psi}_2\ \widetilde{S}=-iEhc\widetilde{\Psi}_3,$$

we can write the first three relationships of (8.6) in the form (8.2). With this formulation the expressions $\widetilde{T}_1$, $\widetilde{T}_2$, $\widetilde{S}$ indeed satisfy system (6.4).

In the preceding section we noted that when the complex Hooke's law is adopted in the form (7.1), the relationships of Hooke's law are retained, while the relationships which introduce the stress functions are distorted. In adopting the complex Hooke's law in the form (8.1) both of these are disrupted, so that, strictly speaking, the real parts of the quantities $\widetilde{u}_1$, $\widetilde{u}_2$, $\widetilde{w}$ obtained from the system of equations (8.3) can no longer be identified with the displacements. In some problems this situation may produce errors in determining the displacements. To avoid this we shall determine the displacements from the complex forces determined from relationships (7.1), i.e., from the system

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{Eh}\operatorname{Re}\left(\widetilde{T}_1 - \nu\widetilde{T}_2\right), \quad \varkappa_1 = -\frac{1}{Ehc}\operatorname{Im}\widetilde{T}_2, \\
\varepsilon_2 &= \frac{1}{Eh}\operatorname{Re}\left(\widetilde{T}_2 - \nu\widetilde{T}_1\right), \quad \varkappa_2 = -\frac{1}{Ehc}\operatorname{Im}\widetilde{T}_1, \\
\omega &= \frac{2(1+\nu)}{Eh}\operatorname{Re}\widetilde{S}, \quad \tau = \frac{1}{Ehc}\operatorname{Im}\widetilde{S}.
\end{aligned} \tag{8.7}$$

In this case it is not necessary to use the last three relationships, because the complex forces are determined by means of equations (8.2), whose imaginary parts turn out to be the last three equations of equalities (8.7). Therefore, in determining displacements we shall make use of a more convenient system of equations

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{Eh}\operatorname{Re}\left(\widetilde{T}_1 - \nu\widetilde{T}_2\right), \quad \varkappa_1 = \operatorname{Re}\widetilde{\varkappa}_1, \\
\varepsilon_2 &= \frac{1}{Eh}\operatorname{Re}\left(\widetilde{T}_2 - \nu\widetilde{T}_1\right), \quad \varkappa_2 = \operatorname{Re}\widetilde{\varkappa}_2, \\
\omega &= \frac{2(1+\nu)}{Eh}\operatorname{Re}\widetilde{S}, \qquad \tau = \operatorname{Re}\widetilde{\tau}.
\end{aligned} \tag{8.8}$$

From the same considerations, when utilizing the proposed method we shall apply the expressions for the complex boundary quantities in the form (7.6). /77

Thus the method of complex displacements in the form considered in the present section is an alternate method to the method of complex forces. In some problems, as we shall show in the second part of the book, it is the preferred method. The accuracy of both methods, as we have just shown, is the same.

9. Formulation of Problems of the Theory of Thin Shells. Summary of Results

In the theory of shells there are several methods of solving problems. We shall pause briefly to consider several of these

(1) The method of displacements consists of the following: the forces and moments in the equilibrium equations are replaced by means of Hooke's law relationships with relationships containing the stress components (II, 2.8). Thus, in determining the displacements u_1, u_2, w, we obtain a system of three equations of the 8-th order, and the static boundary conditions (if they exist) are also expressed in terms of displacements and their derivatives.

The method of displacements is used very seldom in the solution of linear static problems because it leads to substantial mathematical difficulties.

(2) The method of forces-moments is more common. In this method the three equilibrium equations are connected to the continuity equations (II, 2.16), in which the strain components are replaced by forces and moments in accordance with the relationships of Hooke's law. As a result we have a system of six equations of 8-th order to determine the six static quantities, (three forces and three moments). This system is also rather complex. To obtain simple solutions, which can be applied in practice, it must be further simplified. In this connection the static-geometric analogy is used quite frequently together with various physical considerations. In view of the system's awkwardness and the associated low visualization, its simplification is a complicated matter which requires a great deal of skill and a clear understanding of the behavior associated with the specific types of shells. In this respect the complex forms of the theory of shells have certain advantages in view of the lower order (4th) of the principal complex systems of equations and in view of their simpler structure. More details concerning the advan- /78 tages and the fields of application of the complex methods will be presented in the second part of the book.

(3) The method of complex forces. As we have established above, the basic difficulty in utilizing this method is associated with the solution of a system of equations containing complex forces (6.4)

$$\begin{aligned}
&\frac{\partial A_2\widetilde{T}_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_2+ic\frac{A_2}{R_1}\frac{\partial\widetilde{T}}{\partial\alpha_1}=-A_1A_2q_1,\\
&\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_1+ic\frac{A_1}{R_2}\frac{\partial\widetilde{T}}{\partial\alpha_2}=-A_1A_2q_2,\\
&\frac{\widetilde{T}_1}{R_1}+\frac{\widetilde{T}_2}{R_2}-ic\Delta\widetilde{T}=q_n,
\end{aligned}\qquad (9.1)$$

where

$$\Delta=\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right\},$$
$$\widetilde{T}=\widetilde{T}_1+\widetilde{T}_2,\ c=\frac{h}{\sqrt{12(1-\nu^2)}}.$$

The forces and moments are determined in the following manner from the established complex forces

$$T_1 = \operatorname{Re}\widetilde{T}_1,\ T_2 = \operatorname{Re}\widetilde{T}_2,\ S = \operatorname{Re}\widetilde{S},$$
$$M_1 = -c\operatorname{Im}\left\{\widetilde{T}_2 + \nu\widetilde{T}_1\right\},\quad M_2 = -c\operatorname{Im}\left\{\widetilde{T}_1 + \nu\widetilde{T}_2\right\} \tag{9.2}$$
$$H = (1-\nu)\operatorname{Im}\widetilde{S}.$$

The maximum stresses which occur at the upper (+) or lower (-) faces of the shell are obtained from the relationshi

$$\sigma_1^{\pm} = \frac{T_1}{h} \pm \frac{6M_1}{h^2},\quad \sigma_2^{\pm} = \frac{T_2}{h} \pm \quad .\quad \sigma_{12}^{\pm} = \frac{S}{h} \pm \frac{6H}{h^2}. \tag{9.3}$$

The displacements of the middle surface are determined by solving approximately the consistent system of differential equations

$$\tilde{\varepsilon}_1 = \frac{1}{Eh}\left\{\widetilde{T}_1 - \nu\widetilde{T}_2 + \frac{i}{c}M_2^*\right\},\quad \tilde{\varkappa}_1 = \frac{i}{Ehc}\left\{\widetilde{T}_2 - T_2^*\right\},$$
$$\tilde{\varepsilon}_2 = \frac{1}{Eh}\left\{\widetilde{T}_2 - \nu\widetilde{T}_1 + \frac{i}{c}M_1^*\right\},\quad \tilde{\varkappa}_2 = \frac{i}{Ehc}\left\{\widetilde{T}_1 - T_1^*\right\}, \tag{9.4}$$
$$\frac{\tilde{\omega}}{2} = \frac{1}{Eh}\left\{(1+\nu)\widetilde{S} - \frac{i}{c}H^*\right\},\quad \tilde{\tau} = -\frac{i}{Ehc}\left\{\widetilde{S} - S^*\right\},$$

where

$$\tilde{\varepsilon}_1 = \frac{1}{A_1}\frac{\partial\tilde{u}_1}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\tilde{u}_2 + \frac{\tilde{w}}{R_1},$$
$$\tilde{\varepsilon}_2 = \frac{1}{A_2}\frac{\partial\tilde{u}_2}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\tilde{u}_1 + \frac{\tilde{w}}{R_2},$$
$$\tilde{\omega} = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\tilde{u}_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\tilde{u}_1}{A_1}\right),$$
$$\tilde{\varkappa}_1 = \frac{1}{A_1}\frac{\partial\tilde{\vartheta}_2}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\tilde{\vartheta}_2 = \tag{9.5}$$
$$= -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\tilde{w}}{\partial\alpha_1} - \frac{\tilde{u}_1}{R_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\tilde{w}}{\partial\alpha_2} - \frac{\tilde{u}_2}{R_2}\right),$$
$$\tilde{\varkappa}_2 = \frac{1}{A_2}\frac{\partial\tilde{\vartheta}_2}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\tilde{\vartheta}_1 =$$
$$= -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\tilde{w}}{\partial\alpha_2} - \frac{\tilde{u}_2}{R_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\tilde{w}}{\partial\alpha_1} - \frac{\tilde{u}_1}{R_1}\right).$$

$$\tilde{\tau}=\frac{1}{2}\left\{\frac{1}{A_1}\frac{\partial\tilde{\vartheta}_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\tilde{\vartheta}_1+\frac{1}{A_2}\frac{\partial\tilde{\vartheta}_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\tilde{\vartheta}_2+\right.$$
$$\left.+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial\tilde{u}_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\tilde{u}_2\right)+\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial\tilde{u}_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\tilde{u}_1\right)\right\}=$$
$$=-\frac{1}{A_1A_2}\left(\frac{\partial^2\tilde{w}}{\partial\alpha_1\partial\alpha_2}-\frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\tilde{w}}{\partial\alpha_1}-\frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\tilde{w}}{\partial\alpha_2}\right)+$$
$$+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial\tilde{u}_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\tilde{u}_1\right)+\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial\tilde{u}_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\tilde{u}_2\right).$$

and the separation of the real parts in the expressions established for the complex displacements. Sometimes it is more convenient to determine the displacements directly from the system /79

$$\begin{aligned}
&\varepsilon_1=\frac{1}{Eh}\operatorname{Re}\left\{\tilde{T}_1-\nu\tilde{T}_2\right\}, && \varkappa_1=-\frac{1}{Ehc}\operatorname{Im}\tilde{T}_2,\\
&\varepsilon_2=\frac{1}{Eh}\operatorname{Re}\left\{\tilde{T}_2-\nu\tilde{T}_1\right\}, && \varkappa_2=-\frac{1}{Ehc}\operatorname{Im}\tilde{T}_1,\\
&\frac{\omega}{2}=\frac{1+\nu}{Eh}\operatorname{Re}\tilde{S}, && \tau=\frac{1}{Ehc}\operatorname{Im}\tilde{S}.
\end{aligned}\tag{9.6}$$

(4) On the other hand, when using the method of complex displacements it becomes necessary to solve the system of equations containing complex displacements in either one of the following two equivalent forms

$$\begin{aligned}
&\tilde{\varepsilon}_1+ic\left(\tilde{\varkappa}_1+\tilde{\varkappa}_2\right)=\frac{1}{Eh}\left(T^*+\frac{i}{c}M_2^*\right),\\
&\tilde{\varepsilon}_1-\tilde{\varepsilon}_2=\frac{i}{Ehc}\left(M_2^*-M_1^*\right), \quad \left(T^*=T_1^*+T_2^*\right),\\
&\tilde{\omega}=-\frac{i}{Ehc}2H^*;
\end{aligned}\tag{9.7}$$

$$\begin{aligned}
&\tilde{\varepsilon}_2+ic\left(\tilde{\varkappa}_1+\tilde{\varkappa}_2\right)=\frac{1}{Eh}\left(T^*+\frac{i}{c}M_1^*\right),\\
&\tilde{\varepsilon}_1-\tilde{\varepsilon}_2=\frac{i}{Ehc}\left(M_2^*-M_1^*\right),\\
&\tilde{\omega}=-\frac{i}{Ehc}2H^*.
\end{aligned}\tag{9.7'}$$

The complex displacements established from these expressions are used to determine the complex forces

$$\tilde{T}_1=T_1^*-iEhc\tilde{\varkappa}_2,\quad \tilde{T}_2=T_2^*-iEhc\tilde{\varkappa}_1,\quad \tilde{S}=S^*+iEhc\tilde{\tau}.\tag{9.8}$$

The displacements are established by solving the following system of the differential equations

$$\begin{aligned} \varepsilon_1 &= \frac{1}{Eh}\left(\operatorname{Re}\widetilde{T}_1 - \nu \operatorname{Re}\widetilde{T}_2\right), \quad \varkappa_1 = \operatorname{Re}\widetilde{\varkappa}_1, \\ \varepsilon_2 &= \frac{1}{Eh}\left(\operatorname{Re}\widetilde{T}_2 - \nu \operatorname{Re}\widetilde{T}_1\right), \quad \varkappa_2 = \operatorname{Re}\widetilde{\varkappa}_2, \\ \omega &= \frac{2(1+\nu)}{Eh}\operatorname{Re}\widetilde{S}, \qquad\qquad \tau = \operatorname{Re}\widetilde{\tau}. \end{aligned} \tag{9.9}$$

The arbitrary elements (functions and constants), which are contained in the solution of the basic differential equations of the considered methods, are determined from the boundary conditions of the specific problem. The most diverse boundary conditions are encountered in the problems of the theory of elasticity, including the following: conditions of periodicity, finiteness, condition of elastic conjugation, etc. We leave the consideration of the problem concerning the boundary conditions in its general formulation for the second part of the book and present only general information regarding them, which is necessary for understanding the material presented in the first part of the book.

Let us assume that the middle line of the boundary element coincides with the coordinate line α_2 (fig. 13). Since, in this case $\gamma = 0$, according to what has been presented in Chapters II-IV, we may assign along the contour $\alpha_1 = \text{const}$, either the four geometric quantities

$$\begin{gathered} u_1,\ u_2,\ w,\ \vartheta_1 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u_1}{R_1} \\ \left(\vec{U} = u_1\vec{e}_1 + u_2\vec{e}_2 + w\vec{n}\right), \end{gathered} \tag{9.10}$$

or the four deformation quantities

$$\begin{gathered} \varkappa_2,\ \varkappa_{21} = \tau - \frac{\omega}{R_2},\quad \varkappa_{2n} = -\frac{1}{A_1A_2}\left(\frac{\partial A_2\varepsilon_2}{\partial \alpha_1} - \frac{\partial A_1\omega}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1}\varepsilon_1\right),\ \varepsilon_2, \\ \left(\vec{\varkappa}_2 = -\varkappa_2\vec{e}_1 + \varkappa_{21}\vec{e}_2 - \varkappa_{2n}\vec{n}\right); \end{gathered} \tag{9.11}$$

or finally the four static quantities

$$\begin{gathered} T_1,\ Q_{12} = S + \frac{2H}{R_2},\quad Q_{1n} = \frac{1}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial \alpha_1} + 2\frac{\partial A_1H}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1}M_2\right),\ M_1 \\ \left(\vec{Q}_1 = T_1\vec{e}_1 + Q_{12}\vec{e}_2 + Q_{1n}\vec{n}\right). \end{gathered} \tag{9.12}$$

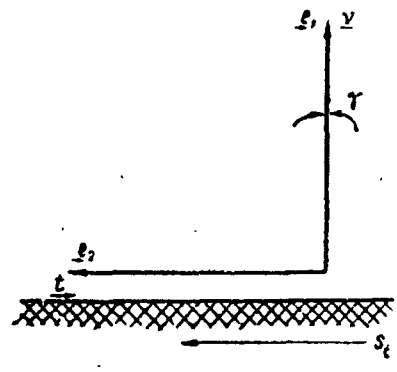

Figure 13

Sometimes the vectors $\vec{U}$, $\vec{\varkappa}_2$, $\vec{Q}_1$ are resolved into other vectors along /81

directions which are more convenient for the particular problem, and the components along these directions are specified.

Boundary conditions may also be mixed when part of the quantities is selected from (9.10) and part from (9.11) and (9.12). In this case the total number of conditions must be equal to four, which corresponds to the order of the basic differential equations.

To clarify what we have just said, we present several versions of boundary conditions which are most frequently used:

(a) fixed edge (geometric condition)

$$u_1 = u_2 = w = \vartheta_1 = 0; \tag{9.13}$$

(b) rigid edge (condition of deformation)

$$\varkappa_2 = \varkappa_{21} = \varkappa_{2\eta} = \varepsilon_2 = 0. \tag{9.14}$$

Condition "b" differs from condition "a" in that the shell can be displaced as a rigid whole. Problems are encountered in which this rigid displacement is substantial. Thus, for example, in a closed cylindrical shell, which is free of surface loading and which is closed along the edges by rigid diaphragms, the value of the occurring stresses is determined by the relative displacement of the diaphragm. In most problems, however, the additional rigid displacement of the edges is insignificant. In these cases condition "b" is preferred, because it can be formulated in terms of the strain components and when it is satisfied it is not necessary to carry out a preliminary determination of the stresses. The term which we have introduced (strain boundary condition) emphasizes this situation. In the future (when possible) we shall use the strain boundary conditions in place of the geometric boundary conditions.

(c) hinged support (mixed condition)

$$T_1 = u_2 = w = M_1 = 0; \tag{9.15}$$

(d) free edge (static condition)

$$T_1 = Q_{12} = Q_{1n} = M_1 = 0. \tag{9.16}$$

The static and strain boundary quantities are associated by the corresponding complex quantities (see section 3) /82

$$\begin{aligned}
&\widetilde{T}_1 = T_1 - iEhc\varkappa_2 = T_1^* - iEhc\widetilde{\varkappa}_2,\\
&\widetilde{Q}_{12} = Q_{12} + iEhc\varkappa_{21} = Q_{12}^* + iEhc\widetilde{\varkappa}_{21},\\
&\widetilde{Q}_{1n} = Q_{1n} - iEhc\varkappa_{2n} = Q_{1n}^* - iEhc\widetilde{\varkappa}_{2n},\\
&\widetilde{M}_1 = M_1 + iEhc\varepsilon_2 = M_1^* + iEhc\widetilde{\varepsilon}_2,\\
&\vec{\widetilde{Q}}_1 = \widetilde{T}_1\vec{e}_1 + \widetilde{Q}_{12}\vec{e}_2 + \widetilde{Q}_{1n}\vec{n}.
\end{aligned} \tag{9.17}$$

The following simplified expressions (see section 7) are valid for the complex boundary quantities

$$\begin{aligned}
&\widetilde{Q}_1 = \widetilde{T}_1, \quad \widetilde{Q}_{12} = \widetilde{S},\\
&\widetilde{Q}_{1n} + i(1+\nu)\frac{c}{A_2}\frac{\partial\widetilde{S}}{\partial\alpha_2} - m_3 - i(1+\nu)cq_1 = \frac{ic}{A_1}\frac{\partial\widetilde{T}}{\partial\alpha_1},\\
&\widetilde{M}_1 + i(1+\nu)c\widetilde{T}_1 = icT.
\end{aligned} \tag{9.18}$$

Relationships (9.17)-(9.18) make it possible to carry out the elastic conjugation of solutions for two shells joined along the common line α_2. In this case, if the shells have the same thickness and the same elastic constants, the coupling is carried out in a complex form without preliminary separation of the real and imaginary parts of the conjugated solutions.

We shall consider first the case of shells joined smoothly without a break in the middle surface. In this case the equations of conjugation (junction) are reduced to eight equalities

$$\begin{aligned}
&T_1^{\mathrm{I}} = T_1^{\mathrm{II}}, \quad \varkappa_2^{\mathrm{I}} = \varkappa_2^{\mathrm{II}},\\
&Q_{12}^{\mathrm{I}} = Q_{12}^{\mathrm{II}}, \quad \varkappa_{21}^{\mathrm{I}} = \varkappa_{21}^{\mathrm{II}},
\end{aligned} \tag{9.19}$$

$$Q_{1n}^{I}=Q_{1n}^{II},\quad \varkappa_{2n}^{I}=\varkappa_{2n}^{II},$$
$$M_1^{I}=M_1^{II},\quad \varepsilon_2^{I}=\varepsilon_2^{II},$$

and these in view of

$$(Ehc)^{I}=(Ehc)^{II} \tag{9.20}$$

may be replaced, according to (9.17) with four complex equalities

$$\widetilde{T}_1^{I}=\widetilde{T}_1^{II},\ \widetilde{Q}_{12}^{I}=\widetilde{Q}_{12}^{II},\ \widetilde{Q}_{1n}^{I}=\widetilde{Q}_{1n}^{II},\ \widetilde{M}_1^{I}=\widetilde{M}_1^{II} \tag{9.21}$$

(the symbols I and II designate the quantities which pertain, respectively, to the first and second shell). The approximate equalities (9.18) show that in the case of smooth junction the following quantities must be continuous along the junction line

$$\begin{gathered}\left(\widetilde{T}_1\right)^{I}=\left(\widetilde{T}_1\right)^{II},\ \left(\widetilde{S}\right)^{I}=\left(\widetilde{S}\right)^{II},\\ \left\{\frac{ic}{A_1}\frac{\partial\widetilde{T}}{\partial\alpha_1}+m_1+ic(1+\nu)q_1\right\}^{I}=\{\ldots\}^{II},\\ \left(\widetilde{T}\right)^{I}=(\ldots)^{II}\qquad\qquad \left(\left(\widetilde{T}_2\right)^{I}=\left(\widetilde{T}_2\right)^{II}\right).\end{gathered} \tag{9.22}$$

When the junction is not smooth, the first three conditions are replaced by one vector condition /83

$$\vec{\widetilde{Q}}_1^{I}=\vec{\widetilde{Q}}_1^{II}, \tag{9.23}$$

and the equality obtained is projected along three selected directions.

Relationships (9.17)-(9.18) may be used in cases when equality (9.20) is not satisfied and also if more than two shells are joined along the junction line. In this case it becomes necessary to utilize eight real conditions (9.19), writing them by means of the real and imaginary parts of the simplified expressions (9.18). The latter may also be used in formulating the static and strain conditions.

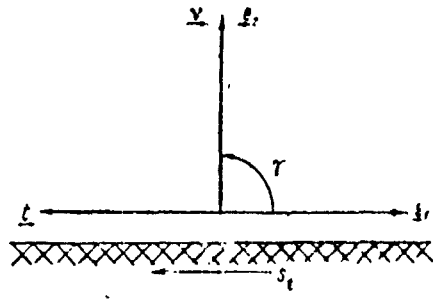

Figure 14

Let us consider another important case when the contour of the region coincides with the coordinate line α_1. In this case (fig. 14) $\gamma = \frac{\pi}{2}$. Since this case is completely analogous to the one just considered, we present the basic relationships without a detailed explanation.

Along the boundary contour (α_2 = const) we can assign either four geometric quantities

$$u_1,\ u_2,\ w,\ \vartheta_2 = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{u_2}{R_2}, \qquad \vec{U} = u_1\vec{e}_1 + u_2\vec{e}_2 + w\vec{n}, \tag{9.24}$$

or four strain quantities

$$\varkappa_1,\ -\varkappa_{12} = \tau - \frac{\omega}{R_1},\quad -\varkappa_{1n} = \frac{1}{A_1A_2}\left(\frac{\partial A_1\varepsilon_1}{\partial \alpha_2} - \frac{\partial A_2\omega}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2}\varepsilon_2\right),\quad \varepsilon_1, \qquad \vec{\varkappa}_1 = -\varkappa_1\vec{e}_2 - \varkappa_{12}\vec{e}_1 - \varkappa_{1n}\vec{n}, \tag{9.25}$$

or four static quantities

$$T_2,\ -Q_{21} = S + \frac{2H}{R_1}, \qquad Q_{2n} = \frac{1}{A_1A_2}\left\{\frac{\partial A_1M_2}{\partial \alpha_2} + 2\frac{\partial A_2H}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2}M_1\right\},\quad M_2, \qquad \vec{Q}_2 = T_2\vec{e}_2 - Q_{21}\vec{e}_1 + Q_{2n}\vec{n}, \tag{9.26}$$

or, finally, four mixed quantities. The complex boundary values have the form /84

$$\begin{aligned}
\widetilde{T}_2 &= T_2 - iEhc\varkappa_1 = T_2^* - iEhc\widetilde{\varkappa}_1,\\
\widetilde{Q}_{21} &= Q_{21} + iEhc\varkappa_{12} = Q_{21}^* + iEhc\vec{\varkappa}_{12},\\
\widetilde{Q}_{2n} &= Q_{2n} - iEhc\varkappa_{1n} = Q_{2n}^* - iEhc\widetilde{\varkappa}_{1n},\\
\widetilde{M}_2 &= M_2 + iEhc\varepsilon_1 = M_2^* + iEhc\widetilde{\varepsilon}_1,\\
\vec{\widetilde{Q}}_2 &= \widetilde{T}_2\vec{e}_2 - \widetilde{Q}_{21}\vec{e}_1 + \widetilde{Q}_{2n}\vec{n}.
\end{aligned} \tag{9.27}$$

The following approximate expressions are valid for them

$$\begin{aligned}
&\widetilde{Q}_2 = \widetilde{T}_2,\ \widetilde{Q}_{21} = -\widetilde{S},\ \widetilde{M}_2 + ic(1+\nu)\widetilde{T}_2 = ic\widetilde{T},\\
&\widetilde{Q}_{2n} + i(1+\nu)\frac{c}{A_1}\frac{\partial\widetilde{S}}{\partial\alpha_1} - m_2 - i(1+\nu)cq_2 = i\frac{c}{A_2}\frac{\partial\widetilde{T}}{\partial\alpha_2}.
\end{aligned} \tag{9.28}$$

The junction conditions for the case of a smooth joining of the shells have the following form

$$\begin{aligned}
&T_2^{\mathrm{I}} = T_2^{\mathrm{II}},\ Q_{21}^{\mathrm{I}} = Q_{21}^{\mathrm{II}},\ Q_{2n}^{\mathrm{I}} = Q_{2n}^{\mathrm{II}},\ M_2^{\mathrm{I}} = M_2^{\mathrm{II}},\\
&\varkappa_1^{\mathrm{I}} = \varkappa_1^{\mathrm{II}},\ \varkappa_{12}^{\mathrm{I}} = \varkappa_{12}^{\mathrm{II}},\ \varkappa_{1n}^{\mathrm{I}} = \varkappa_{1n}^{\mathrm{II}}\ \ \varepsilon_1^{\mathrm{I}} = \varepsilon_1^{\mathrm{II}}.
\end{aligned} \tag{9.29}$$

For shells of uniform thickness and having the same elastic constants, the conditions are replaced by four complex conditions

$$\widetilde{T}_2^{\mathrm{I}} = \widetilde{T}_2^{\mathrm{II}},\ \widetilde{Q}_{21}^{\mathrm{I}} = \widetilde{Q}_{21}^{\mathrm{II}},\ \widetilde{Q}_{2n}^{\mathrm{I}} = \widetilde{Q}_{2n}^{\mathrm{II}},\ \widetilde{M}_2^{\mathrm{I}} = \widetilde{M}_2^{\mathrm{II}}, \tag{9.30}$$

which according to (9.28) can be reduced to the form

$$\begin{aligned}
&\left(\widetilde{T}_2\right)^{\mathrm{I}} = \left(\widetilde{T}_2\right)^{\mathrm{II}},\ \left(\widetilde{S}\right)^{\mathrm{I}} = \left(\widetilde{S}\right)^{\mathrm{II}},\\
&\left\{\frac{ic}{A_2}\frac{\partial\widetilde{T}}{\partial\alpha_2} + m_2 + i(1+\nu)q_2\right\}^{\mathrm{I}} = \{\ldots\}^{\mathrm{II}},\\
&(\widetilde{T})^{\mathrm{I}} = (\widetilde{T})^{\mathrm{II}}\ \ \left((\widetilde{T}_1)^{\mathrm{I}} = (\widetilde{T}_1)^{\mathrm{II}}\right).
\end{aligned} \tag{9.31}$$

If the junction is not smooth, the first three conditions of (9.30) are replaced by a single vector condition

$$\vec{\widetilde{Q}}_2^{\mathrm{I}} = \vec{\widetilde{Q}}_2^{\mathrm{II}}, \tag{9.32}$$

which is projected along three selected directions.

In the relationships which we have used the values of the static system (T_1^*, T_2^*, ..., H^*) represent a particular solution of the system of equi- /85

librium equations

$$\begin{aligned}
&\frac{\partial A_2 T_1^*}{\partial \alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2 S^*}{\partial \alpha_2}-\frac{\partial A_2}{\partial \alpha_1}T_2^*+\frac{1}{R_1}\left(\frac{\partial A_2 M_1^*}{\partial \alpha_1}+\frac{1}{A_1}\cdot\frac{\partial A_1^2 H^*}{\partial \alpha_2}-\right.\\
&\left.-\frac{\partial A_2}{\partial \alpha_1}M_2^*\right)+\frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{R_1}H^*\right)+\frac{1}{R_2}\frac{\partial A_1}{\partial \alpha_2}H^*=-A_1A_2q_1,\\
&\frac{\partial A_1 T_2^*}{\partial \alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2 S^*}{\partial \alpha_1}-\frac{\partial A_1}{\partial \alpha_2}T_1^*+\frac{1}{R_2}\left(\frac{\partial A_1 M_2^*}{\partial \alpha_2}+\frac{1}{A_2}\cdot\frac{\partial A_2^2 H^*}{\partial \alpha_1}-\right.\\
&\left.-\frac{\partial A_1}{\partial \alpha_2}M_1^*\right)+\frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{R_2}H^*\right)+\frac{1}{R_1}\frac{\partial A_2}{\partial \alpha_1}H^*=-A_1A_2q_2,\\
&\frac{T_1^*}{R_1}+\frac{T_2^*}{R_2}-\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{1}{A_1}\left(\frac{\partial A_2 M_1^*}{\partial \alpha_1}+\frac{1}{A_1}\cdot\frac{\partial A_1^2 H^*}{\partial \alpha_2}-\frac{\partial A_2}{\partial \alpha_1}M_2^*\right)+\right.\\
&\left.+\frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left(\frac{\partial A_1 M_2^*}{\partial \alpha_2}+\frac{1}{A_2}\cdot\frac{\partial A_2^2 H^*}{\partial \alpha_1}-\frac{\partial A_1}{\partial \alpha_2}M_1^*\right)\right\}=q_n.
\end{aligned}\tag{9.33}$$

which give the same principal vector and principal moment as the unknown solution along each particular contour of the boundary region. This provides for the single value of the stress functions $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$. For the static system in the

first part of the book we shall frequently adopt the membrane solution which satisfies the following system of equations

$$\begin{aligned}
&\frac{\partial A_2 T_1^*}{\partial \alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2 S^*}{\partial \alpha_2}-\frac{\partial A_2}{\partial \alpha_1}T_2^*=-A_1A_2q_1,\\
&\frac{\partial A_1 T_2^*}{\partial \alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2 S^*}{\partial \alpha_1}-\frac{\partial A_1}{\partial \alpha_2}T_1^*=-A_1A_2q_2,\\
&\frac{T_1^*}{R_1}+\frac{T_2^*}{R_2}=q_n.
\end{aligned}\tag{9.34}$$

CHAPTER V. THEORY OF SHELLS OF REVOLUTION

1. Shells of Revolution

For shells of revolution (shells whose middle surface is formed by the rotation of a two-dimensional curve around a straight line lying in the same plane) we select the angles $\theta = \alpha_1$, $\varphi = \alpha_2$ (fig. 15) as the curvilinear /86

coordinates. In this case the coordinate lines (we shall call them, respectively, the meridians and parallel circles) coincide with the lines of curvature. This is easy to see if we note from figure 15 that, as we move along the coordinate lines, the increment in the orthogonal unit vector of the normal to the middle surface coincides (in the limit) in direction with the tangents to the coordinate lines (see Chapter I, end of section 1). It is also easy to see that

$$ds_1 = R_1 d\theta, \quad ds_2 = R_2 \sin\theta d\varphi, \tag{1.1}$$

from which it follows that

$$A_1 = R_1, \qquad A_2 = R_2 \sin\theta. \tag{1.2}$$

Here R_1 and R_2 are the principal radii of curvature. Due to the axial symmetry

of the middle surface both radii of curvature as well as the Lamé parameters do not depend on the angle φ. The Codazzi-Gauss equations (I, 1.31) are reduced to a single relationship

$$\frac{dR_2 \sin\theta}{d\theta} = R_1 \cos\theta. \tag{1.3}$$

In considering the shells of revolution it is convenient to utilize the quantities r and z, where r is the distance of a point on the middle surface from the axis of rotation and z is its distance from some initial plane which is perpendicular to the axis. From figure 15 we can see that

$$r = R_2 \sin\theta. \tag{1.4}$$

Further, by projecting an arc element of the meridian ds_1 along the vertical and horizontal directions, we obtain (fig. 16) /87

$$\begin{aligned} dz &= -\sin\theta ds_1 = -R_1 \sin\theta d\theta, \\ dr &= \cos\theta ds_1 = R_1 \cos\theta d\theta, \end{aligned} \tag{1.5}$$

from which it follows that

$$\frac{dr}{d\theta} = R_1 \cos\theta, \quad \frac{dz}{d\theta} = -R_1 \sin\theta, \tag{1.6}$$

$$z - z_0 = -\int_{\theta_0}^{\theta} R_1 \sin\theta d\theta. \tag{1.7}$$

The first of the equalities which have been obtained coincides with the Gauss equation (1.3) and in essence means that an infinitely small increment in the arc of the meridian ds_1 corresponds to an infinitely small increment in the distance to the axis of rotation, $dr = ds_1 \cos\theta$. Thus in the considered case there is a direct provision for the Codazzi-Gauss equation to be a condition for the continuity of the middle surface.

We introduce unit vectors $\vec{e}_\rho$, $\vec{e}_z$ along the horizontal and vertical directions. They are related to the orthogonal unit vectors, which were determined earlier, by the following relationships (fig. 16)

$$\begin{aligned} \vec{e}_\rho &= \cos\theta \vec{e}_1 + \sin\theta \vec{n}, \quad \vec{e}_1 = \cos\theta \vec{e}_\rho - \sin\theta \vec{e}_z, \\ \vec{e}_z &= -\sin\theta \vec{e}_1 + \cos\theta \vec{n}, \quad \vec{n} = \sin\theta \vec{e}_\rho + \cos\theta \vec{e}_z. \end{aligned} \tag{1.8}$$

Using them to resolve the displacement vector ($u_1 = u$, $u_2 = v$)

$$\vec{U} = u\vec{e}_1 + v\vec{e}_2 + w\vec{n} = u_\rho \vec{e}_\rho + v\vec{e}_2 + u_z \vec{e}_z \tag{1.9}$$

along the horizontal and vertical directions, we obtain the following expressions for its components along the new directions /88

$$\begin{aligned} u_\rho &= \cos\theta u + \sin\theta w, \\ u_z &= -\sin\theta u + \cos\theta w \end{aligned} \quad \left(\begin{aligned} u &= \cos\theta u_\rho - \sin\theta u_z \\ w &= \sin\theta u_\rho + \cos\theta u_z \end{aligned} \right). \tag{1.10}$$

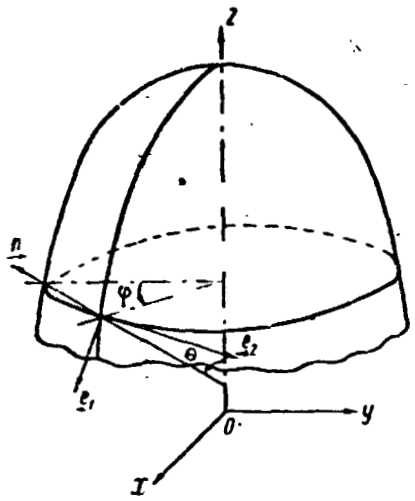

Figure 15

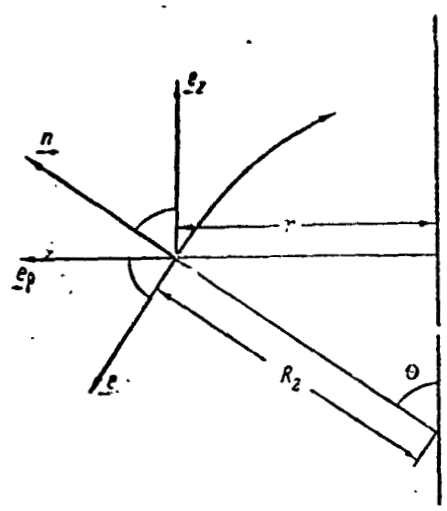

Figure 16

We use these relationships to find the following expressions for the angles of rotation ($\vartheta_1 = \vartheta$, $\vartheta_2 = \psi$) from (II, 1.15, 1.8)

$$\begin{aligned}
\vartheta &= -\frac{1}{R_1}\left(\frac{\partial w}{\partial \vartheta} - u\right) = -\frac{1}{R_1}\left(\sin\theta\frac{\partial u_\rho}{\partial\theta} + \cos\theta\frac{\partial u_z}{\partial\theta}\right),\\
\psi &= -\frac{1}{R_2\sin\theta}\left(\frac{\partial w}{\partial\varphi} - \sin\theta\, v\right) =\\
&= -\frac{1}{r}\left(\sin\theta\frac{\partial u_\rho}{\partial\varphi} + \cos\theta\frac{\partial u_z}{\partial\varphi} - \sin\theta\, v\right),\\
\omega_1 &= \frac{1}{R_1}\frac{\partial v}{\partial\theta},\quad \omega_2 = \frac{1}{R_2\sin\theta}\left(\frac{\partial u}{\partial\varphi} - \cos\theta\, v\right) =\\
&= \frac{1}{r}\left(\cos\theta\cdot\frac{\partial u_\rho}{\partial\varphi} - \sin\theta\frac{\partial u_z}{\partial\varphi} - \cos\theta\, v\right)
\end{aligned} \tag{1.11}$$

and the strain components

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{R_1}\left(\frac{\partial u}{\partial\theta} + w\right) = \frac{1}{R_1}\left(\cos\theta\frac{\partial u_\rho}{\partial\theta} - \sin\theta\frac{\partial u_z}{\partial\theta}\right) =\\
&= \frac{1}{R_1\cos\theta}\frac{\partial u_\rho}{\partial\theta} + \operatorname{tg}\theta\vartheta = -\frac{1}{R_1\sin\theta}\frac{\partial u_z}{\partial\theta} - \operatorname{ctg}\theta\, v,\\
\varepsilon_2 &= \frac{1}{R_2\sin\theta}\left(\frac{\partial v}{\partial\varphi} + \cos\theta u + \sin\theta\, w\right) = \frac{1}{r}\left(\frac{\partial v}{\partial\varphi} + u_\rho\right),\\
\omega &= \frac{R_2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{v}{R_2\sin\theta}\right) + \frac{1}{R_2\sin\theta}\frac{\partial u}{\partial\varphi} =\\
&= \frac{r}{R_1}\frac{\partial}{\partial\theta}\left(\frac{v}{r}\right) + \frac{1}{r}\left(\cos\theta\frac{\partial u_\rho}{\partial\varphi} - \sin\theta\frac{\partial u_z}{\partial\varphi}\right),\\
\varkappa_1 &= -\frac{1}{R_1}\frac{\partial}{\partial\theta}\left[\frac{1}{R_1}\left(\frac{\partial w}{\partial\vartheta} - u\right)\right] = \frac{1}{R_1}\frac{\partial\vartheta}{\partial\theta},
\end{aligned} \tag{1.12}$$

$$\varkappa_2 = -\frac{1}{(R_2\sin\theta)^2}\left(\frac{\partial^2 w}{\partial\varphi^2} - \sin\theta\frac{\partial v}{\partial\varphi}\right) - \frac{\cos\theta}{R_1R_2\sin\theta}\left(\frac{\partial w}{\partial\theta} - u\right) =$$
$$= -\frac{1}{r^2}\left(\sin\theta\frac{\partial^2 u_\rho}{\partial\varphi^2} + \cos\theta\frac{\partial^2 u_z}{\partial\varphi^2} - \sin\theta\frac{\partial v}{\partial\varphi}\right) + \frac{1}{r}\cos\theta\vartheta,$$
$$\tau = -\frac{1}{R_1}\frac{\partial}{\partial\theta}\left[\frac{1}{R_2\sin\theta}\left(\frac{\partial w}{\partial\varphi} - \sin\theta v\right)\right] + \frac{1}{R_1R_2\sin\theta}\left(\frac{\partial u}{\partial\varphi} - \cos\theta v\right) =$$
$$= -\frac{\sin\theta}{R_1}\frac{\partial^2}{\partial\theta\partial\varphi}\left(\frac{u_\rho}{r}\right) - \frac{\cos\theta}{R_1}\frac{\partial^2}{\partial\theta\partial\varphi}\left(\frac{u_z}{r}\right) + \frac{\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{v}{r}\right).$$

(Here and below we use the Codazzi-Gauss equations without any special reference to this fact.)

The equilibrium equations for the shells of revolution (IV, 9.33) are written in the form /89

$$\begin{aligned}
&\frac{\partial R_2\sin\theta}{\partial\theta} + R_1\frac{\partial S^*}{\partial\varphi} - R_1\cos\theta T_2^* + \\
&+\frac{1}{R_1}\left[\frac{\partial R_2\sin\theta M_1^*}{\partial\theta} + 2R_1\frac{\partial H^*}{\partial\varphi} - R_1\cos\theta M_2^*\right] = -R_1R_2\sin\theta q_1, \\
&R_1\frac{\partial T_2^*}{\partial\varphi} + \frac{1}{R_2\sin\theta}\frac{\partial R_2^2\sin^2\theta S^*}{\partial\theta} + \\
&+\frac{1}{R_2}\left[R_1\frac{\partial M_2^*}{\partial\varphi} + 2R_2\sin\theta\frac{\partial H^*}{\partial\theta} + 2(R_1+R_2)\cos\theta H^*\right] = \\
&= -R_1R_2\sin\theta q_2, \\
&\frac{T_1^*}{R_1} + \frac{T_2^*}{R_2} - \frac{1}{R_1R_2\sin\theta}\left\{\frac{\partial}{\partial\theta}\frac{1}{R_1}\left(\frac{\partial R_2\sin\theta M_1^*}{\partial\theta} + R_1\frac{\partial H^*}{\partial\varphi} - R_1\cos\theta M_2^*\right) + \right. \\
&\left. + \frac{1}{R_2\sin\theta}\left(R_1\frac{\partial^2 M_2^*}{\partial\varphi^2} + \frac{1}{R_2\sin\theta}\frac{\partial^2 R_2^2\sin^2\theta H^*}{\partial\theta\partial\varphi}\right)\right\} = q_n.
\end{aligned} \tag{1.13}$$

The system expressed in terms of the complex forces (IV, 9.1) assumes the following form

$$\begin{aligned}
&\frac{\partial R_2\sin\theta\widetilde{T}_1}{\partial\theta} + R_1\frac{\partial\widetilde{S}}{\partial\varphi} - R_1\cos\theta\widetilde{T}_2 + ic\frac{R_2\sin\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta} = -R_1R_2\sin\theta q_1, \\
&R_1\frac{\partial\widetilde{T}_2}{\partial\varphi} + \frac{1}{R_2\sin\theta}\frac{\partial R_2^2\sin^2\theta\widetilde{S}}{\partial\theta} + ic\frac{R_1}{R_2}\frac{\partial\widetilde{T}}{\partial\varphi} = -R_1R_2\sin\theta q_2, \\
&\frac{\widetilde{T}_1}{R_1} + \frac{\widetilde{T}_2}{R_2} - ic\Delta\widetilde{T} = q_n,
\end{aligned} \tag{1.14}$$

where we now have

$$\Delta = \frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\right) + \frac{1}{(R_2\sin\theta)^2}\frac{\partial^2}{\partial\varphi^2}.$$

According to the relationships presented in Chapter II, section 3, the strain boundary values along the parallel circle are written in the following way

$$\begin{aligned}
&\varkappa_2 = \frac{1}{R_2 \sin\theta}\left(\frac{\partial\psi}{\partial\varphi} + \cos\theta\,\vartheta\right),\\
&\varkappa_{21} = \frac{1}{R_2 \sin\theta}\left(\frac{\partial\vartheta}{\partial\varphi} - \cos\theta\,\psi - \sin\theta\,\omega_z\right) = \tau - \frac{\omega}{R_2},\\
&\varkappa_{2n} = \frac{1}{R_2 \sin\theta}\left(\frac{\partial\omega_2}{\partial\varphi} + \sin\theta\,\vartheta\right) =\\
&= -\frac{1}{R_1 R_2 \sin\theta}\left(\frac{\partial R_2 \sin\theta\,\varepsilon_2}{\partial\theta} - R_1\frac{\partial\omega}{\partial\varphi} - R_1\cos\theta\,\varepsilon_1\right),\\
&\vec{\varkappa}_2 = -\varkappa_2\vec{e}_1 + \varkappa_{21}\vec{e}_2 - \varkappa_{2n}\vec{n};\ \varepsilon_2.
\end{aligned} \tag{1.15}$$

Let us resolve the vector $\vec{\varkappa}_2$ along new directions

$$\vec{\varkappa}_2 = -\varkappa_\rho\vec{e}_\rho + \varkappa_{21}\vec{e}_2 - \varkappa_z\vec{e}_z. \tag{1.16}$$

Then it follows from equality (1.8) that /90

$$\varkappa_\rho = \cos\theta\,\varkappa_2 + \sin\theta\,\varkappa_{2n},\quad \varkappa_z = -\sin\theta\,\varkappa_2 + \cos\theta\,\varkappa_{2n}. \tag{1.17}$$

Using the last two relationships in (1.11) and also (1.15) and (1.12) we obtain

$$\begin{aligned}
&\varkappa_\rho = \frac{1}{r}\left\{\vartheta - \frac{1}{r}\frac{\partial^2 u_z}{\partial\varphi^2}\right\},\quad \varkappa_{21} = \frac{1}{r}\left\{\frac{\partial}{\partial\varphi}\left(\vartheta + \frac{1}{r}u_z\right)\right\},\\
&\varkappa_z = \frac{1}{r}\left\{-\frac{\partial v}{\partial\varphi} + \frac{\partial^2 u_\rho}{\partial\varphi^2}\right\},\quad \varepsilon_2 = \frac{1}{r}\left\{\frac{\partial v}{\partial\varphi} + u_\rho\right\}.
\end{aligned} \tag{1.18}$$

In a similar manner we obtain

$$\vec{Q}_1 = Q_\rho\vec{e}_\rho + Q_{12}\vec{e}_2 - Q_z\vec{e}_z, \tag{1.19}$$

where the quantities

$$Q_\rho = \cos\theta\,T_1 + \sin\theta\,Q_{1n},\quad Q_z = \sin\theta\,T_1 - \cos\theta\,Q_{1n} \tag{1.20}$$

are the horizontal and vertical generalized boundary forces.

For complex boundary values we find the following expression

$$\begin{aligned}
\widetilde{Q}_\rho &= Q_\rho - iEhc\varkappa_\rho = Q_\rho^* - iEhc\widetilde{\varkappa}_\rho,\\
\widetilde{Q}_z &= Q_z + iEhc\varkappa_z = Q_z^* + iEhc\widetilde{\varkappa}_z,\\
\widetilde{Q}_{12} &= Q_{12} + iEhc\varkappa_{21} = Q_{12}^* + iEhc\widetilde{\varkappa}_{21},\\
\widetilde{M}_1 &= M_1 + iEhc\varepsilon_2 = M_1^* + iEhc\widetilde{\varepsilon}_2,\\
\vec{\widetilde{Q}}_1 &= \widetilde{Q}_\rho\vec{e}_\rho + \widetilde{Q}_{12}\vec{e}_2 - \widetilde{Q}_z\vec{e}_z,
\end{aligned} \tag{1.21}$$

where

$$\begin{aligned}
&\widetilde{\varkappa}_\rho = \frac{1}{r}\left\{\widetilde{\vartheta} - \frac{1}{r}\frac{\partial^2\widetilde{u}_z}{\partial\varphi^2}\right\}, \quad \varkappa_{21} = \frac{1}{r}\left\{\frac{\partial}{\partial\varphi}\left(\widetilde{\vartheta} + \frac{\widetilde{u}_z}{r}\right)\right\},\\
&\widetilde{\varkappa}_z = \frac{1}{r^2}\left\{-\frac{\partial\widetilde{v}}{\partial\varphi} + \frac{\partial^2\widetilde{u}_\rho}{\partial\varphi^2}\right\}, \quad \widetilde{\varepsilon}_2 = \frac{1}{r}\left\{\frac{\partial\widetilde{v}}{\partial\varphi} + \widetilde{u}_\rho\right\};
\end{aligned} \tag{1.22}$$

$$Q_{12}^* = S^* + \frac{2H^*}{R_2};$$

$$Q_{1n}^* = \frac{1}{R_1R_2\sin\theta}\left(\frac{\partial R_2\sin\theta M_1^*}{\partial\theta} + R_1\frac{\partial 2H^*}{\partial\varphi} - R_1\cos\theta M_2^*\right), \tag{1.23}$$

$$\begin{aligned}
Q_\rho^* &= \cos\theta T_1^* + \sin\theta Q_{1n}^*,\\
Q_z^* &= \sin\theta T_1^* - \cos\theta Q_{1n}^*.
\end{aligned}$$

As far as the simplified expressions for the complex forces are concerned, they take on the following form in accordance with relationships (IV, 9.18)

$$\begin{aligned}
&\widetilde{Q}_{12} = \widetilde{S},\\
&\widetilde{Q}_\rho + \underline{i(1+\nu)\frac{c}{R_2}\frac{\partial\widetilde{S}}{\partial\varphi} - \sin\theta\,[m_1 + i(1+\nu)cq_1]} =\\
&\qquad = \cos\theta\widetilde{T}_1 + ic\frac{\sin\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta},\\
&\widetilde{Q}_z - \underline{i(1+\nu)\frac{c\operatorname{ctg}\theta}{R_2}\frac{\partial\widetilde{S}}{\partial\varphi} + \cos\theta\,[m_1 + i(1+\nu)cq_1]} =\\
&\qquad = \sin\theta\widetilde{T}_1 - ic\frac{\cos\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta},\\
&\widetilde{M}_1 = ic\left\{\widetilde{T}_2 - \nu\widetilde{T}_1\right\}.
\end{aligned} \tag{1.24}$$

In the overwhelming majority of problems the underlined terms are small. /91
We drop them and transform to simpler expressions for the boundary complex values

$$\widetilde{Q}_{12}=\widetilde{S}, \qquad \widetilde{Q}_{\rho}=\cos\theta\widetilde{T}_1+ic\frac{\sin\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta},$$
$$\widetilde{M}_1=ic\left\{\widetilde{T}_2-\nu\widetilde{T}_1\right\}, \quad \widetilde{Q}_z=\sin\theta\widetilde{T}_1-ic\frac{\cos\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta}. \tag{1.25}$$

Along the boundary element which coincides with the meridian, the strain boundary quantities (IV, 9.25) assume the form

$$\varkappa_1=\frac{1}{R_1}\frac{\partial\vartheta}{\partial\theta},$$
$$-\varkappa_{12}=\frac{1}{R_1}\left(\frac{\partial\psi}{\partial\theta}-\omega_1\right)=\tau-\frac{\omega}{R_1}, \tag{1.26}$$
$$-\varkappa_{1n}=-\frac{1}{R_1}\left(\frac{\partial\omega_1}{\partial\theta}+\psi\right)=\frac{1}{R_1R_2\sin\theta}\left(R_1\frac{\partial\varepsilon_1}{\partial\varphi}-\frac{\partial R_2\sin\theta\,\omega}{\partial\theta}\right),$$
$$\vec{\varkappa}_1=-\varkappa_{12}\vec{e}_1-\varkappa_1\vec{e}_2-\varkappa_{1n}\vec{n}; \quad \varepsilon_1.$$

For the static and complex boundary values the following relationships are valid

$$M_2,\ \vec{Q}_2=-Q_{21}\vec{e}_1+T_2\vec{e}_2+Q_{2n}\vec{n}, \tag{1.27}$$

where

$$Q_{21}=S+\frac{2H}{R_1},$$
$$Q_{2n}=\frac{1}{R_1R_2\sin\theta}\left(R_1\frac{\partial M_2}{\partial\varphi}+2\frac{\partial R_2\sin\theta H}{\partial\theta}\right),$$
$$\widetilde{T}_2=T_2-iEhc\varkappa_1=T_2^*-iEhc\widetilde{\varkappa}_1, \tag{1.28}$$
$$\widetilde{Q}_{21}=Q_{21}+iEhc\varkappa_{12}=Q_{21}^*+iEhc\widetilde{\varkappa}_{12},$$
$$\widetilde{Q}_{2n}=Q_{2n}-iEhc\varkappa_{1n}=Q_{2n}^*-iEhc\widetilde{\varkappa}_{1n},$$
$$\widetilde{M}_2=M_2+iEhc\varepsilon_1=M_2^*+iEhc\widetilde{\varepsilon}_1.$$

The latter quantities, according to (IV, 9.28) may be used in the following simplified form /92

$$\widetilde{Q}_2=\widetilde{T}_2, \quad Q_{21}=-\widetilde{S},$$
$$\widetilde{Q}_{2n}+i(1+\nu)\frac{c}{R_1}\frac{\partial\widetilde{S}}{\partial\theta}-[m_2+i(1+\nu)q_2]=i\frac{c}{R_2\sin\theta}\frac{\partial\widetilde{T}}{\partial\varphi}, \tag{1.29}$$
$$\widetilde{M}_2=ic\left\{\widetilde{T}_1-\nu\widetilde{T}_2\right\},$$

and in this case the underlined terms are usually small and can be dropped.

2. Transformation of the System of Equations Containing Complex Forces

Following V. V. Novozhilov, we reduce the system of equations expressed in terms of complex forces (1.14) to a more symmetric and simple system, consisting of two equations of the function $\widetilde{T}$ and

$$\widetilde{U}=R_2\sin^2\theta\,\widetilde{T}_1-ic\frac{R_2\sin\theta}{R_1}\cos\theta\frac{\partial\widetilde{T}}{\partial\theta}. \tag{2.1}$$

In addition we introduce one other function

$$\widetilde{V}=(R_2\sin\theta)^2\widetilde{S}. \tag{2.2}$$

We begin the transformation of the system by replacing the function $\widetilde{T}_2$ in the second equation of (1.14) with the expression

$$\widetilde{T}_2=\widetilde{T}-\widetilde{T}_1. \tag{2.3}$$

Then, by using equalities (2.1) and (2.2) we eliminate $\widetilde{T}_1$ and $\widetilde{S}$, and arrive at the equation

$$\frac{\partial\widetilde{V}}{\partial\theta}-\frac{R_1}{\sin\theta}\frac{\partial\widetilde{U}}{\partial\varphi}-icR_2\cos\theta\frac{\partial^2\widetilde{T}}{\partial\varphi\partial\theta}+R_1(R_2+ic)\sin\theta\frac{\partial^2\widetilde{T}}{\partial\varphi^2}=-R_1R_2^2\sin^2\theta\,q_2. \tag{2.4}$$

Furthermore, we multiply the first equation by $\sin\theta$, the third one by $R_1R_2\sin\theta\cos\theta$ and add them. After simple transformations we obtain the second equation

$$\frac{\partial\widetilde{U}}{\partial\theta}+\frac{R_1}{R_2^2\sin\theta}\frac{\partial\widetilde{V}}{\partial\varphi}-ic\frac{R_1\cos\theta}{R_2\sin\theta}\frac{\partial^2\widetilde{T}}{\partial\varphi^2}=(q_n\cos\theta-q_1\sin\theta)R_1R_2\sin\theta. \tag{2.5}$$

We obtain the third equation by eliminating $\widetilde{T}_2$ and $\widetilde{T}_1$ from the last equation of system (1.14) by means of equalities (2.3) and (2.1)

$$-ic\left\{\frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta}\right)+\frac{1}{R_2\sin^2\theta}\frac{\partial^2\widetilde{T}}{\partial\varphi^2}\right\}+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\widetilde{U}}{\sin^2\theta}+$$
$$+\widetilde{T}=R_2q_n.$$

If we add to the latter equation the equation which was obtained by eliminating the functions $\widetilde{V}$ from relationships (2.4) and (2.5), we arrive at the desired system of equations /93

$$\begin{aligned} G(\widetilde{U})-\left[1-ic\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\frac{\partial^2\widetilde{T}}{\partial\varphi^2}&=f(\theta,\varphi),\\ -icG(\widetilde{T})+\widetilde{T}+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\widetilde{U}&=R_2q_n, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} G&=\frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\right)+\frac{1}{R_2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},\\ f(\theta,\varphi)&=\frac{1}{R_1R_2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[(q_n\cos\theta-q_1\sin\theta)R_2^3\sin^2\theta\right]+\right.\\ &\left.+\frac{\partial q_2}{\partial\varphi}R_1R_2^2\sin^2\theta\right\}. \end{aligned} \tag{2.8}$$

We note that for a spherical shell ($R_1 = R_2 = R_0$) the resulting system is substantially simplified and its solution is reduced to the successive solution of two second order equations.

By using the functions $\widetilde{T}$ and $\widetilde{U}$, obtained from the basic system, it is easy to determine all force components. Thus from (2.1) we find

$$\widetilde{T}_1=\frac{\widetilde{U}}{R_2\sin^2\theta}+ic\frac{\operatorname{ctg}\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta}. \tag{2.9}$$

From this it follows, according to (2.3), that

$$\widetilde{T}_2=-\frac{\widetilde{U}}{R_2\sin^2}+\widetilde{T}-ic\frac{\operatorname{ctg}\theta}{R_1}\frac{\partial\widetilde{T}}{\partial\theta}. \tag{2.10}$$

Furthermore, from equations (2.4) and (2.5) we determine the derivatives of $\widetilde{V}$ with respect to θ and φ. Finally, from (2.2) we obtain

$$\widetilde{S}=\frac{\widetilde{V}}{(R_2\sin\theta)^2}. \tag{2.11}$$

For the forces which have been established the determination of complex displacements is reduced to the solution of the approximately compatible relationships of the complex Hooke's law (IV, 9.5), which, for shells of revolution, take the following form

$$\begin{aligned}
&(R_1\widetilde{\varepsilon}_1)\quad \frac{\partial\widetilde{u}}{\partial\theta}+\widetilde{w}=\frac{1}{\cos\theta}\frac{\partial\widetilde{u}_\rho}{\partial\theta}+R_1\operatorname{tg}\theta\,\widetilde{\vartheta}=\frac{R_1}{Eh}\left\{\widetilde{T}_1-\nu\widetilde{T}_2+\frac{i}{c}M_2^*\right\},\\
&(R_2\sin\theta\,\widetilde{\varepsilon}_2)\quad \frac{\partial\widetilde{v}}{\partial\varphi}+\cos\theta\,\widetilde{u}+\sin\theta\,\widetilde{w}=\frac{\partial\widetilde{v}}{\partial\varphi}+\widetilde{u}_\rho=\\
&\qquad=\frac{R_2\sin\theta}{Eh}\left\{\widetilde{T}_2-\nu\widetilde{T}_1+\frac{i}{c}M_1^*\right\},\\
&(\widetilde{\omega})\quad \frac{R_2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\widetilde{v}}{R_2\sin\theta}\right)+\frac{1}{R_2\sin\theta}\frac{\partial\widetilde{u}}{\partial\varphi}=\frac{r}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\widetilde{v}}{r}\right)+\\
&\quad+\frac{1}{r}\left(\cos\theta\frac{\partial\widetilde{u}_\rho}{\partial\varphi}-\sin\theta\frac{\partial\widetilde{u}_z}{\partial\varphi}\right)=\frac{1}{Eh}\left\{2(1+\nu)\widetilde{S}-\frac{i}{c}2H^*\right\},\\
&(\widetilde{\varkappa}_1)\quad -\frac{1}{R_1}\frac{\partial}{\partial\theta}\left[\frac{1}{R_1}\left(\frac{\partial\widetilde{w}}{\partial\theta}-\widetilde{u}\right)\right]=\frac{1}{R_1}\frac{\partial\widetilde{\vartheta}}{\partial\theta}=\frac{i}{Ehc}\left\{\widetilde{T}_2-T_2^*\right\},\\
&(\widetilde{\varkappa}_2)\quad -\frac{1}{(R_2\sin\theta)^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial\widetilde{w}}{\partial\varphi}-\sin\theta\,\widetilde{v}\right)-\frac{\cos\theta}{R_1R_2\sin\theta}\left(\frac{\partial\widetilde{w}}{\partial\theta}-\widetilde{u}\right)=\\
&=-\frac{1}{r^2}\left(\sin\theta\frac{\partial^2\widetilde{u}_\rho}{\partial\varphi^2}+\cos\theta\frac{\partial^2\widetilde{u}_z}{\partial\varphi^2}-\sin\theta\frac{\partial\widetilde{v}}{\partial\varphi}\right)+\frac{\cos\theta}{r}\widetilde{\vartheta}=\frac{i}{Ehc}(\widetilde{T}_1-T_1^*),\\
&(\widetilde{\tau})\quad -\frac{1}{R_1}\frac{\partial}{\partial\theta}\left[\frac{1}{R_2\sin\theta}\left(\frac{\partial\widetilde{w}}{\partial\varphi}-\sin\theta\,\widetilde{v}\right)+\frac{1}{R_1R_2\sin\theta}\left(\frac{\partial\widetilde{u}}{\partial\varphi}-\cos\theta\,\widetilde{v}\right)\right]=\\
&\qquad=-\frac{\sin\theta}{R_1}\frac{\partial^2}{\partial\theta\partial\varphi}\left(\frac{\widetilde{u}_\rho}{r}\right)-\frac{\cos\theta}{R_1}\frac{\partial^2}{\partial\theta\partial\varphi}\left(\frac{\widetilde{u}_z}{r}\right)+\frac{\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\widetilde{v}}{r}\right)=\\
&\qquad=-\frac{i}{Ehc}(\widetilde{S}-S^*).
\end{aligned} \tag{2.12}$$

Thus, all quantities of interest to us are expressed in terms of the basic complex functions $\widetilde{U}$, $\widetilde{V}$ and $\widetilde{T}$. The functions $\widetilde{U}$ and $\widetilde{V}$ introduced by us have a definite static-geometric meaning. Indeed, according to $(1.25)_4$ and (1.21) /94

$$\widetilde{U}=R_2\sin\theta\,\widetilde{Q}_z=r(Q_z+iEhc\varkappa_z). \tag{2.13}$$

A vertical force acts along the arc element ds_2 of the parallel circle. Its value is

$$dF_z=-Q_z ds_2=-R_2\sin\theta\,Q_z d\varphi=-U d\varphi\quad(U=\operatorname{Re}\widetilde{U}). \tag{2.14}$$

From this we can see that the real part of function $\widetilde{U}$ represents an axial (vertical) force, acting on the parallel circle per unit angle. The imaginary part, on the other hand, with an accuracy up to the multiplier, is equal to the change in the curvature of the parallel ring in the horizontal plane. In the same way, according to $(1.25)_1$ and (2.2),

$$\widetilde{V} = (R\sin\theta)^2 \widetilde{Q}_{12} = r^2 (Q_{12} + iEhc\varkappa_{21}). \tag{2.15}$$

The value of the axial (twisting) moment applied to the arc element ds_2 of the parallel ring, is equal to (fig. 19)

$$dB_z = Q_{12} R_1 \sin\theta\,(R_2 \sin\theta d\varphi) = V d\varphi \quad (V = \operatorname{Re}\widetilde{V}). \tag{2.16}$$

Thus the real part of the function $\widetilde{V}$ represents the twisting moment acting on the parallel ring again per unit value of angle φ. The imaginary part, with an accuracy up to the multiplier, is equal to the twisting of the normal element, determined by the parallel ring.

The basic difficulty in designing shells of revolution is associated /95 with the solution of the characteristic complex system (2.7). The latter consists of a system of equations of fourth order with partial derivatives and, in general, with variable coefficients. Its solution is a rather complex mathematical problem. The inclination to obtain simple solutions which can be used in practice requires that we find sufficiently effective particular methods for solving the system. One such method is the separation of variables considered in the next section.

3. Method of the Separation of Variables

The method in which variables are separated is widely used in the design of shells of revolution. The essence of this well-known method is that all quantities, both unknown and assigned, are represented as expansions in terms of the values (harmonics) of the angle φ, i.e., in the form

$$\Phi(\theta, \varphi) = \sum_{k=0}^{\infty} \{\Phi_k'(\theta)\cos k\varphi + \Phi_k^*(\theta)\sin k\varphi\}. \tag{3.1}$$

Since each term of the sum is a periodic function of φ, the proposed method is particularly convenient in analyzing shells which are closed in φ.

The coefficients of all relationships obtained above do not depend on φ. Therefore, if expansions of the form (3.1) are substituted into them and if the quantities $\begin{matrix}\cos k\varphi \\ \sin k\varphi\end{matrix}$ which enter as common multipliers are reduced, we obtain

independent relationships for each coefficient of the expansion. The basic difficulty of separating the variables is associated with the solution of a system of ordinary differential equations pertaining to each harmonic. The general solution is obtained by adding the particular solutions for all values of k.

The solutions corresponding to individual harmonics are also of independent interest. Thus, many problems associated with the design of shells of revolution are completely exhausted by considering the case k = 0. This case, of most practical interest, is called the symmetric case. The second one from the standpoint of significance is the antisymmetric case (k = 1). It is also called the case of wind loading. This is associated with the fact that when domes are analyzed for wind loading, the latter is simulated by means of the expression

$$q_n = p \sin\theta \cos\varphi.$$

Let us consider the case of the k-th harmonic, i.e., the case when the quantities of interest to us depend on the angle φ in the following manner

$$\begin{gathered}
u = u_{,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad v = v_{,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix}, \quad w = w_{,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad u_\rho = u_{\rho,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix},\\
u_z = u_{z,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \varepsilon_1 = \varepsilon_{1,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \varepsilon_2 = \varepsilon_{2,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \omega = \omega_{,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix},\\
\varkappa_1 = \varkappa_{1,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \varkappa_2 = \varkappa_{2,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \tau = \tau_{,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix}, \quad T_1 = T_{1,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix},\\
T_2 = T_{2,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad S = S_{,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix}, \quad M_1 = M_{1,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix},\\
M_2 = M_{2,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad H = H_{,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix},\\
Q_\rho = Q_{\rho,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad Q_z = Q_{z,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad Q_{12} = Q_{12,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix}, \quad \vartheta = \vartheta_{,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix},\\
\varkappa_\rho = \varkappa_{\rho,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \varkappa_z = \varkappa_{z,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad \varkappa_{21} = \varkappa_{21,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix},\\
U = U_{,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad V = V_{,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix}, \quad q_1 = q_{1,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix},\\
q_n = q_{n,k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, \quad q_2 = q_{2,k}\begin{matrix}\sin k\varphi\\ \cos k\varphi\end{matrix}.
\end{gathered} \tag{3.2}$$

Double functions have been introduced in this case to shorten the writing. /96 The case when the upper functions are left in all of the relationships will be called the first case of the k-th harmonic, and the lower functions will correspond to the second case. The same meaning is attributed to the double signs $\pm$ ($\mp$) used below. In addition, we shall assume that the quantities which differ from these above by the presence of additional signs will have the same relationship as a function of the angle φ.

Substituting the quantities (3.2) into the relationships of the preceding two sections, we obtain, first of all, the characteristic system which consists of ordinary differential equations

$$\begin{aligned} G_k[\widetilde{U}_{,k}] + k^2\left[1 - ic\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\widetilde{T}_{,k} &= f_k(\theta), \\ -icG_k[\widetilde{T}_{,k}] + \widetilde{T}_{,k} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\widetilde{U}_{,k} &= R_2 q_{n,k}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} G_k &= \frac{1}{R_1 R_2 \sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2 \sin\theta}{R_1}\frac{d}{d\theta}\right) - \frac{k^2}{R_2 \sin^2\theta}, \\ f_k(\theta) &= \frac{1}{R_1 R_2 \sin\theta}\left\{\frac{d}{d\theta}\left[\left(q_{n,k}\cos\theta - q_{1,k}\sin\theta\right)R_2^3 \sin^2\theta\right] \pm \right. \\ &\left. \pm kq_{2,k}R_1 R_2^2 \sin^2\theta\right\}. \end{aligned}$$

The complex forces are expressed in the following manner in terms of the values $\widetilde{U}_{,k}$ and $\widetilde{T}_{,k}$ obtained from this system

$$\begin{aligned} \widetilde{T}_{1,k} &= \frac{\widetilde{U}_{,k}}{R_2 \sin^2\theta} + ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,k}}{d\theta}, \\ \widetilde{T}_2 &= -\frac{\widetilde{U}_{,k}}{R_2 \sin^2\theta} + \widetilde{T}_{,k} - ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,k}}{d\theta}. \end{aligned} \tag{3.4}$$

As far as $\widetilde{V}_{,k}$ is concerned, (when $k \neq 0$) it is determined from the follow- /97
ing equality in accordance with (2.5)

$$\begin{aligned} \pm k\widetilde{V}_{,k} = &-\frac{R_2^2 \sin\theta}{R_1}\frac{d\widetilde{U}_{,k}}{d\theta} - ick^2 R_2 \cos\theta\,\widetilde{T}_{,k} + \\ &+ \left(q_{r,k}\cos\theta - q_{1,k}\sin\theta\right)R_2^3 \sin^2\theta. \end{aligned}$$

from which it follows that

$$\begin{aligned} \pm k\widetilde{S}_{,k} = &-\frac{1}{R_1 \sin\theta}\frac{d\widetilde{U}_{,k}}{d\theta} - ick^2\frac{\operatorname{ctg}\theta}{R_2 \sin\theta}\widetilde{T}_{,k} + \\ &+ R_2\left(q_{n,k}\cos\theta - q_{1,k}\sin\theta\right). \end{aligned} \tag{3.5}$$

(The symmetric case ($k = 0$) will be considered in a special manner in section 5.)

We have the following approximately consistent system of equations for determining the complex displacements

$$
\begin{aligned}
&(R_1\tilde{\varepsilon}_{1,k})\ \frac{d\tilde{u}_{,k}}{d\theta}+\tilde{w}_{,k}=-\frac{1}{\sin\theta}\frac{d\tilde{u}_{z,k}}{d\theta}-R_1\operatorname{ctg}\theta\ \tilde{\vartheta}_{,k}=\frac{1}{\cos\theta}\frac{d\tilde{u}_{\rho,k}}{d\theta}+\\
&\quad+R_1\operatorname{tg}\theta\ \tilde{\vartheta}_{,k}=\frac{R_1}{Eh}\left\{\tilde{T}_{1,k}-\nu\tilde{T}_{2,k}+\frac{i}{c}M^*_{2,k}\right\},\\
&(R_2\sin\theta\ \tilde{\varepsilon}_{2,k})\quad\pm k\tilde{v}_{,k}+\cos\theta\ \tilde{u}_{,k}+\sin\theta\,\tilde{w}_{,k}=\pm k\tilde{v}_{,k}+\tilde{u}_{\rho,k}\\
&\quad=\frac{R_2\sin\theta}{Eh}\left\{\tilde{T}_{2,k}-\nu\tilde{T}_{1,k}+\frac{i}{c}M^*_{1,k}\right\},\\
&(\tilde{\omega}_{,k})\quad\frac{R_2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{v}_{,k}}{R_2\sin\theta}\right)\mp\frac{k\tilde{u}_{,k}}{R_2\sin\theta}=\frac{r}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{v}_{,k}}{r}\right)+\\
&\mp\frac{k}{r}\left(\cos\theta\ \tilde{u}_{\rho,k}-\sin\theta\ \tilde{u}_{z,k}\right)=\frac{1}{Eh}\left\{2(1+\nu)\tilde{S}_{,k}-\frac{i}{c}2H^*_{,k}\right\},\\
&(\tilde{\varkappa}_{1,k})\quad-\frac{1}{R_1}\frac{d}{d\theta}\left[\frac{1}{R_1}\left(\frac{d\tilde{w}_{,k}}{d\theta}-\tilde{u}_{,k}\right)\right]=\frac{1}{R_1}\frac{d\tilde{\vartheta}_{,k}}{d\theta}=\frac{i}{Ehc}\left[\tilde{T}_{2,k}-T^*_{2,k}\right],\qquad(3.6)\\
&(\tilde{\varkappa}_{2,k})\quad\frac{k^2\tilde{w}_{,k}\pm k\sin\theta\,\tilde{v}_{,k}}{(R_2\sin\theta)^2}-\frac{\cos\theta}{R_1R_2\sin\theta}\left(\frac{d\tilde{w}_{,k}}{d\theta}-\tilde{u}_{,k}\right)=\\
&=\frac{1}{r^2}\left[k^2(\sin\theta\ \tilde{u}_{\rho,k}+\cos\theta\ \tilde{u}_{z,k})\pm k\sin\theta\ \tilde{v}_{,k}+\cos\theta\ r\tilde{\vartheta}_{,k}\right]=\\
&\quad=\frac{i}{Ehc}\left[\tilde{T}_{1,k}-T^*_{1,k}\right],\\
&(\tilde{\tau}_{,k})\quad\frac{1}{R_1}\frac{d}{d\theta}\left[\frac{\pm k\tilde{w}_{,k}+\sin\theta\ \tilde{v}_{,k}}{R_2\sin\theta}\right]-\frac{\cos\theta\ \tilde{v}_{,k}\pm k\tilde{u}_{,k}}{R_1R_2\sin\theta}=\\
&=\pm k\left[\frac{\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{u}_{\rho,k}}{r}\right)+\frac{\cos\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{u}_{z,k}}{r}\right)\right]+\frac{\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{v}_{,k}}{r}\right)=\\
&\quad=-\frac{i}{Ehc}\left(\tilde{S}_{,k}-S^*_{,k}\right).
\end{aligned}
$$

The complex boundary quantities assume the form /98

$$
\begin{aligned}
&\tilde{Q}_{\rho,k}=Q_{\rho,k}-iEhc\varkappa_{\rho,k}=Q^*_{\rho,k}-iEhc\tilde{\varkappa}_{\rho,k},\\
&\tilde{Q}_{z,k}=Q_{z,k}+iEhc\varkappa_{z,k}=Q^*_{z,k}+iEhc\tilde{\varkappa}_{z,k},\qquad(3.7)\\
&\tilde{Q}_{12,k}=Q_{12,k}+iEhc\varkappa_{21,k}=Q^*_{12,k}+iEhc\tilde{\varkappa}_{21,k},\\
&\tilde{M}_{1,k}=M_{1,k}+iEhc\varepsilon_{2,k}=M^*_{1,k}+iEhc\tilde{\varepsilon}_{2,k},
\end{aligned}
$$

where

$$
\begin{aligned}
&Q_{12,k}=S_{,k}+\frac{2H_{,k}}{R_2},\\
&Q_{\rho,k}=\cos\theta T_{1,k}+\sin\theta\ Q_{1n,k},\qquad(3.8)\\
&Q_{z,k}=\sin\theta T_{1,k}-\cos\theta\ Q_{1n,k};
\end{aligned}
$$

$$Q_{1n,k}=\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2s\quad)M_{1,k}}{d\theta}\pm kR_1\,2H_{,k}-R_1\cos\theta\;M_{2,k}\right),$$
$$\varkappa_{\rho,k}=\frac{1}{r}\left\{\vartheta_{,k}+\frac{k^2}{r}u_{z,k}\right\},\quad \varkappa_{21,k}=\frac{1}{r}\left\{\mp k\left(\vartheta_{,k}+\frac{u_{z,k}}{r}\right)\right\},\tag{3.9}$$
$$\varkappa_{z,k}=\frac{1}{r^2}\left\{\mp kv_{,k}-k^2u_{,k}\right\},\quad \varepsilon_{2,k}=\frac{1}{r}\left\{\pm kv_{,k}+u_{\rho,k}\right\}.$$

The following approximate expressions are valid for these quantities

$$\widetilde{Q}_{\rho,k}=\cos\theta\;\widetilde{T}_{1,k}+ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}_{,k}}{d\theta},$$
$$\widetilde{Q}_{z,k}=\sin\theta\;\widetilde{T}_{1,k}-ic\frac{\cos\theta}{R_1}\frac{d\widetilde{T}_{,k}}{d\theta},\tag{3.10}$$
$$\widetilde{M}_{1,k}=ic\left\{\widetilde{T}_{2,k}-\nu\widetilde{T}_{1,k}\right\},\quad \widetilde{Q}_{12,k}=\widetilde{S}_{,k}.$$

We note that there is a whole class of shells of revolution (pointed out by V. V. Novozhilov) for which the basic system of equations (3.3) breaks down into two independent equations. This rather broad class is characterized by the following relationships between the radii of curvature

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}=A=\text{const}.\tag{3.11}$$

It is easy to verify this by introducing a new auxiliary equation

$$\widetilde{W}_k=\widetilde{U}_{,k}+\alpha\widetilde{T}_{,k},\tag{3.12}$$

where α is a constant coefficient which must be determined. In view of conditions (3.11), the principal system (3.3) is written in the form

$$G_k\left\{\widetilde{U}_{,k}\right\}+k^2(1-icA)\widetilde{T}_{,k}=f_k(\theta),$$
$$-icG_k\left\{\widetilde{T}_{,k}\right\}+\widetilde{T}_{,k}+A\widetilde{U}_{,k}=R_2q_{n,k}.\tag{3.13}$$

Let us multiply the second equation by $i\frac{\alpha}{c}$ and add it to the first. By making use of the introduced function $\widetilde{W}_k$, this equation may be written in the form /99

$$G_k\left\{\widetilde{W}_k\right\}+i\frac{\alpha}{c}A\widetilde{W}_k+\left\{i\frac{\alpha}{c}(1-\alpha A)+k^2(1-icA)\right\}\widetilde{T}_{,k}=$$
$$=f_k(\theta)+i\frac{\alpha}{c}R_2q_{n,k}.\tag{3.14}$$

Let us select the constant α based on the condition that it makes the expression in the brackets equal zero, i.e., that it be a solution of the quadratic equation

$$A\alpha^2 - \alpha + ick^2(1 - icA) = 0.$$

Let α_1 and α_2 be the roots of this equation. Substituting them into (3.14) we obtain two equations

$$\begin{aligned} G_k\{\widetilde{W}_k^{(1)}\} + i\frac{\alpha_1}{c} A\widetilde{W}_k^{(1)} &= f_k(\theta) + i\frac{\alpha_1}{c} R_2 q_{n,k} \\ G_k\{\widetilde{W}_k^{(2)}\} + i\frac{\alpha_2}{c} A\widetilde{W}_k^{(2)} &= f_k(\theta) + i\frac{\alpha_2}{c} R_2 q_{n,k}, \end{aligned} \tag{3.15}$$

for the functions

$$\widetilde{W}_k^{(1)} = \widetilde{U}_{,k} + \alpha_1 \widetilde{T}_{,k}, \quad \widetilde{W}_k^{(2)} = \widetilde{U}_{,k} + \alpha_2 \widetilde{T}_{,k}.$$

(When A = 0, there is one root; however, this case corresponds to a spherical shell for which, as we have already pointed out, the integration of the system is reduced to a successive determination of $\widetilde{T}_{,k}$ and $\widetilde{U}_{,k}$.)

This class is considered in detail in (ref. 31). This reference also shows that the approach in question of separating a system is applicable, as an approximation, to an arbitrary shell of revolution.

4. Membrane Theory of Shells of Revolution

The most radical method of simplifying the relationships of the theory of shells is the adoption of the membrane theory. This is the name given to a theory which neglects the moments and cutting forces arising in the shell. The theory which takes them into account is called the general or moment theory.

Assuming that in the system of equilibrium equations (1.13) $M_1^* = M_2^* = H^* = 0$, we obtain the basic system of equations of the membrane theory

$$\begin{aligned} \frac{\partial R_2 \sin\theta\, T_1^*}{\partial\theta} + R_1\frac{\partial S^*}{\partial\varphi} - R_1\cos\theta\, T_2^* &= -R_1R_2\sin\theta\, q_1, \\ R_1\frac{\partial T_2^*}{\partial\varphi} + \frac{1}{R_2\sin\theta}\frac{\partial R_2^2\sin^2\theta\, S^*}{\partial\theta} &= -R_1R_2\sin\theta\, q_2, \\ \frac{T_1^*}{R_1} + \frac{T_2^*}{R_2} &= q_n. \end{aligned} \tag{4.1}$$

Comparing it with the basic system of equations of the general theory /100 (1.14), we see that the equations of the membrane theory may be obtained quite formally by setting c = 0 in the equations which contain the complex forces. This situation also has a simple physical explanation. Indeed, by comparing the quantities $D = \frac{Eh^3}{12\,(1-\nu^2)}$--flexural rigidity (cylindrical rigidity) and $\frac{Eh}{1-\nu^2}$--tensional rigidity, contained in the expression of Hooke's law, we can see that as the thickness h of the shell is decreased, the first quantity decreases much faster than the second. Thus when the thickness is decreased, the role played by the moment terms decreases very rapidly. It is said that the thin shell (membrane) does not support moments.

All these statements also refer to all relationships of the theory of shells. Taking this into account, we shall obtain the relationships of the membrane theory from the corresponding complex relationships, assuming that in these c = 0. Thus the complex system (2.7) transforms into

$$\begin{aligned} G[U^*] - \frac{\partial^2 T^*}{\partial \varphi^2} &= f(\theta, \varphi), \\ T^* + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{U^*}{\sin^2\theta} &= R_2 q_n. \end{aligned} \tag{4.2}$$

Eliminating T^* from these equations, we obtain an equation for determining the basic auxiliary function U^* of the membrane theory of shells of revolution

$$G[U^*] + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\frac{\partial^2 U^*}{\partial \varphi^2} = f(\theta, \varphi) + R_2\frac{\partial^2 q_n}{\partial \varphi^2}. \tag{4.3}$$

or, in expanded form,

$$\begin{aligned} &\frac{1}{R_1 R_2 \sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2^2 \sin\theta}{R_1}\frac{\partial U^*}{\partial\theta}\right) + \frac{1}{R_1 \sin^2\theta}\frac{\partial^2 U^*}{\partial\varphi^2} = \frac{1}{R_1 R_2 \sin\theta} \times \\ &\times \frac{\partial}{\partial\theta}\left[(q_n \cos\theta - q_1 \sin\theta) R_2^3 \sin^2\theta\right] + R_2\left(\frac{\partial^2 q_n}{\partial\varphi^2} + \sin\theta\frac{\partial q_2}{\partial\varphi}\right). \end{aligned} \tag{4.4}$$

The forces T_1^* and T_2^*, according to (3.4), are determined in terms of U^* by means of the following relationships

$$T_1^* = \frac{U^*}{R_2 \sin^2\theta}, \quad T_2^* = R_2 q_n - \frac{U^*}{R_1 \sin^2\theta}. \tag{4.5}$$

For the derivatives of V^* we obtain the expressions

$$\frac{\partial v^*}{\partial \theta} = \frac{R_1}{\sin\theta}\frac{\partial U^*}{\partial \varphi} - R_1 R_2 \sin\theta \frac{\partial T^*}{\partial \varphi} - R_1 R_2^2 \sin^2\theta q_2,$$
$$\frac{\partial V^*}{\partial \varphi} = -\frac{R_2^2 \sin\theta}{R_1}\frac{\partial U^*}{\partial \theta} + (q_3 \cos\vartheta - q_1 \sin\theta) R_2^3 \sin^2\theta. \tag{4.6}$$

Having used these to determine V^* and dividing the latter by $(R_2 \sin\theta)^2$, we obtain S^*. It is easy to see that the quantities U^*, V^* have the same static meaning as the real parts of the quantities $\widetilde{U}$, $\widetilde{V}$.

In order to determine the displacements, it is necessary to solve the following system of equations /101

$$(R_1\varepsilon_1^*) \qquad \frac{\partial u^*}{\partial \theta} + w^* = \frac{R_1}{Eh}(T_1^* - \nu T_2^*),$$
$$(\varepsilon_2^*) \quad \frac{\partial}{\partial \varphi}\left(\frac{v^*}{R_2 \sin\theta}\right) + \frac{\cos\theta\, u^*}{R_2 \sin\theta} + \frac{\sin\theta\, w^*}{R_2 \sin\theta} = \frac{1}{Eh}(T_2^* - \nu T_1^*), \tag{4.7}$$
$$(\omega^*) \quad \frac{R_2 \sin\theta}{R_1}\frac{\partial}{\partial \theta}\left(\frac{v^*}{R_2 \sin\theta}\right) + \frac{1}{R_2 \sin\theta}\frac{\partial u^*}{\partial \varphi} = \frac{2(1+\nu)}{Eh} S^*.$$

This system is reduced to a second order equation with the same form as (4.4). Indeed, by eliminating w^* from the first two equations, we obtain a new equation

$$-R_2 \frac{\partial}{\partial \varphi}\left(\frac{v^*}{R_2 \sin\theta}\right) + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{u^*}{\sin\theta}\right) = \frac{1}{Eh}\left[R_1(T_1^* - \nu T_2^*) - R_2(T_2^* - \nu T_1^*)\right]. \tag{4.8}$$

By introducing two new unknown functions

$$\xi = \frac{u^*}{\sin\theta}, \quad \zeta = \frac{v^*}{R_2 \sin\vartheta}, \tag{4.9}$$

we write equations (4.8) and $(4.7)_3$ in the form

$$\frac{R_2^2 \sin\theta}{R_1}\frac{\partial \zeta}{\partial \theta} + \frac{\partial \xi}{\partial \varphi} = \frac{1}{Eh}\{2(1+\nu)R_2 S^*\},$$
$$\frac{R_2}{\sin\theta}\frac{\partial \zeta}{\partial \varphi} - \frac{\partial \xi}{\partial \theta} = \frac{1}{Eh}\left\{\frac{1}{\sin\vartheta}\left[(R_1 + \nu R_2)T_1^* - (R_2 + \nu R_1)T_2^*\right]\right\}.$$

Eliminating ξ from these equations we obtain the following equation

$$\frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{R_2^2\sin\theta}{R_1}\frac{\partial\zeta}{\partial\theta}\right]+\frac{1}{R_1\sin^2\theta}\frac{\partial^2\zeta}{\partial\varphi^2}=-\frac{1}{Eh}\frac{1}{R_1R_2\sin\theta}\times$$
$$\times\left\{2(1+\nu)\frac{\partial R_2S^*}{\partial\theta}-\frac{1}{\sin\theta}\left[(R_1+\nu R_2)\frac{\partial T_1^*}{\partial\varphi}-(R_2+\nu R_1)\frac{\partial T_2^*}{\partial\varphi}\right]\right\}, \tag{4.10}$$

which differs from (4.4) only in the form of the right side. Thus, when we determine forces and displacements we are in essence solving the same mathematical problem.

Equation (4.4) belongs to a number of well-investigated equations of mathematical physics. It is easy to establish its type. For this purpose we write only the senior derivatives and obtain

$$\frac{\partial^2U^*}{R_1^2\partial\theta^2}+\left(\frac{1}{R_1R_2}\right)\frac{\partial^2U^*}{\partial\varphi^2}+\ldots=\ldots$$

From this we can see that the type of equation is determined by the sign of the quantity $K=\frac{1}{R_1R_2}$, which represents the Gaussian curvature of the middle surface. Therefore, for shells with a positive Gaussian curvature, the equation will be elliptic, while for shells with zero curvature it will be parabolic and for shells with negative curvature it will be hyperbolic. /102

Of the shells with positive Gaussian curvatures the most useful are shells whose middle surface is produced by the rotation of second order curves around the axis of symmetry. For these

$$R_1=\frac{R_0}{(1+\gamma\sin^2\theta)^{3/2}},\quad R_2=\frac{R_0}{(1+\gamma\sin^2\theta)^{1/2}}, \tag{4.11}$$

in which case

$\gamma = 0$ corresponds to a sphere
$\gamma = -1$ to a paraboloid
$\gamma > -1$ to an ellipsoid
$\gamma < -1$ to a hyperboloid

For the family of shells of revolution (4.11) the basic equation takes on a particularly simple form

$$\frac{R_2^2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{\partial U^*}{\partial\theta}\right)+\frac{R_2^3}{R_1}\frac{\partial^2U^*}{\partial\varphi^2}=\frac{R_2^2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\times$$
$$\times\left[(q_n\cos\theta-q_1\sin\theta)R_2^3\sin^2\theta\right]+R_2^4\sin^2\theta\left(\frac{\partial^2q_n}{\partial\varphi^2}+\sin\theta\frac{\partial q_2}{\partial\varphi}\right). \qquad (4.12)$$

It is significant that in view of equality (4.11) the coefficient before the second derivative with respect to φ is a constant. If we introduce a new independent variable ξ associated with θ by the relationship

$$\xi=R_0\int_{\theta_0}^{\theta}\frac{R_1d\theta}{R_2^2\sin\theta}=\int_{\theta_0}^{\theta}\frac{d\theta}{\sin\theta\sqrt{1+\gamma\sin^2\theta}}=\left\{\ln\left(-\frac{\sqrt{1+\gamma}\sin\theta}{\gamma\sqrt{1+\gamma\sin^2\theta}+\cos\theta}\right)\right\}\Bigg|_{\theta_0}^{\theta}, \qquad (4.13)$$

the homogeneous equation which corresponds to (4.12) is reduced to a harmonic equation

$$\frac{\partial^2U^*}{\partial\xi^2}+\frac{\partial^2U^*}{\partial\varphi^2}=0.$$

As a result, when computing shells of this class by means of the membrane theory, it is possible to apply the methods of the theory associated with functions of a complex variable (in the complex plane $z=\xi+i\varphi$). We note that if we use functions of a complex variable, it is simpler to proceed directly from system (4.6). Eliminating T* from the first equation of the system by means of equality $(4.2)_1$, we can write this equation in the following form

$$\frac{R_2^2\sin\theta}{R_1}\frac{\partial U^*}{\partial\theta}+\frac{\partial V^*}{\partial\varphi}=(q_n\cos\theta-q_1\sin\theta)R_2^3\sin^2\theta,$$
$$\frac{R_2^2\sin\theta}{R_1}\frac{\partial V^*}{\partial\theta}-\frac{R_2^3}{R_1}\frac{\partial U^*}{\partial\varphi}=-R_2^4\sin^2\theta\left(\sin\theta q_2+\frac{\partial q_n}{\partial\varphi}\right). \qquad (4.14)$$

By replacing the independent variable (4.13) we obtain a system which transforms to the known Cauchy-Gauss equations when surface loading is absent. /103

When we use the method of separating variables, the basic equation for the k-th harmonic is written in the form

$$\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left[\frac{R_2^2\sin\theta}{R_1}\frac{dU_{,k}^*}{d\theta}\right]-\frac{k^2}{R_1\sin^2\theta}U_{,k}^*=$$
$$=\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left[(q_{n,k}\cos\theta-q_{1,k}\sin\theta)R_2^2\sin^2\theta\right]-$$
$$-R_2(k^2q_{n,k}\mp k\sin\theta\,q_{2,k}). \qquad (4.15)$$

When $k \neq 0$, the quantity $V^*_{,k}$ is determined from the relationship

$$\pm kV^*_{,k} = -\frac{R_2^2 \sin\theta}{R_1}\frac{dU^*_{,k}}{d\theta} + (q_{n,k}\cos\theta - q_{1,k}\sin\theta) R_2^2 \sin^2\theta. \tag{4.16}$$

Finally, the forces are given by the following expressions

$$T^*_{1,k} = \frac{U^*_{,k}}{R_2 \sin^2\theta}, \quad T^*_{2,k} = R_2 q_{n,k} - \frac{U^*_{,k}}{R_1 \sin^2\theta}, \quad S^*_{,k} = \frac{V^*_{,k}}{(R_2 \sin\theta)^2}. \tag{4.17}$$

As far as the displacements are concerned, they are obtained from the system

$$\begin{aligned}
&(R_1\varepsilon^*_{1,k})\ \frac{du^*_{,k}}{d\theta} + w^*_{,k} = \frac{1}{\cos\theta}\frac{du^*_{\rho,k}}{d\theta} + R_1 \operatorname{tg}\theta\,\theta^*_{,k} = \frac{R_1}{Eh}(T^*_{1,k} - \nu T^*_{2,k}),\\
&(R_2 \sin\theta\,\varepsilon^*_{2,k})\ \pm kv^*_{,k} + \cos\theta\, u^*_{,k} + \sin\theta\, w^*_{,k} = \pm kv^*_{,k} + u^*_{\rho,k} =\\
&\qquad = \frac{r}{Eh}(T^*_{2,k} - \nu T^*_{1,k}),\\
&(\omega^*_{,k})\ \frac{R_2 \sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{v^*_{,k}}{R_2 \sin\theta}\right) \mp \frac{ku^*_{,k}}{R_2 \sin\theta} = \frac{r}{R_1}\frac{d}{d\theta}\left(\frac{v^*_{,k}}{r}\right) +\\
&\qquad \mp \frac{k}{r}(\cos\theta\, u^*_{\rho,k} + \sin\theta\, u^*_{z,k}) = \frac{2(1+\nu)}{Eh} S^*.
\end{aligned} \tag{4.18}$$

Along with one of its partial solutions the expression for displacements also contains the solution of the homogeneous system

$$\varepsilon^*_{1,k} = \varepsilon^*_{2,k} = \omega^*_{,k} = 0. \tag{4.19}$$

The latter may make the components of flexural deformation $\varkappa_1$, $\varkappa_2$, τ become equal to zero. In this case they are rigid displacements of the shell. In the opposite case they consist of purely flexural displacements, i.e., displacements which do not produce tension and shear in the middle surface. Thus the displacements, obtained from the membrane relationships legitimately, contain displacements due to pure flexure.

In system (4.1) the number of equations is equal to the number of unknown functions. Thus, when we determine the membrane forces we need not use the equations of continuity. As a result the latter turn out to be disrupted /104 to some degree. Thus, when considering examples (see section 17), we shall show that in places where there is a sharp change in the geometry of the shell (discontinuity) the values of displacements determined by means of the membrane theory also lose their discontinuity. Additionally, the lower order of the membrane equations does not satisfy boundary conditions of the general form.

Furthermore, the smaller number of boundary conditions which can be satisfied cannot be selected arbitrarily. Thus, having dropped the moments and the cutting forces in the formulation of the membrane theory we have naturally eliminated the possibility of assigning them along the edge of the shell. These considerations, it would seem, tend to negate the value of the membrane solution. However, this is not so. Below we shall show that for most problems of practical interest the membrane solution gives a good particular solution of the general problem when moments are taken into account.

In addition, it is frequently possible to select a shell geometry and to produce such conditions at the fixed edges, that the shell, when loaded uniformly, will operate in a state of stress close to the membrane state. If we add to everything we have said the fact that, from the standpoint of making rational use of shell material, the membrane state of stress is most advantageous, the practical importance of the membrane theory becomes clear.

In the first part of the book we shall limit the consideration of the membrane theory to specific classes of problems, with the thought of returning to it from a more general position in the second part of this work.

5. The Second Symmetric Case

The lower functions and signs with k = 0 in the preceding sections pertain to the second symmetrical case. The examination of this case is quite elementary. First of all, it follows from expressions (3.2) that

$$u_{,0} = w_{,0} = T_{1,0} = T_{2,0} = M_{1,0} = M_{2,0} = q_{1,0} = q_{n,0} = 0. \tag{5.1}$$

Relationship (3.5) is unsuitable for determining the function $\widetilde{V}_{,0}$. Therefore it must be determined directly from equation (2.4), which takes the form

$$\frac{d\widetilde{V}_0}{d\theta} = -R_1R_2^2\sin^2\theta\, q_{2,0}. \tag{5.2}$$

It follows that

$$\widetilde{V}_{,0} = -\int_0^\theta R_1R_2^2\sin^2\theta\, q_{2,0}\,d\theta + \widetilde{C}_2. \tag{5.3}$$

We note first of all that the function $\widetilde{V}_{,0}$ is real. Indeed, in accordance with $(3.10)_4$, $(3.7)_3$, (2.2) and $(3.9)_2$ /105

$$\operatorname{Im} \widetilde{V}_{,0} = r^2 Ehc\varkappa_{21,0} = 0.$$

It is also easy to establish the geometric meaning of the real function $\widetilde{V}_{,0}$.

Indeed, in the general case the increment of the vector of displacements during the tracing of the parallel circle is equal, in accordance with equality (II, 3.24'), to

$$\vec{U}(\theta, 2\pi) - \vec{U}(\theta, 0) = \int_0^{2\pi} \left\{ \left[\left[\int_0^{\varphi} \vec{\varkappa}_2 R_2 \sin\theta \, d\varphi, \vec{e}_2 \right] + \varepsilon_2 \vec{e}_2 \right\} \times \right. \\ \left. \times R_2 \sin\theta \, d\varphi, \right. \tag{5.4}$$

where

$$\vec{\varkappa}_2 = -\varkappa_2 \vec{e}_1 + \varkappa_{21} \vec{e}_2 - \varkappa_{2n} \vec{n}.$$

Since we are considering the second symmetric case we should let $\varkappa_2 = \varkappa_{2n} = \varepsilon_2 = 0$, $\varkappa_{21} = \varkappa_{21}(\theta)$. By substituting $\vec{\varkappa}_2 = \varkappa_{21}(\theta)\,\vec{e}_2$ into the expression under the integral sign and noting that $\vec{e}_2 = \frac{1}{\cos\theta} \frac{d\vec{e}_1}{d\varphi}$, $\left[\vec{e}_1, \vec{e}_2\right] = \vec{n}$, we obtain

$$\vec{U}(\theta, 2\pi) - \vec{U}(\theta, 0) = 2\pi (R_2 \sin\theta)^2 \varkappa_{21} \vec{e}_z.$$

Taking the scalar product of this equation and $\vec{e}_z$, we obtain the desired equality

$$u_z(\theta, 2\pi) - u_z(\theta, 0) = 2\pi (R_2 \sin\theta)^2 \varkappa_{21}.$$

From this we can see that the requirement $\varkappa_{21} = 0$ (or, what is equivalent, the requirement $\operatorname{Im} \widetilde{V}_{,0} = 0$) makes the vertical displacement of the points on the middle surface single-valued. When we considered the second symmetric case we accepted, a priori, a periodic form for the displacement component. It is for this reason that we have arrived at the real function $\widetilde{V}_{,0}$. When the displacements are not single-valued or, as they are called, are dislocation displacements (see Part II), the quantity $\widetilde{V}_{,0}$ may be complex.

It follows from the fact that $\tilde{V}_{,0}$ is real and from equalities (5.3) and (2.16) that the arbitrary constant $\tilde{C}_2$ is also real and its relation to the twisting axial moment is given by

$$\tilde{C}_2 = C_2 = V_{,0}(\theta_0) = \frac{1}{2\pi} B_z(\theta). \tag{5.5}$$

The quadrature (5.3) has a clear static meaning. To clarify this meaning we shall consider part of the shell which is bounded by two parallel circles (θ_0 and θ). We write (5.3) in the form

$$2\pi V_{,0}(\theta) = 2\pi V_{,0}(\theta_0) - \int_{\theta_0}^{\theta}\int_{0}^{2\pi} |R_2 \sin\theta\, q_{z,0}(R_1 d\theta R_2 \sin\theta\, d\varphi)|. \tag{5.3'}$$

Remembering the static meaning of the function V (see (2.16)) and noting that the quantity $R_1 d\theta\, R_2 \sin\theta\, d\varphi$ represents the area of an element from the middle surface, we see that relationship (5.3') provides the condition whereby the axial component of the principal moment of edge reactions and of the surface loading acting on the isolated part of the shell is equal to zero. /106

Now we determine $S_{,0}$ from (5.3)

$$S_{,0} = \frac{C_2 - \int_{\theta_0}^{\theta} R_1 R_2^2 \sin^2\theta\, q_{z,0}\, d\theta}{(R_2 \sin\theta)^2}. \tag{5.6}$$

In regard to the displacement vector component $v_{,0}$ which is the only one different from 0, its value, according to $(3.6)_3$, is equal to

$$v_{,0} = R_2 \sin\theta \left\{ \frac{2(1+\nu)}{Eh} \int_{\theta_0}^{\theta} S_{,0} \frac{R_1 d\theta}{R_2 \sin\theta} + a_2 \right\}. \tag{5.7}$$

Assuming that the membrane solution corresponds to the same twisting moment as the moment solution, we obtain $V^*_{,0} = V_{,0}$. From this it follows that $S_{,0} = S^*_{,0}$, $v_{,0} = v^*_{,0}$.

The coincidence of results obtained by means of the membrane theory and by means of the moment theory is explained by the negligibly small effect (in this case) of the twisting moment on the value of stresses and displacements

in the shell. Indeed, according to $(2.12)_{3,6}$ and in accordance with the generalized Hooke's law

$$\omega_{,0}=\frac{R_2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{v_{,0}}{R_2\sin\theta}\right),\quad \tau_{,0}=\frac{\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{v_{,0}}{R_2\sin\theta}\right)=\frac{\omega_{,0}}{R_2},$$
$$S_{,0}=\frac{Eh}{2(1+\nu)}\omega_{,0},\quad H_{,0}=\frac{Eh^3}{12(1+\nu)}\tau_{,0}=\frac{h^2}{6R_2}S_{,0}. \tag{5.8}$$

If we formulate the boundary shear force on the basis of (1.23) and (5.8) we obtain

$$Q_{12,0}=S_{,0}+\frac{2H_{,0}}{R_2}=S_{,0}\left(1+\frac{h^2}{3R_2^2}\right).$$

From this we can see that the role of the twisting moment in the static boundary condition is insignificant.

Next, by determining the stresses

$$\sigma_S=\frac{S_{,0}}{h},\quad \sigma_H=\frac{6H_{,0}}{h^2}=\frac{h}{R_2}\frac{S_{,0}}{h}=\frac{h}{R_2}\sigma_S \tag{5.9}$$

we find that when we determine the stresses we should neglect the twisting element. Thus, indeed, the role of the twisting moment in the second symmetric case is small, and the difference between the membrane and the moment theories is eliminated.

6. The First Symmetric Case

For this case we have the upper functions and signs for k = 0 in the relationships of sections 3 and 4. In this case $v_{,0} = \omega_{,0} = S_{,0} = H_{,0} = q_{2,0} = 0$ (below, for the sake of simplicity the sign "0" will be dropped). /107

The first equation of the basic system (3.3) permits two quadratures. We shall not solve it because $\tilde{U}$ can be determined in a simpler manner directly from equation (2.5), which takes the form

$$\frac{d\tilde{U}}{d\theta}=(q_n\cos\theta-q_1\sin\theta)R_1R_2\sin\theta. \tag{6.1}$$

It follows that

$$\widetilde{U}=\widetilde{C}_1+\int_{\theta_0}^{\theta}(q_n\cos\theta-q_1\sin\theta)\,R_1R_2\sin\theta\,d\theta \tag{6.2}$$

As in the second symmetric case, it is easy to see that $\widetilde{U}$ is a real function. Indeed, according to (3.4)-(3.10)

$$\operatorname{Im}\widetilde{U}=Ehcr\varkappa_z=0.$$

By using transformations analogous to those of the preceding sections we establish the equality

$$v(\theta,2\pi)-v(\theta,0)=2\pi r^2\varkappa_z,$$

from which it follows that the real value of the function $\widetilde{U}$ provides for the single value of displacement v.

From the fact that the function $\widetilde{U}$ is real and from its meaning established in section 2, it follows that the constant $\widetilde{C}_1$ is real and that

$$\widetilde{C}_1=C_1=U(\theta_0)=-\frac{1}{2\pi}F_z(\theta_0). \tag{6.3}$$

Let us clarify the static meaning of quadrature (6.2) by rewriting it initially in the form

$$2\pi U(\theta)=2\pi U(\theta_0)+\int_0^{2\pi}\int_{\theta_0}^{\theta}\{(q_n\cos\theta-q_1\sin\theta)(R_1d\theta\,R_2\sin\theta\,d\varphi)\}. \tag{6.2'}$$

Noting that the quantity $(q_n\cos\theta - q_1\sin\theta) = q_z$ is the vertical component of the vector representing the surface loading, it is easy to see that relationship (6.2') is the condition which states that the axial component of the principal vector for the edge reactions and the axial component of the load /108 acting on the considered finite region of the shell are equal to 0.

Substituting the expression which has been obtained for U

$$U = C_1 + \int_{\theta_0}^{\theta} (q_n \cos\theta - q_1 \sin\theta) R_1 R_2 \sin\theta \, d\theta \tag{6.2''}$$

into the second equation of (3.13), we find the basic equation which determines $\widetilde{T}$

$$\frac{d^2\widetilde{T}}{d\theta^2} + \left[\left(2\frac{R_1}{R_2} - 1\right)\operatorname{ctg}\theta - \frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{d\widetilde{T}}{d\theta} + i\frac{R_1^2}{cR_2}\widetilde{T} = i\frac{R_1^2}{R_2}F(\theta).$$

$$F(\theta) = R_2 q_n - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{U}{\sin^2\theta} = \tag{6.4}$$

$$= R_2 q_n - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\left[C_1 + \int_{\theta_0}^{\theta}(q_n\cos\theta - q_1\sin\theta) R_1 R_2 \sin\theta \, d\theta\right].$$

The complex forces are expressed in terms of $\widetilde{T}$ and U, which have been determined above by the following relationships (see (3.4))

$$\widetilde{T}_1 = \frac{U}{R_2\sin^2\theta} + ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}}{d\theta},$$
$$\widetilde{T}_2 = -\frac{U}{R_2\sin^2\theta} + \widetilde{T} - ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}}{d\theta}. \tag{6.5}$$

Let us determine the complex displacements. First we select a static system (it need not necessarily be a membrane system) such that the vertical force which corresponds to it is the same for the unknown solution.

For this purpose it is necessary to let

$$U^* = R_2 \sin\theta \, Q_z^* = U. \tag{6.6}$$

Then, in order to determine the complex displacements, we obtain four equations in accordance with equality (3.6)

$$\frac{1}{\cos\theta}\frac{d\widetilde{u}_\rho}{d\theta} + R_1 \operatorname{tg}\theta\,\widetilde{\vartheta} = \frac{R_1}{Eh}\left\{-\nu\widetilde{T} + i(1+\nu)\frac{c}{R_1}\operatorname{ctg}\theta\frac{d\widetilde{T}}{d\theta} + \right.$$
$$\left. + \frac{(1+\nu)}{R_2\sin^2\theta}U + \frac{i}{c}M_2^*\right\},$$
$$\widetilde{u}_\rho = \frac{R_2\sin\theta}{Eh}\left\{\widetilde{T} - i(1+\nu)\frac{c}{R_1}\operatorname{ctg}\theta\frac{d\widetilde{T}}{d\theta} - \frac{(1+\nu)}{R_2\sin^2\theta}U + \frac{i}{c}M_1^*\right\}, \tag{6.7}$$

$$\frac{d\tilde{\vartheta}}{d\theta} = i\frac{R_1}{Ehc}\left\{\tilde{T} - ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\tilde{T}}{d\theta} - \left(\frac{U}{R_2\sin^2\theta} + T_2^*\right)\right\}.$$

$$\tilde{\vartheta} = \frac{i}{Ehc}\left\{ic\frac{R_2}{R_1}\frac{d\tilde{T}}{d\theta} + \frac{R_2\sin\theta}{\cos\vartheta}\left(\frac{U}{R_2\sin^2\theta} - T_1^*\right)\right\}.$$

By using equation (6.4) it is easy to verify the approximate consistency of the specified system. /109

From the second equation we obtain

$$u_p = \frac{R_2\sin\theta}{Eh}\left\{\operatorname{Re}\tilde{T} + (1+\nu)c\frac{\operatorname{ctg}\theta}{R_1}\operatorname{Im}\frac{d\tilde{T}}{d\theta} - \frac{1+\nu}{R_2\sin^2\theta}U\right\}. \tag{6.8}$$

Next, by using an identity which is easy to verify

$$\frac{d\tilde{u}_z}{d\theta} = -R_1\left(\cos\theta\,\tilde{\vartheta} + \sin\theta\,\tilde{\varepsilon}_1\right) \tag{6.9}$$

we find

$$\tilde{u}_z = \tilde{a}_1 - \int_{\theta_0}^{\theta} R_1\left(\cos\theta\,\tilde{\vartheta} + \sin\theta\,\tilde{\varepsilon}_1\right)d\vartheta. \tag{6.10}$$

By using equalities $(6.7)_1$ and $(6.7)_4$ we have

$$-R_1\left(\sin\theta\,\tilde{\varepsilon}_1 + \cos\theta\,\tilde{\vartheta}\right) = \frac{R_1}{Eh}\left\{\nu\sin\theta\,\tilde{T} - \underline{i(1+\nu)\frac{c}{R_1}\cos\theta\frac{d\tilde{T}}{d\theta}} - \right.$$
$$\left. - \underline{\frac{1+\nu}{R_2\sin\theta}U} - \underline{\frac{i}{c}\sin\theta\,M_2^*} + \frac{R_2}{R_1}\cos\theta\frac{d\tilde{T}}{d\theta} - \underline{\frac{i}{c}R_2\sin\theta\left(\frac{U}{R_2\sin^2\theta} - T_1^*\right)}\right\}.$$

By dropping the underlined small terms in this expression and by separating the real part we obtain, from (6.10),

$$u_z = a_1 +$$
$$+\frac{1}{Eh}\int_{\theta_0}^{\theta}\left\{\operatorname{Re}\left(\frac{R_2\cos\theta}{R_1}\frac{d\tilde{T}}{d\theta} + \nu\sin\theta\,\tilde{T}\right) - \frac{(1+\nu)}{R_2\sin\theta}U\right\}R_1 d\vartheta. \tag{6.11}$$

Here a_1 is the rigid axial displacement of the shell. Proceeding further we find that since, according to (3.9),

$$\varkappa_\rho = \frac{\vartheta}{r}, \quad \varepsilon_2 = \frac{u_\rho}{r}, \quad \varkappa_z = 0, \tag{6.12}$$

the complex boundary expressions (3.7)-(3.9) take the form

$$\begin{aligned} \widetilde{Q}_\rho &= Q_\rho - iEhc\frac{\vartheta}{r} = Q_\rho^* - iEhc\frac{\widetilde{\vartheta}}{r}, \\ \widetilde{M}_1 &= M_1 + iEhc\frac{u_\rho}{r} = M_1^* + iEhc\frac{\widetilde{u_\rho}}{r}. \\ \widetilde{Q}_z &= Q_z = Q_z^* = \frac{U}{R_2\sin\theta}. \end{aligned} \tag{6.13}$$

The approximate expressions for these are obtained from relationships (3.10)

$$\begin{aligned} \widetilde{Q}_\rho &= \cos\theta\,\widetilde{T}_1 + ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}}{d\theta}, \\ \widetilde{M}_1 &= ic\left\{\widetilde{T}_2 - \nu\widetilde{T}_1\right\}, \\ \widetilde{Q}_z &= \sin\theta\,\widetilde{T}_1 - ic\frac{\cos\theta}{R_1}\frac{d\widetilde{T}}{d\theta} = \frac{U}{R_2\sin\theta}. \end{aligned} \tag{6.14}$$

By using the third relationship, the first can be transformed into the form /110

$$Q_\rho = i\frac{c}{R_1\sin\theta}\frac{d\widetilde{T}}{d\theta} + \frac{\operatorname{ctg}\theta}{R_2\sin\theta}U. \tag{6.15}$$

When shells of equal thickness and with the same elastic constants are joined, the conditions of junction at the parallel ring are formulated in the following manner

$$\left\{\widetilde{Q}_z\right\}^{\mathrm{I}} = \left\{\widetilde{Q}_z\right\}^{\mathrm{II}}, \quad \left\{\widetilde{Q}_\rho\right\}^{\mathrm{I}} = \left\{\widetilde{Q}_\rho\right\}^{\mathrm{II}}, \quad \left\{\widetilde{M}_1\right\}^{\mathrm{I}} = \left\{\widetilde{M}_1\right\}^{\mathrm{II}}. \tag{6.16}$$

The first of these places a limitation on the selection of static systems for shells which are joined (it is required that the vertical forces be equal at the junction line). This condition is satisfied automatically, if (as we have

stipulated above) we select the elastic systems so that they are statistically equivalent to the sou[illegible] solu[illegible].

In accordance with expressions (6.13) the remaining two conditions are reduced to a requirement that the horizontal forces, bending moments, angular rotations and horizontal displacements be continuous along the line of junction. Taking into account equalities (6.15) and (6.5), they may be written in the following form

$$\left\{\frac{\operatorname{ctg}\theta}{r}U+\frac{ic}{\sin\theta}\frac{d\widetilde{T}}{ds_1}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}}\quad(ds_1=R_1d\theta),$$
$$\left\{\widetilde{T}-ic(1+\nu)\operatorname{ctg}\theta\frac{d\widetilde{T}}{ds_1}-(1+\nu)\frac{U}{r\sin\theta}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}}. \tag{6.17}$$

In the case of shallow shells which are close to a circular plate it is convenient to replace these relationships with equivalent relationships ($(6.14)_1$)

$$\left\{\cos\theta\,\widetilde{T}_1+ic\sin\theta\frac{d\widetilde{T}}{ds_1}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},$$
$$\left\{\widetilde{T}-(1+\nu)\widetilde{T}_1\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}} \tag{6.18}$$
$$\left(\left\{\widetilde{T}_2-\nu\widetilde{T}_1\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}}\right).$$

In the case of smooth joining, conditions (6.17)-(6.18) are replaced by simpler conditions

$$\left\{\widetilde{T}\right\}^{\mathrm{I}}=\left\{\widetilde{T}\right\}^{\mathrm{II}},\quad\left\{\frac{d\widetilde{T}}{ds_1}\right\}^{\mathrm{I}}=\left\{\frac{d\widetilde{T}}{ds_1}\right\}^{\mathrm{II}}. \tag{6.19}$$

The formulation of the junction conditions in complex form makes it possible to utilize to the fullest the advantages of the complex method, because it becomes unnecessary to separate, a priori, the real and the imaginary parts of the junction solutions.

After expressing all quantities of interest to us in terms of the basic complex function $\widetilde{T}$ we return to the equation which determines it.

7. Asymptotic Solution of the Principal Complex Equation

The basic characteristic equation for the symmetric case

$$\frac{d^2\widetilde{T}}{d\theta^2}+\left[2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\Big]\frac{d\widetilde{T}}{d\theta}+i\frac{R_1^2}{cR_2}\widetilde{T}=i\frac{R_1^2}{cR_2}F(\theta), \tag{7.1}$$

where

$$F(\theta)=R_2q_n-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{U}{\sin^2\theta}=R_2q_n-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\times$$
$$\times\left[C_1+\int_{\theta_0}^{\theta}(q_n\cos\theta-q_1\sin\theta)\,R_1R_2\sin\theta\,d\theta\right].$$

may, of course, be integrated by any of the known general methods used in the theory of differential equations. However, the presence of the factor $\frac{R_1^2}{cR_2}$ in this equation makes it possible to apply the asymptotic methods, which lead to relatively simple solutions. /111

We begin with a homogeneous equation, writing it in the form

$$\frac{d^2\widetilde{T}^0}{d\theta^2}+p(\theta)\frac{d\widetilde{T}^0}{d\theta}+q(\theta)\widetilde{T}^0=0. \tag{7.2}$$

where

$$p(\theta)=\left(2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}, \tag{7.3}$$
$$q(\theta)=i\frac{R_1^2}{cR_2}.$$

Let R_0 be some characteristic linear dimension of the middle surface. We introduce the dimensionless distance from the axis

$$\rho=\frac{r}{R_0}. \tag{7.4}$$

In accordance with (1.3) and (1.4) we have

$$R_2=R_0\frac{\rho}{\sin\theta}, \tag{7.5}$$
$$R_1=R_0\frac{\rho'}{\cos\theta}. \tag{7.6}$$

From the last equality we obtain

$$\frac{dR_1}{d\theta} = R_0\left(\frac{\rho''}{\cos\theta} + \frac{\sin\theta}{\cos^2\theta}\rho'\right). \tag{7.7}$$

(Here and below the prime indicates a derivative with respect to θ.)

By using the relationships which we have just obtained we find /112

$$\begin{aligned} p(\theta) &= -\frac{\rho''}{\rho'} + 2\frac{\rho'}{\rho} - \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} = \left(\ln\frac{\rho^2\cos\theta}{\rho'\sin\theta}\right)', \\ q(\theta) &= i\frac{R_0}{c}\frac{\sin\theta\rho''}{\cos^2\theta\rho}. \end{aligned} \tag{7.8}$$

We shall seek a particular solution of the homogeneous equation (7.2) in the following form

$$\widetilde{T}^0 = \varphi(\theta)\, e^{\psi(\theta)}. \tag{7.9}$$

In regard to $\varphi(\theta)$ and $\psi(\theta)$, we shall assume that they are smooth functions and that $\psi(\theta)$ also contains a large multiplier. Under this condition, the first cofactor represents a slowly varying function, while the second represents a rapidly varying function. Differentiating $\widetilde{T}^0$ with respect to θ, we obtain

$$\begin{aligned} \frac{d\widetilde{T}^0}{d\theta} &= \{\varphi\psi' + \varphi'\}\, e^{\psi(\theta)}, \\ \frac{d^2\widetilde{T}^0}{d\theta^2} &= \{\varphi\psi'^2 + (2\varphi'\psi' + \varphi\psi'') + \varphi''\}\, e^{\psi(\theta)}. \end{aligned} \tag{7.10}$$

Substituting these expressions into equation (7.2), we obtain an equality

$$\left\{\psi'^2 + i\frac{R_0}{c}\frac{\sin\theta\rho''}{\cos^2\theta\rho}\right\}\varphi + \{2\psi'\varphi' + (\psi'' + p\psi')\varphi\} + \\ + \{\varphi'' + p\varphi'\} = 0. \tag{7.11}$$

We shall require that the expression in the first bracket be equal to zero. This determines the function ψ

$$\psi = \pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{R_0}{c}}\int_0^\theta \sqrt{\frac{\sin\theta\rho''}{\cos^2\theta\rho}}\, d\theta. \tag{7.12}$$

We divide the expression in the second brackets by the quantity $\varphi\psi'$ and set it equal to zero. As a result we obtain an equality which determines the function φ

$$2\frac{\varphi'}{\varphi}+\frac{\psi''}{\psi'}+p(\theta)=0.$$

Substituting (7.8) into this expression, we obtain

$$\ln\left\{\varphi^2\psi'\frac{\rho^2\cos\theta}{\rho'\sin\theta}\right\}=C'.$$

Replacing ψ by expression (7.12) and raising the resultant expression to a higher power, we find

$$\varphi=C\sqrt[4]{\frac{\sin\theta}{\rho^3}}. \tag{7.13}$$

We must now return to the old designations in these expressions. As a result we obtain the following expression for the general solution of the homogeneous equation /113

$$\widetilde{T}^0=\frac{1}{R_2^{3/4}\sin^{1/2}\theta}\left\{\widetilde{C}_1e^{-\frac{1-i}{\sqrt{2}}\int_{\theta_0}^{\theta}\frac{R_1d\theta}{\sqrt{cR_1}}}+\widetilde{C}_2e^{\frac{1-i}{\sqrt{2}}\int_{\theta_0}^{\theta}\frac{R_1d\theta}{\sqrt{cR_1}}}\right\}. \tag{7.14}$$

Let us establish to what degree the obtained expression may be considered as a solution of equation (7.2). The function ψ and its derivative, according to (7.12), are quantities of order $\sqrt{\frac{R_0}{c}}$, while φ and its derivatives are quantities of the order of unity.

Thus, by setting to zero the expressions in the first and second brackets of equality (7.11), we eliminate quantities of the order $\frac{R_0}{c}$ and $\sqrt{\frac{R_0}{c}}$ in this expression, retaining terms of the order of unity. Therefore, we can state that the solution which has been established satisfies equation (7.2) with an accuracy which neglects the quantities of order c/R_0 compared with unity.

We can proceed further and seek the particular solution of the homogeneous

equation in the form of an asymptotic series in powers of the small parameter $\sqrt{\frac{2cR_2}{R_1^2}}$,

$$\tilde{T}^0 = \left\{\varphi_0(\vartheta) + \sqrt{\frac{2cR_2}{R_1^2}}\,\varphi_1(\vartheta) + \left(\frac{2cR_2}{R_1^2}\right)\varphi_2(\vartheta) + \ldots\right\} e^{\lambda\vartheta}.$$

Our solution corresponds to the case when the first term φ_0 is retained in this series. By retaining the subsequent terms we may, in general, refine the solution. There is, however, no great need for such refinement, because the first approximation which we established produces an error which does not exceed the usual error incurred in the theory of shells (c/R_0 compared with unity).

Let us now turn to the consideration of the nonhomogeneous equation. Let us assume that the components of the surface load and the radii of curvature are sufficiently smooth functions which do not increase substantially during differentiation. Then the right side of equation (7.1) should also be considered smooth. If in this equation we drop the terms under the integral sign, we obtain as a particular solution the following expression

$$\tilde{T}^{\text{partial}} = F(\theta). \tag{7.15}$$

As a measure of error, which is permissible in the adoption of a particular solution having the form (7.15), we can use the value of the ratio of the underlined (remaining) part to the eliminated part

$$\left|\frac{\frac{d^2F}{d\theta^2} + \left[\left(2\frac{R_1}{R_2} - 1\right)\operatorname{ctg}\theta - \frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{dF}{d\theta}}{\frac{R_1^2}{cR_2}F}\right|. \tag{7.16}$$

Because $F(\theta)$ is smooth it will be of the same order as the numerator of the ratio; then the presence of a large quantity of the order R_0/c in the denominator makes this ratio small (of the order c/R_0). The small value of ratio (7.16) (it can always be computed for a specific shell under consideration) confirms the possibility of adopting a particular solution in the form (7.15). Let us assume that the particular solution can be taken in the form (7.15). In this case, we take the solution given by the membrane theory ($M_1^* = M_2^* = H^* = 0$) as the static system. Then, according to (7.1) and (4.17)

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$$F(\theta)=R_2 q_n-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{U}{\sin^2\theta}=T^*. \tag{7.17}$$

This means that the particular solution in the case considered may be taken from the membrane theory.

Substituting the general solution of equation (7.1)

$$\widetilde{T}=T^*+\widetilde{T}^0 \tag{7.18}$$

into expression (6.5) we obtain, by using (4.5)

$$\widetilde{T}_1=T_1^*+\underline{ic\frac{\operatorname{ctg}\theta}{R_1}\frac{dT^*}{d\theta}}+ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0}{d\theta},$$

$$\widetilde{T}_2=-T_1^*+T^*-\underline{ic\frac{\operatorname{ctg}\theta}{R_1}\frac{dT^*}{d\theta}}+\widetilde{T}^0-ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0}{d\theta}.$$

Because T^* is smooth, its derivative with respect to θ will be a quantity of the same order as the function itself. Then the presence of a small factor c/R_1 in the underlined terms makes it possible to drop them. As a result we find

$$\begin{aligned}\widetilde{T}_1&=T_1^*+ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0}{d\theta},\\ \widetilde{T}_2&=T_2^*+\widetilde{T}^0-ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0}{d\theta}.\end{aligned} \tag{7.19}$$

In a similar fashion, from (6.8), (6.11), (6.13) and (6.15) we obtain

$$\begin{aligned}u_\varphi&=\frac{R_2\sin\theta}{Ek}\left\{T_2^*-\nu T_1^*+(1+\nu)c\frac{\operatorname{ctg}\theta}{R_1}\operatorname{Im}\frac{d\widetilde{T}^0}{d\theta}+\operatorname{Re}\widetilde{T}^0\right\},\\ u_z&=a_1+\frac{1}{Ek}\int_0^\theta\left\{\frac{R_2\cos\theta}{R_1}\frac{dT^*}{d\theta}-\sin\theta\left(T_1^*-\nu T_2^*\right)+\right.\\ &\qquad\left.+\operatorname{Re}\left(\frac{R_2\cos\theta}{R_1}\frac{d\widetilde{T}^0}{d\theta}+\nu\sin\theta\,\widetilde{T}^0\right)\right\}R_1 d\theta,\\ \vartheta&=\vartheta^*-\frac{1}{Ek}\frac{R_2}{R_1}\operatorname{Re}\frac{d\widetilde{T}^0}{d\theta}\quad\left(\vartheta^*=-\frac{1}{Ek}\frac{R_2}{R_1}\frac{dT^*}{d\theta}\right),\\ Q_\varphi&=Q_\varphi^*-\frac{c}{R_1\sin\theta}\operatorname{Im}\frac{d\widetilde{T}^0}{d\theta}\quad\left(Q_\varphi^*=\cos\theta\,T_1^*\right).\end{aligned} \tag{7.20}$$

Let us consider a shell of revolution bounded by two parallel rings /115
$\theta = \theta_1$ and $\theta = \theta_2$. For this shell the coordinate θ varies within the limits $\theta_1 \leq \theta \leq \theta_2$. Multiplying the first term contained in the figure brackets (7.14) by $e^{-(1-i)\int_{\theta_1}^{\theta_2} \frac{R_1 d\theta}{\sqrt{2cR_2}}}$, and the second by $e^{(1-i)\int_{\theta_2}^{\theta_2} \frac{R_1 d\theta}{\sqrt{2cR_2}}}$, we write the solution of the homogeneous equation in the form

$$\widetilde{T}^0 = \frac{1}{R_2^{3/4}\sin^{1/2}\theta}\left\{\widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}\right\}, \qquad \beta_1 = \int_{\theta_1}^{\theta} \frac{R_1 d\theta}{\sqrt{2cR_2}}, \quad \beta_2 = \int_{\theta_2}^{\theta} \frac{R_1 d\theta}{\sqrt{2cR_2}}. \tag{7.21}$$

The transformation which has been carried out in connection with the multiplication by constant factors is reflected only in the constants contained in solution (7.14). These constants are arbitrary (undetermined) before the solution is subjected to boundary conditions. Therefore, the form of the solution given by (7.21) is entirely equivalent to the preceding form (7.14).

Each of the exponential functions contained in (7.21) has important properties. The first one along the edge of the shell ($\theta = \theta_1$) assumes its maximum value, which is equal to unity, as the angle increases, i.e., it decreases very rapidly as it moves away from its edge. The rate of decrease is determined by the presence of a large factor $\sqrt{\frac{R_0}{c}}$ in the exponent. In exactly the same manner the second term associated with the second edge also decreases very rapidly moving away from this edge. This explains why a solution of this type is called the edge effect.

It is possible now to formulate equality (7.18) in the following manner: for smooth shells of revolution, which are under the action of a smooth /116
external load, the general solution may be obtained by superposing a solution of the edge effect type on the membrane solution.

It is easy to see that in equation (7.11) the term which has been dropped becomes infinitely large at the apex of the shell because $\lim_{\theta \to 0} p(\theta) = \infty$. The same is true in regard to the numerator of ratio (7.16). Furthermore, shells of revolution are encountered where the term containing the large factor becomes equal to zero at certain points on the meridian. At these points of the meridian (in the future we shall call them singular points), the elementary approaches for finding the solutions which have been presented, as a rule, are

not applicable. To find solutions which are valid in the neighborhood of singular points, it becomes necessary to use more complex methods. We shall discuss these in the second part of the book.

The points along the meridian, where the solution can be obtained by the elementary method considered above, will be called ordinary points.

8. Further Simplification. Compliance Coefficients

We have shown that the accuracy of solution (7.21) is quite sufficient within the framework of the theory of thin shells. A cruder approximation is used in practice, based on the fact that in the solution the factor $\frac{1}{R_2^{3/4}\sin^{1/2}\theta}$ varies at a slower rate than the exponential factors contained within the brackets. Therefore, if we neglect the variation in this factor and include it into the constants of integration, we obtain the following simple form of the solution

$$\widetilde{T}^0 = \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}, \qquad \beta_1 = \int_{\theta_1}^{\theta} \frac{R_1 d\theta}{\sqrt{2cR_2}}, \qquad \beta_2 = \int_{\theta_1}^{\theta} \frac{R_1 d\theta}{\sqrt{2cR_2}}. \tag{8.1}$$

It is not difficult to determine the asymptotic error of this solution. Since the function $\psi(\theta)$ is retained, while the function $\varphi(\theta)$ is changed, only the expression in the first brackets with quantities of the order of R_0/c becomes equal to zero in equality (7.11). The terms within the second bracket, whose values are of the order $\sqrt{\frac{R_0}{c}}$ are not eliminated. If, as a measure of asymptotic error (as above), we use the order of the ratio of remaining terms to the eliminated terms, we arrive at the conclusion that the solution satisfies the homogeneous equation with an accuracy determined by the elimination of terms of order $\sqrt{\frac{c}{R_0}}$ with respect to unity. The simplified form of the solution given by (8.1) is frequently called the first approximation and is sometimes referred to as the Gekkeler solution. /117

Since we have adopted the simplified form of the solution based on the neglect of terms which vary slowly, there is no basis for taking into account the variation of other smooth functions. In addition, we can simplify the expressions obtained earlier by dropping the function $\widetilde{T}^0$ in them compared with its derivative. Indeed, when $\widetilde{T}^0$ is differentiated, it acquires a large factor

$\sqrt{\frac{R_0}{c}}$. Therefore, by neglecting the function compared with its derivative we again neglect a quantity whose order of magnitude is $\sqrt{\frac{c}{R_0}}$ compared with unity. Thus, substituting (8.1) into (7.19) we obtain

$$\widetilde{T}_1 = T_1^* - \frac{1+i}{\sqrt{2}}\sqrt{\frac{c}{R_2}}\operatorname{ctg}\theta\left[\widetilde{C}_1 e^{-(1-i)\beta_1} - \widetilde{C}_2 e^{(1-i)\beta_2}\right],$$
$$\widetilde{T}_2 = T_2^* + \widetilde{C}_1\left[1 + \frac{1+i}{\sqrt{2}}\sqrt{\frac{c}{R_2}}\operatorname{ctg}\theta\right]e^{-(1-i)\beta_1} +$$
$$+ \widetilde{C}_2\left[1 - \frac{1+i}{\sqrt{2}}\sqrt{\frac{c}{R_2}}\operatorname{ctg}\theta\right]e^{(1-i)\beta_2}.$$

Following the accepted principle of simplification we should drop the underlined terms and therefore should use the complex forces in the following simplified form

$$\widetilde{T}_1 = T_1^*,$$
$$\widetilde{T}_2 = T_2^* + \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}. \tag{8.2}$$

(The underlined term in the first equation is dropped on the basis that the stress which corresponds to it is smaller by a factor $\sqrt{\frac{c}{R_0}}$ than the stress corresponding to the terms retained in the second equation.)

To determine the forces and moments we introduce the real constants

$$\widetilde{C}_1 = A_1 + iB_1, \quad \widetilde{C}_2 = A_2 + iB_2$$

and replace the complex exponential functions in (8.2) by their expressions

$$e^{-(1-i)\beta_1} = e^{-\beta_1}(\cos\beta_1 + i\sin\beta_1), \quad e^{(1-i)\beta_2} = e^{\beta_2}(\cos\beta_2 - i\sin\beta_2).$$

Now, by separating (8.2) into real and imaginary parts we obtain from (IV, 9.2)

$$T_2 = T_2^* + [(A_1\cos\beta_1 - B_1\sin\beta_1)e^{-\beta_1} +$$
$$+ (A_2\cos\beta_2 + B_2\sin\beta_2)e^{\beta_2}],$$
$$M_1 = -c[(B_1\cos\beta_1 + A_1\sin\beta_1)e^{-\beta_1} + \tag{8.3}$$

$$+(B_2\cos\beta_2 - A_2\sin\beta_2)\,e^{\beta_2}],$$

$$M_2 = \nu M_1,$$

$$T_1 = T_1^*.$$

Furthermore, in accordance with (7.20) we find

$$\begin{aligned}
\vartheta &= \vartheta^* + \frac{1}{Eh}\sqrt{\frac{R_2}{2c}}\Big\{[(A_1+B_1)\cos\beta_1 + \\
&\quad + (A_1-B_1)\sin\beta_1]\,e^{-\beta_1} + [-(A_2+B_2)\cos\beta_2 + \\
&\quad + (A_2-B_2)\sin\beta_2]\,e^{\beta_2}\Big\}, \\
Q_\rho &= Q_\rho^* + \sqrt{\frac{c}{2R_2}}\frac{1}{\sin\theta}\Big\{[(-A_1+B_1)\cos\beta_1 + (A_1+B_1)\sin\beta_1]\,e^{-\beta_1} + \\
&\quad + [(A_2-B_2)\cos\beta_1 + (A_2+B_2)\sin\beta_2]\,e^{\beta_2}\Big\}, \\
u_\rho &= u_\rho^* + \frac{R_2\sin\theta}{Eh}[(A_1\cos\beta_1 - B_1\sin\beta_1)e^{-\beta_1} + \\
&\quad + (A_2\cos\beta_2 + B_2\sin\beta_2)e^{\beta_2}]
\end{aligned} \tag{8.4}$$

$$\left(\vartheta^* = -\frac{1}{Eh}\frac{R_2}{R_1}\frac{dT_1^*}{d\theta},\ u_\rho^* = \frac{R_2\sin\theta}{Eh}(T_2^* - \nu T_1^*),\ Q_\rho^* = \cos\theta\, T_1^*\right).$$

Neglecting the variation in the slowly changing factors we may consider the following approximate equality to be valid /118

$$\int_{\theta_1}^{\theta}\left(\frac{R_2\cos\theta}{R_1}\frac{d\widetilde{T}^0}{d\theta} + \nu\sin\theta\,\widetilde{T}^0\right)R_1 d\theta \approx R_2\cos\theta\,\widetilde{T}^0, \tag{8.5}$$

By using this equality, expression $(7.20)_2$ for u_z may be expressed in a simpler form

$$\begin{aligned}
u_z &= u_z^* + \frac{R_2\cos\theta}{Eh}[(A_1\cos\beta_1 - B_1\sin\beta_1)e^{-\beta_1} + \\
&\quad + (A_2\cos\beta_2 + B_2\sin\beta_2)e^{\beta_2}]
\end{aligned} \tag{8.6}$$

$$\left(u_z^* = a_1 + \frac{1}{Eh}\int_{\theta_1}^{\theta}\left[\frac{R_2\cos\theta}{R_1}\frac{dT^*}{d\theta} - \sin\theta(T_1^* - \nu T_2^*)\right]R_1 d\theta\right).$$

Finally, from equalities (8.6) and (1.10) we obtain

$$u = u^*,$$
$$w = w^* + \frac{R_2}{Eh}\left[(A_1 \cos \beta_1 - B_1 \sin \beta_1)\, e^{-\beta_1} + \right. \tag{8.7}$$
$$\left. + (A_2 \cos \beta_2 + B_2 \sin \beta_2)\, e^{\beta_2}\right].$$
$$(u^* = \cos \theta\, u_\rho^* - \sin \theta\, u_z^*,\ w^* = \sin \theta\, u_\rho^* + \cos \theta\, u_z^*).$$

For a complete determination of these quantities it is necessary to determine the values of constants A_1, B_1, A_2, B_2. The latter are obtained from /119

the boundary conditions of the particular problem. In cases when the edges of the shell are sufficiently removed from each other, the determination of the arbitrary constants is substantially simplified. Indeed, let us assume that we have stipulated some boundary conditions along the edge $\theta = \theta_1$. Since, in this case, $\beta_1 = 0$, A_1 and B_1 enter this condition with coefficients which are of the order of unity. As far as A_2 and B_2 are concerned, their factors are $\sin \beta_0 e^{-\beta_0}$, $\cos \beta_0 e^{-\beta_0}$, where $\beta_0 = \int_{\theta_1}^{\theta_2} \frac{R_1 d\theta}{\sqrt{2cR_2}}$. If the boundaries are far enough from each other, β_0 is a large quantity and the factors preceding A_2 and B_2 are small. This makes it possible for us to drop the terms in the boundary condition at the considered edge ($\theta = \theta_1$), when such terms are associated with the edge effect at the other edge, and to satisfy the condition by selecting two constants A_1 and B_1. The same is true concerning the formulation of boundary conditions at the other edge.

By satisfying the boundary conditions separately it is much easier to determine the arbitrary constants, because, in this case, instead of a system for four algebraic equations we have to solve two systems of the second order. The question arises as to how far the edges of the shell should be from each other, so that the specified approach of satisfying the boundary conditions separately is justified. We shall consider this question by using a spherical shell as an example.

For the spherical shell $R_1 = R_2 = R_0$

$$\beta_1 = \sqrt{\frac{R_0}{2c}}(\theta - \theta_1) = \sqrt[4]{3(1-\nu^2)}\sqrt{\frac{R_0}{h}}(\theta - \theta_1). \tag{8.8}$$

We assign a degree of accuracy for the required solution. For example, let this accuracy be 10 percent. From figure 17, which presents the graphs of functions $e^{-\beta}\cos\beta$ and $e^{-\beta}\sin\beta$ we find that when $\beta > \beta_0 = 2.1$ the functions of interest to us do not exceed 0.1. Let us determine the aperture angle corresponding to $\beta_0 = 2.1$. From (8.8) we find $\Delta\theta = \theta_0 - \theta_1 = \frac{\beta_0}{\sqrt[4]{3(1-\nu^2)}}\sqrt{\frac{h}{R_0}}$. The quantity $\Delta\theta$ may be called the width of the zone where the edge effect is damped. As we can see from the last relationship, the width of the zone depends on the accepted degree of accuracy for the solution and on the thinness of the shell's walls (on the ratio R_0/h). /120

Thus, for example, for a shell with a ratio $R/h = 30$ and with $\nu = 0.3$ /121 the width of the damping zone is equal to $\Delta\theta = \frac{2.1}{1.285}\sqrt{\frac{1}{30}} = 0.30 \approx 17^\circ$. It is easy to see that as the thickness of the shell is decreased, the damping zone of the edge effect is decreased.

Let us assume that the shell is sufficiently long so that it is possible to satisfy the boundary conditions separately. We write down the quantities which refer to the first edge ($\theta = \theta_1$)

$$
\begin{aligned}
T_2 &= T_2^* + (A_1\cos\beta_1 - B_1\sin\beta_1)\,e^{-\beta_1},\\
M &= -c\,(B_1\cos\beta_1 + A_1\sin\beta_1)\,e^{-\beta_1},\\
\vartheta &= \vartheta^* + \frac{1}{Eh}\sqrt{\frac{R_2}{2c}}\,[(A_1+B_1)\cos\beta_1 + (A_1-B_1)\sin\beta_1]\,e^{-\beta_1},\\
Q_\varphi &= Q_\varphi^* + \sqrt{\frac{c}{2R_2\sin\theta}}\,\frac{1}{}\,[(-A_1+B_1)\cos\beta_1 + (A_1+B_1)\sin\beta_1]\,e^{-\beta_1}\\
u_\rho &= u_\rho^* + \frac{R_2\sin\theta}{Eh}\,(A_1\cos\beta_1 - B_1\sin\beta_1)\,e^{-\beta_1},\\
u_z &= u_z^* + \frac{R_2\cos\theta}{Eh}\,(A_1\cos\beta_1 - B_1\sin\beta_1)\,e^{-\beta_1}.
\end{aligned}
\tag{8.9}
$$

To determine the two constants contained in this equation we must assign two of the following quantities at the boundary ($\theta = \theta_1$)

$$Q_\varphi,\ M_1,\ u_\rho,\ \vartheta. \tag{8.10}$$

It is very convenient to replace the constants A_1 and B_1 in terms of the boundary values given by two of the quantities in (8.10) before trying to satisfy the boundary conditions of the problem. This is true because the quantities (8.10) have a clearly defined static (or geometric) meaning. Let us do this.

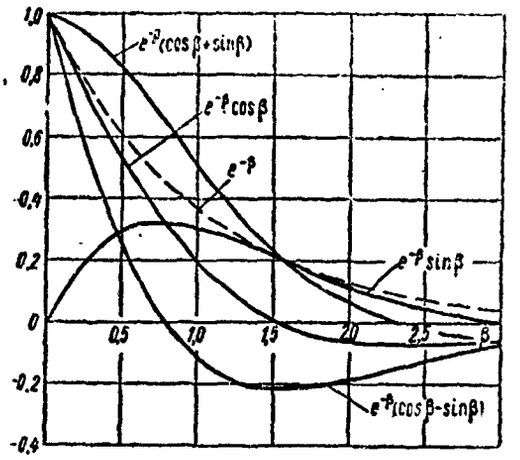

Figure 17

TABLE 1

β	$e^{-\beta}$	$e^{-\beta}\cos\beta$	$e^{-\beta}\sin\beta$	$e^{-\beta}(\cos\beta-\sin\beta)$	$e^{-\beta}(\cos\beta+\sin\beta)$
0	1	1	0	1	1
0,25	0,779	0,755	0,192	0,563	0,947
0,50	0,606	0,532	0,291	0,241	0,823
0,75	0,472	0,316	0,321	0,025	0,637
1,00	0,368	0,199	0,310	—0,111	0,509
1,25	0,287	0,090	0,272	—0,182	0,362
1,50	0,223	0,015	0,222	—0,207	0,237
1,75	0,174	—0,031	0,171	—0,202	0,140
2,00	0,135	—0,056	0,123	—0,179	0,067
2,25	0,105	—0,066	0,082	—0,148	0,016
2,50	0,082	—0,066	0,049	—0,115	—0,017
2,75	0,064	—0,059	0,024	—0,083	—0,035
3,00	0,050	—0,050	0,007	—0,057	—0,043

NOTE: Commas in tables represent decimal points.

From (8.9) we find (β_1 (θ_1) = 0)

$$M_0 = -cB_1,\ Q_0 = Q_0^* + \sqrt{\frac{c}{2R_2^0 \sin\theta_1}}(-A_1 + B_1) \qquad (8.11)$$
$$(Q_0 = Q_0(\theta_1),\ M_0 = M_1(\theta_1))$$

(Here and below the small zeros are used to designate quantities assigned at the edge.)

Solving relationships (8.11) for the arbitrary constants, we obtain

$$B_1 = -\frac{1}{c} M_0,$$
$$A_1 = -\frac{1}{c} M_0 - \sqrt{\frac{2R_2^0}{c}} \sin\theta_1 (Q_0 - Q_0^*).$$

Furthermore, by substituting these constants into expression (8.9) we obtain the following expressions for the quantities which are of interest to us /122

$$\begin{aligned}
T_1 &= T_2^* + \left[-\frac{1}{c}(\cos\beta_1 - \sin\beta_1) M_0 - \right. \\
&\left. - \sqrt{\frac{2R_2^0}{c}} \sin\theta_1 \cos\beta_1 (Q_0 - Q_0^*)\right] e^{-\beta_1}, \\
M_1 &= \left[(\cos\beta_1 + \sin\beta_1) M_0 + \right. \\
&\left. + c\sqrt{\frac{2R_2^0}{c}} \sin\theta_1 \sin\beta_1 (Q_0 - Q_0^*)\right] e^{-\beta_1}, \\
M_2 &= \nu M_1, \quad T_1 = T_1^*, \\
\vartheta &= \vartheta^* + \frac{1}{Eh}\sqrt{\frac{R_2}{2c}}\left[-\frac{2}{c}\cos\beta_1 M_0 - \right. \\
&\left. - \sqrt{\frac{2R_2^0}{c}} \sin\theta_1 (\cos\beta_1 + \sin\beta_1)(Q_0 - Q_0^*)\right] e^{-\beta_1}, \\
u_\rho &= u_\rho^* + \frac{R_2 \sin\theta}{Eh}\left[-\frac{1}{c}(\cos\beta_1 - \sin\beta_1) M_0 - \right. \\
&\left. - \sqrt{\frac{2R_2^0}{c}} \sin\theta_1 \cos\beta_1 (Q_0 - Q_0^*)\right] e^{-\beta_1}, \\
u_z &= u_z^* + \frac{R_2 \cos\theta}{Eh}\left[-\frac{1}{c}(\cos\beta_1 - \sin\beta_1) M_0 - \right. \\
&\left. - \sqrt{\frac{2R_2^0}{c}} \sin\theta_1 \cos\beta_1 (Q_0 - Q_0^*)\right] e^{-\beta_1}.
\end{aligned} \tag{8.12}$$

Carrying out the same transformations with quantities associated with the other edge we obtain analogous expressions

$$\begin{aligned}
T_2 &= T_2^* + \left[-\frac{1}{c}(\cos\beta_2 + \sin\beta_2) M_0 + \right. \\
&\left. + \sqrt{\frac{2R_2^0}{c}} \sin\theta_2 \cos\beta_2 (Q_0 - Q_0^*)\right] e^{\beta_2}, \\
M_1 &= \left[(\cos\beta_2 - \sin\beta_2) M_0 + \right. \\
&\left. + c\sqrt{\frac{2R_2^0}{c}} \sin\theta_2 \sin\beta_2 (Q_0 - Q_0^*)\right] e^{\beta_2}, \\
M_2 &= \nu M_1, \quad T_1 = T_1^*, \\
\vartheta &= \vartheta^* + \frac{1}{Eh}\sqrt{\frac{R_2}{2c}}\left[\frac{2}{c}\cos\beta_2 M_0 - \right.
\end{aligned} \tag{8.13}$$

$$-\sqrt{\frac{2R_2^0}{c}}\sin\theta_2(\cos\beta_2-\sin\beta_2)(Q_0-Q_0^*)\Big]e^{\beta_2}$$

$$u_1=u_\rho^*+\frac{R_2\sin\theta}{Eh}\Big[-\frac{1}{c}(\cos\beta_1+\sin\beta_1)M_0+$$

$$+\sqrt{\frac{2R_2^0}{c}}\sin\theta_2\cos\beta_2(Q_0-Q_0^*)\Big]e^{\beta_2},$$

$$w_1=w_2^*+\frac{R_2\cos\theta}{Eh}\Big[-\frac{1}{c}(\cos\beta_1+\sin\beta_1)M_0+$$

$$+\sqrt{\frac{2R_2^0}{c}}\sin\theta_2\cos\beta_2(Q_0-Q_0^*)\Big]e^{\beta_2}.$$

(Here the small zeros designate the quantities which are assigned along the edge $\theta=\theta_2$.) We recall that in both cases /123

$$u_\rho^*=\frac{R_2\sin\theta}{Eh}(T_2^*-\nu T_1^*),\quad \vartheta^*=-\frac{1}{Eh}\frac{R_2}{R_1}\frac{dT^*}{d\theta}.$$

$$u_2=a_1+\frac{1}{Eh}\int_{\theta_1}^{\theta}\Big[\frac{R_2\cos\theta}{R_1}\frac{dT^*}{d\theta}-\sin\theta(T_1^*-\nu T_2^*)\Big]R_1d\theta. \tag{8.14}$$

From these relationships, in particular, we can obtain expressions for the edge values of the angular displacement and the horizontal displacement. For this purpose it is necessary to substitute, respectively, $\theta=\theta_1$ and $\theta=\theta_2$ in the expressions for these quantities. Combining the results for both edges we obtain

$$u_0=u_2^*+\alpha_{11}(Q_0-Q_0^*)+\alpha_{12}M_0,$$
$$\vartheta_0=\vartheta_0^*+\alpha_{21}(Q_0-Q_0^*)+\alpha_{22}M_0. \tag{8.15}$$

where

$$\alpha_{11}=\mu 2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)^{3/2}\sqrt{\sin\theta_0}\frac{1}{Eh},$$

$$\alpha_{12}=\alpha_{21}=-2\sqrt{3(1-\nu^2)}\left(\frac{r_0}{h}\right)\frac{1}{Eh},$$

$$\alpha_{22}=\mu 4[3(1-\nu^2)]^{3/4}\sqrt{\frac{r_0}{h}}\frac{1}{\sqrt{\sin\theta_0}}\frac{1}{Eh^2}. \tag{8.16}$$

Here u_0, ϑ_0 are the values of the quantities u_ρ, ϑ along the considered edge, $\mu = -1$ at the edge $\theta = \theta_1$ and $\mu = +1$ at the edge $\theta = \theta_2$.

Relationships (8.15) and (8.16) give us expressions for the boundary values of the generalized displacements u_0, ϑ_0 in terms of the generalized forces Q_0, M_0 corresponding to them. The quantities α_{11}, α_{12}, α_{21}, α_{22} which depend both on the elastic properties of the shell material and on the form of the middle surface are usually called the edge compliance coefficients (sometimes the coefficients of rigidity). Their symmetric property $\alpha_{12} = \alpha_{21}$ follows from the Betti reciprocity law (Part II) and may serve as an additional confirmation that the above transformations have been carried out properly. Relationships (8.15) may be solved for the generalized edge forces, because the determinant of the transformation

$$D = \alpha_{11}\alpha_{22} - \alpha_{12}^2 = \frac{12(1-\nu^2)}{(Eh)^2}\left(\frac{r_0}{h}\right)^2 \tag{8.17}$$

is not equal to zero.

Let us show that we can drop the quantity ϑ_0^* in equalities (8.15). This will mean that in computing the angular displacement of the shell's edge we need not take into account the angular displacement corresponding to the membrane solution. To evaluate the effect of ϑ_0^* on the value of the edge /124

moment and on the value of the cutting force we assume that the edge of the shell is fixed.

We shall consider two cases:

I. $u_0^* \neq 0$, $\vartheta_0^* = 0$.

The values of the edge moment and of the edge cutting force in this case are determined in accordance with (8.15) from the system

$$u_0 = 0 = u_0^* + \alpha_{11}\left(Q_0^{(1)} - Q_0^{*(1)}\right) + \alpha_{12}M_0^{(1)},$$
$$\vartheta_0 = 0 = \alpha_{21}\left(Q_0^{(1)} - Q_0^{*(1)}\right) + \alpha_{22}M_0^{(1)}.$$

Solving this system we obtain

$$Q_0^{(1)} - Q_0^{*(1)} = -\frac{\alpha_{22}}{D}u_0^*, \quad M_0^{(1)} = \frac{\alpha_{21}}{D}u_0^*.$$

II. $u_0^* = 0$, $\vartheta_0^* \neq 0$.

$Q_0^{(2)} - Q_0^{*(2)}$, $M_0^{(2)}$ which are determined from the system

$$u_0 = 0 = \alpha_{11}\left(Q_0^{(2)} - Q_0^{*(2)}\right) + \alpha_{12}M_0^{(2)},$$
$$\vartheta_0 = 0 = \vartheta_0^* + \alpha_{21}\left(Q_0^{(2)} - Q_0^{*(2)}\right) + \alpha_{22}M_0^{(2)},$$

are equal to

$$Q_0^{(2)} - Q_0^{*(2)} = -\frac{\alpha_{12}}{D}\vartheta_0^*, \quad M^{(2)} = \frac{\alpha_{11}}{D}\vartheta_0^*.$$

Let us use equality (8.16) to compute the ratios corresponding to both cases of values. We obtain

$$\left|\frac{Q_0^{(2)} - Q_0^{*(2)}}{Q_0^{(1)} - Q_0^{*(1)}}\right| = \left|\frac{\alpha_{12}}{\alpha_{22}}\frac{\vartheta_0^*}{u_0^*}\right| = 0.39\sqrt{\frac{h}{R_2^0}}\left|\frac{r_0\vartheta_0^*}{u_0^*}\right|,$$
$$\left|\frac{M_0^{(2)}}{M_0^{(1)}}\right| = \left|\frac{\alpha_{11}}{\alpha_{21}}\frac{\vartheta_0^*}{u_0^*}\right| = 0.78\sqrt{\frac{h}{R_2^0}}\left|\frac{r\vartheta_0^*}{u_0^*}\right|.$$

For the membrane solution the ratio $\left|\frac{r_0\vartheta_0^*}{u_0^*}\right|$ is a quantity of the order of unity. Therefore, the edge values of the bending moment and of the cutting force corresponding to the membrane angular displacement ϑ_0^* are less by a factor $\sqrt{\frac{h}{R_2^0}}$ than the corresponding quantities determined by the membrane radial displacement. It follows from this that if we remain within the framework of errors inherent in the first approximation, the quantity ϑ_0^* in relationship (8.15) should be dropped. We note that in many cases the equality $\vartheta_0^* = 0$ is satisfied exactly.

We introduce a new function β, assuming that $\beta = \beta_1$ when we consider the edge $\theta = \theta_1$ and $\beta = -\beta_2$ for the second edge $\theta = \theta_2$. By using this function we can write the first four relationships of (8.12) and (8.13) in a single form /125

$$T_2 = T_2^* + \left[-\frac{2\sqrt{3(1-\nu^2)}}{h}M_0(\cos\beta - \sin\beta) + \right.$$
$$\left. + \mu 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{R_2^0}{h}}\sin\theta_0(Q_0 - Q_0^*)\cos\beta\right]e^{-\beta} \qquad (8.18)$$

$$M_1 = [(\cos\beta + \sin\beta) M_0 -$$
$$- \mu \frac{h}{\sqrt[4]{3(1-\nu^2)}} \sqrt{\frac{R_2^0}{h}} \sin\theta_0 (Q_0 - Q_0^*) \sin\beta] e^{-\beta}.$$
$$M_2 = \nu M_1, \quad T_1 = T_1^*.$$

Next we introduce the following quantities

$$\sigma_{M_0} = \frac{6M_0}{h^2}, \quad \sigma_k = 2\sqrt{3(1-\nu^2)} \sqrt{\frac{r_0}{h}} \sqrt{\sin\theta_0} \frac{Q_0 - Q_0^*}{h}. \tag{8.19}$$

The meaning of the first quantity is quite clear: it is the maximum flexural stress which occurs when a bending moment M_0 is applied along the edge of the shell.

Furthermore, acccrding to (1.20)

$$Q_{1n} = \sin\theta\, Q_\rho - \cos\theta\, Q_z. \tag{8.20}$$

In the symmetric case

$$Q_z = Q_z^*,$$

and, additionally,

$$Q_\rho^* = \cos\theta\, T_1^*, \quad Q_z^* = \sin\theta\, T_1^*.$$

From this we obtain the equality

$$\cos\theta\, Q_z^* = \sin\theta\, Q_\rho^*.$$

Replacing it by means of the second term in (8.20), we obtain

$$Q_{1n} = \sin\theta\,(Q_\rho - Q_\rho^*).$$

From the last relationship it becomes clear that the quantity $\sin\theta_0\ (Q_0 - Q_0^*)$ represents a cutting force which extends (compresses) the edge of the shell like a ring. By determining the tangenti·' stress $\sigma_{T_3} = \frac{T_2}{h}$ (it is frequently called

a ring stress) we see that, in accordance with $(8.18)_1$, σ_k is the boundary ring stress produced by the action of the cutting force applied along the edge of the shell.

Now, by determining from (8.18) the tangential (ring) and meridian flexural stress $\sigma_{M_1} = \frac{6M_1}{h^2}$, we obtain

$$\begin{gathered}\sigma_{M_1} = \sigma_{M_0}(\cos\beta + \sin\beta)e^{-\beta} - \mu\left(\frac{3}{\sqrt{3(1-\nu^2)}}\right)\sigma_k \sin\beta\, e^{-\beta},\\ \sigma_{T_1} = \sigma_{T_2}^{\circ} + \mu\sigma_k \cos\beta\, e^{-\beta} - \left(\frac{\sqrt{3(1-\nu^2)}}{3}\right)\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta},\\ \sigma_{M_2} = \nu\sigma_{M_1},\quad \sigma_{T_1} = \sigma_{T_1}^{\circ}.\end{gathered} \tag{8.22}$$

The graphs of the functions which are used here are shown in figure 17. /126
The compliance coefficients can also be used during the elastic joining of shells, and this must be done in cases when complex joining is impossible (when shells of different thickness or shells of different materials are joined or when more than two shells are joined along the junction line).

Let us assume that the membrane solution in which C_1 and a_1 have already been determined is such that $u_0 = u_0^*$, $\vartheta_0 = \vartheta_0^*$. Then, since the determinant (8.17) is not equal to zero, it follows from (8.15) that $M_0 = (Q_0 - Q_0^*)$ $\sin\theta_0 = 0$. This means that neither a bending moment nor a cutting force is applied to the edge of the shell. In addition it follows from the relationships that all quantities coincide with those determined from the membrane theory. Usually, however, the boundary conditions of the problem cannot be satisfied by a proper selection of the membrane solution. As we can see from the relationships which have been obtained, in this general case the solution is composed of two parts. The first represents a membrane solution. Its role is reduced so that it satisfies the equilibrium conditions of the shell as a whole and the removal of the surface load. The membrane solution which is obtained is associated, as a rule, with boundary values which do not satisfy the boundary values of the considered problem. The purpose of the second part (a solution of the edge effect type) is to eliminate the incompatibility in the boundary conditions. Physically this is achieved by applying a bending moment M_0 and a cutting force $(Q_0 - Q_0^*)\sin\theta_0$ to the edge of the shell.

By using the relationships which have been derived it is possible to completely solve (in the considered special case when a shell of revolution is symmetrically deformed) a rather complex problem which has been formulated in a general way: what are the boundary conditions that can be assigned in the theory of shells?

Three pairs of quantities play a role in the relationships obtained above (the generalized displacements and the corresponding generalized forces)

$$\begin{matrix} u_0 & \vartheta_0 & (u_z)_0 \\ , & , & , \\ Q_0 & M_0 & (Q_z)_0 \end{matrix} \tag{8.23}$$

where according to (8.12) and (8.13)

$$\begin{aligned} (u_z)_0 &= (u_z^*)_0 + \gamma_1 (Q_0 - Q_0^*) + \gamma_2 M_0, \\ \gamma_1 &= \mu 2\sqrt[4]{3(1-\nu^2)} \left(\frac{r_0}{h}\right)^{3/2} \frac{\cos\theta_0}{\sqrt{\sin\theta_0}} \frac{1}{E}, \\ \gamma_2 &= -2\sqrt[4]{3(1-\nu^2)} \left(\frac{r_0}{h}\right) \operatorname{ctg}\theta_0 \frac{1}{Eh}. \end{aligned} \tag{8.24}$$

Frequently it is assumed that only one quantity from each pair can be assigned (either the generalized displacement or the corresponding generalized force). We shall show that this is not the case. Let us assume, for example, that two quantities from one pair u_0 and Q_0 are assigned at the edge. We can /127

see from relationships (8.15) that the established boundary conditions do not contradict them. The assignment of u_0 and Q_0 determines M_0 in accordance

with $(8.15)_1$. Then ϑ_0 is determined from $(8.15)_2$.

Thus, by considering a specific example we have become convinced that it is possible to assign both the generalized displacements as well as the generalized forces at the edge of the shell. However, this situation which is possible in principle is seldom achieved in practice, because we usually know more or less reliably either the value of the generalized displacement or the value of the generalized force. Above we have decided to call such conditions ordinary.

The quantities from the first two pairs may be assigned independently at each edge. The situation is different in regard to the third pair. Q_z con-

tains one arbitrary constant C_1 and may be assigned only at one edge. This is

associated with the fact that the assignment of a vertical force which is not self-balanced at some horizontal cross section of the shell determines completely its values at any other cross section. The presence of quadrature (6.2") is also associated with this. By selecting a constant a_1, we can assign u_z at one edge. This fixes completely the rigid axial displacement of the shell.

Of course u_z can be assigned on both ends of the shell. This determines the

values of both components and consequently the value of Q_z.

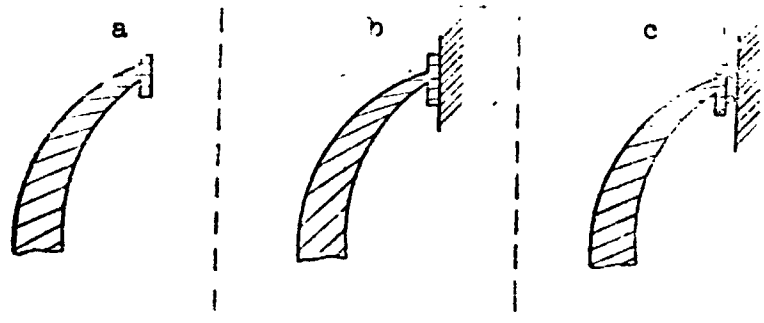

Figure 18

For a shell of revolution, the following three versions of the ordinary boundary conditions are most important:

(a) free edge (fig. 18a)

$$Q_0 = M_0 = (Q_z)_0 = 0, \tag{8.25}$$

(b) fixed edge (fig. 18b)

$$u_0 = \vartheta_0 = (u_z)_0 = 0, \tag{8.26}$$

(c) sliding constraint (fig. 18c)

$$u_0 = \vartheta_0 = 0, \tag{8.27}$$

while $(Q_z)_0$ is an assigned quantity.

In conclusion we present the expressions for displacements and for angles of rotation /128

$$\begin{aligned}
u_\rho &= u_\rho^* + \\
&+ \frac{R_2 \sin\theta}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \mu\sigma_k \cos\beta\, e^{-\beta}\right\}, \\
u_z &= u_z^* + \\
&+ \frac{R_2 \cos\theta}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \mu\sigma_k \cos\beta\, e^{-\beta}\right\}, \\
w &= w^* + \\
&+ \frac{R_2}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \mu\sigma_k \cos\beta\, e^{-\beta}\right\},
\end{aligned} \tag{8.28}$$

$$u = u^*,$$

$$\vartheta = \vartheta^* + \frac{\sqrt[4]{3(1-\nu^2)}}{E}\sqrt{\frac{R_2}{h}}\left\{\mu\frac{2\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}\cos\beta\, e^{-\beta}\right.$$
$$\left. -\sigma_k(\cos\beta+\sin\beta)\,e^{-\beta}\right\}.$$

Here u_ρ^*, u_z^*, ϑ^* are quantities determined by equalities (8.14), while

$$w^* = \sin\theta\, u_\rho^* + \cos\theta\, u_z^*,\quad u^* = \cos\theta\, u_\rho^* - \sin\theta\, u_z^*.$$

9. The Antisymmetric Case (k = 1)

We shall consider simultaneously the first and the second antisymmetric case. According to (3.3) the principal complex system takes on the form

$$\begin{aligned} &G_1\{\widetilde{U}_{.1}\} + \left[1 - ic\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\widetilde{T}_{.1} = f_1(\theta),\\ &-icG_1\{\widetilde{T}_{.1}\} + \widetilde{T}_{.1} + \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\widetilde{U}_{.1} = R_2 q_{n.1}, \end{aligned} \tag{9.1}$$

where

$$\begin{aligned} G_1 &= \frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left[\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\right] - \frac{1}{R_2\sin^2\theta},\\ f_1(\theta) &= \frac{1}{R_1R_2\sin\theta}\left\{\frac{d}{d\theta}\left[(q_{n.1}\cos\theta - q_{1.1}\sin\theta)R_2^3\sin^2\theta\right] \pm\right.\\ &\left.\pm q_{2.1}R_1R_2^2\sin^2\theta\right\}. \end{aligned} \tag{9.2}$$

As in the symmetric case, the basic system must permit two quadratures. To establish these we subtract the second equation from the first. If we introduce a new auxiliary function /129

$$\widetilde{W} = \widetilde{U}_{.1} + ic\widetilde{T}_{.1}, \tag{9.3}$$

the equation which has been obtained can be written in the form

$$G_1\{\widetilde{W}\} - \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\widetilde{W} = f_1(\theta) - R_2 q_{n.1}. \tag{9.4}$$

Expanding this equation we find

$$\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left[\frac{R_2^2\sin\theta}{R_1}\frac{d\widetilde{W}}{d\theta}\right]-\frac{1}{R_1\sin^2\theta}\widetilde{W}=f_1(\theta)-R_2q_{n,1}. \tag{9.5}$$

We introduce another auxiliary function, which is associated with $\widetilde{W}$ by means of the following relationship

$$\widetilde{Y}=R_2\sin\theta\,\widetilde{W}. \tag{9.6}$$

Using this relationship and carrying out simple transformations, equation (9.5) is reduced to the following form

$$\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1\sin\theta}\frac{d\widetilde{Y}}{d\theta}\right)=f_1(\theta)-R_2q_{n,1}.$$

From this is follows that

$$\frac{1}{R_1\sin\theta}\frac{d\widetilde{Y}}{d\theta}=\int_{\theta_0}^{\theta}\frac{1}{R_2\sin\theta}\frac{d}{d\theta}\left[\left(q_{n,1}\cos\theta-q_{1,1}\sin\theta\right)R_2^3\sin^2\theta\right]d\theta-$$
$$-\int_{\theta_0}^{\theta}\left(q_{n,1}\mp q_{2,1}\sin\theta\right)R_1R_2d\theta+\frac{\widetilde{C}_3}{\widetilde{C}_5}.$$

Integrating the first integral in the right side of the equality by parts, we obtain

$$\frac{d\widetilde{Y}}{d\theta}=\frac{\widetilde{C}_3}{\widetilde{C}_5}\cdot R_1\sin\theta+\Phi(\theta)R_1\sin\theta, \tag{9.7}$$

where

$$\Phi(\theta)=\left(q_{n,1}\cos\theta-q_{1,1}\sin\theta\right)R_2^2\sin\theta-$$
$$-\int_{\theta_0}^{\theta}\left(q_{n,1}\sin\theta+q_{1,1}\cos\theta-q_{2,1}\right)R_1R_2\sin\theta d\theta.$$

Integrating once again we obtain the second quadrature

$$\widetilde{Y}=\frac{\widetilde{C}_4}{\widetilde{C}_6}+\frac{\widetilde{C}_3}{\widetilde{C}_5}\cdot\int_{\theta_0}^{\theta}R_1\sin\theta d\theta+\int_{\theta_0}^{\theta}\Phi(\theta)R_1\sin\theta d\theta.$$

From this we obtain an expression for the function $\widetilde{W}$

$$\widetilde{W}=\frac{1}{R_2\sin\theta}\left\{\frac{\widetilde{C}_4}{\widetilde{C}_6}+\frac{\widetilde{C}_3}{\widetilde{C}_5}\cdot\int_{\theta_0}^{\theta}R_1\sin\theta d\theta+\int_{\theta_0}^{\theta}\Phi(\theta)R_1\sin\theta d\theta\right\}. \tag{9.8}$$

Let us establish the real values of the constants $\widetilde{C}_3$, $\widetilde{C}_4$ and their static meaning. First of all, according to (3.9), $\widetilde{C}_5$ $\widetilde{C}_6$ /130

$$\begin{aligned}&\varkappa_{\rho,1}=\frac{1}{r}\left(\vartheta_{,1}+\frac{u_{z,1}}{r}\right),\quad \varkappa_{\gamma_1,1}=\mp\frac{1}{r}\left(\vartheta_{,1}+\frac{u_{z,1}}{r}\right),\\&\varkappa_{z,1}=-\frac{1}{r^2}(u_{\rho,1}\pm v_{,1}),\quad \varepsilon_{2,1}=\frac{1}{r}(u_{\rho,1}\pm v_{,1}).\end{aligned} \tag{9.9}$$

By using these expressions the complex boundary quantities (3.7) can be written in the form

$$\begin{aligned}&\widetilde{Q}_{\rho,1}=Q_{\rho,1}-i\frac{Ehc}{r}\left(\vartheta_{,1}+\frac{u_{z,1}}{r}\right)=Q^*_{\rho,1}-i\frac{Ehc}{r}\left(\widetilde{\vartheta}_{,1}+\frac{\widetilde{u}_{z,1}}{r}\right),\\&\widetilde{Q}_{12,1}=Q_{12,1}\mp i\frac{Ehc}{r}\left(\vartheta_{,1}+\frac{u_{z,1}}{r}\right)=Q^*_{12,1}\mp i\frac{Ehc}{r}\left(\widetilde{\vartheta}_{,1}+\frac{\widetilde{u}_{z,1}}{r}\right),\\&\widetilde{Q}_{z,1}=Q_{z,1}-i\frac{Ehc}{r^2}(u_{\rho,1}\pm v_{,1})=Q^*_{z,1}-i\frac{Ehc}{r^2}\left(\widetilde{u}_{\rho,1}\pm\widetilde{v}_{,1}\right),\\&\widetilde{M}_{1,1}=M_{1,1}+i\frac{Ehc}{r}(u_{\rho,1}\pm v_{,1})=M^*_{1,1}+i\frac{Ehc}{r}\left(\widetilde{u}_{\rho,1}+\widetilde{v}_{,1}\right).\end{aligned} \tag{9.10}$$

Here

$$\begin{aligned}&Q_{12,1}=S_{,1}+\frac{2H_{,1}}{R_2},\\&Q_{\rho,1}=\cos\theta T_{1,1}+\sin\theta Q_{1n,1},\quad Q_{z,1}=\sin\theta T_{1,1}-\cos\theta Q_{1n,1},\\&Q_{1n,1}=\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2\sin\theta M_{1,1}}{d\theta}\pm 2R_1H_{,1}-R_1\cos\theta M_{2,1}\right).\end{aligned} \tag{9.11}$$

In addition, the following approximate expressions are valid

$$\widetilde{Q}_{12,1}=\widetilde{S}_{,1}\quad \widetilde{M}_{1,1}=ic\left(\widetilde{T}_{2,1}-\nu\widetilde{T}_{1,1}\right),$$
$$\widetilde{Q}_{\rho,1}=\cos\theta\widetilde{T}_{1,1}+ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}, \tag{9.12}$$
$$\widetilde{Q}_{z,1}=\sin\theta\widetilde{T}_{1,1}-ic\frac{\cos\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}.$$

Finally, in accordance with equalities $(3.4)_1$ and (3.5),

$$\widetilde{U}_{,1}=R_2\sin^2\theta\widetilde{T}_{1,1}-ic\frac{R_2\sin\theta\cos\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}, \tag{9.13}$$

$$\pm\widetilde{S}=-\frac{1}{R_1\sin\theta}\frac{d\widetilde{U}_{,1}}{d\theta}-ic\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}_{,1}+ \tag{9.14}$$
$$+R_2(q_{n,1}\cos\theta-q_{1,1}\sin\theta).$$

From this it follows that

$$\widetilde{W}=\widetilde{U}_{,1}+ic\widetilde{T}_{,1}=\widetilde{T}_{1,1}R_2\sin^2\theta+$$
$$+ic\left\{\widetilde{T}_{,1}-\frac{R_2\sin\theta\cos\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right\}. \tag{9.15}$$

However, according to (9.12),

$$\widetilde{M}_{1,1}+r\widetilde{Q}_{z,1}=\widetilde{T}_{1,1}R_2\sin^2\theta+$$
$$+ic\left\{\widetilde{T}_{,1}-\underline{(1+\nu)\widetilde{T}_{1,1}}-\frac{R_2\sin\theta\cos\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right\}.$$

If in the above equation we neglect small (underlined) terms and consider /131 equalities (9.10) and (9.15), we obtain

$$\widetilde{W}=\widetilde{M}_{1,1}+r\widetilde{Q}_{z,1}=M_{1,1}+rQ_{z,1}=M^*_{1,1}+rQ^*_{z,1}=W^*. \tag{9.16}$$

From this it follows that the constants $\widetilde{c}_3$, $\widetilde{c}_4$ are real. Further, by drawing attention to (9.8), we obtain

$\widetilde{c}_5$ $\widetilde{c}_6$

$$\begin{matrix} C_4 \\ C_6 \end{matrix} = [r(M_{1,1} + rQ_{z,1})]_{\theta=\theta_0}. \tag{9.17}$$

As in the symmetric cases we can show that the real property of the function $\widetilde{W}$ provides for the periodic nature of the respective displacements and angles of rotation.

We now rewrite equality $(9.12)_2$ with the aid of $(3.4)_1$ in the form

$$\widetilde{Q}_{\rho,1} = \frac{\operatorname{ctg}\theta}{R_2 \sin\theta}\widetilde{U}_{,1} + ic\frac{1}{R_1 \sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta}. \tag{9.18}$$

It follows from equality (9.10) that

$$\widetilde{Q}_{\rho,1} \mp \widetilde{Q}_{12,1} = Q_{\rho,1} \mp Q_{12,1} = Q^*_{\rho,1} \mp Q^*_{12,1}. \tag{9.19}$$

However, by using equalities (9.18), $(9.12)_1$ and (9.14) this quantity can be represented in the following form

$$\begin{aligned} \widetilde{Q}_{\rho,1} \mp \widetilde{Q}_{12,1} &= \frac{\operatorname{ctg}\theta}{R_2 \sin\theta}\widetilde{U}_{,1} + \frac{ic}{R_1 \sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta} + \\ &+ \frac{1}{R_1 \sin\theta}\frac{d\widetilde{U}_{,1}}{d\theta} + ic\frac{\operatorname{ctg}\theta}{R_2 \sin\theta}\widetilde{T}_{,1} - R_2(q_{n,1}\cos\theta - q_{1,1}\sin\theta) = \\ &= \frac{1}{r}\left\{\frac{1}{R_1 \sin\theta}\frac{dR_2 \sin\theta W}{d\theta} - R_2^2 \sin\theta(q_{n,1}\cos\theta - q_{1,1}\sin\theta)\right\}. \end{aligned} \tag{9.20}$$

Comparing the above expression with (9.7) - (9.6) we find

$$\begin{matrix} C_3 \\ C_5 \end{matrix} = [r(Q_{\rho,1} \mp Q_{12,1})]_{\theta_0}. \tag{9.21}$$

Let us determine the principal vector and the principal moment of forces and moments which are applied to the parallel ring of the shell (fig. 19). First of all, /132

$$\begin{aligned} B_x &= \int_0^{2\pi}[(-\sin\varphi)M_1 - (r\sin\varphi)Q_z]\,r\,d\varphi = \\ &= r\int_0^{2\pi}\left\{-\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}M_{1,1} - r\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}Q_{z,1}\right\}d\varphi. \end{aligned}$$

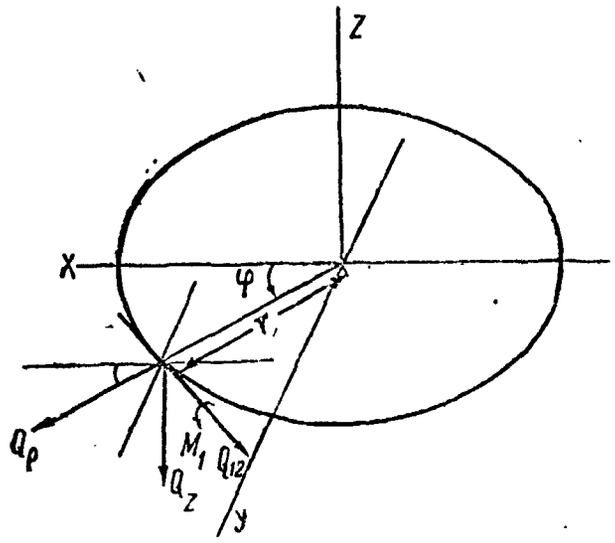

Figure 19

Integrating over φ we obtain

$$\mathbf{B}_x = \begin{cases} 0 \\ -\pi r\left(M_{1,1} + rQ_{z,1}\right) \end{cases}. \qquad (9.22)_1$$

Here, as we have already stated, the upper function refers to the first symmetric case, while the lower one refers to the second symmetric case. The remaining components of the principal vector and of the principal moment, which are different from 0, are determined in exactly the same way

$$\mathbf{B}_y = \begin{cases} \pi r\left(M_{1,1} + rQ_{z,1}\right) \\ 0 \end{cases},$$

$$\mathbf{F}_x = \begin{cases} \pi r\left(Q_{\rho,1} - T_{12,1}\right) \\ 0 \end{cases}, \qquad (9.22)_{2,\ 3,\ 4}$$

$$\mathbf{F}_y = \begin{cases} 0 \\ \pi r\left(Q_{\rho,1} + T_{12,1}\right) \end{cases}.$$

We recall that the components $\mathbf{B}_z$ and $\mathbf{F}_z$ pertain to the symmetric cases /133

(see sections 5 and 6). Thus, the general case when the load, which is not self-balanced, is assigned at the parallel ring may be considered by means of relationships obtained for the symmetric and the antisymmetric cases. First of all, according to (9.16), it follows from the expressions which have been obtained that

$$W=\frac{1}{\pi r}\frac{B_y}{B_x}. \tag{9.23}$$

Thus the function W determines the value of the principal moment of all forces and moments applied at the cross section perpendicular to the axis of the shell. Furthermore, the static meaning of the following constants becomes clear from equalities (9.21) and (9.17)

$$\begin{aligned} C_3&=\frac{1}{\pi}(F_x)_{\theta_0}, \quad C_5=\frac{1}{\pi}(F_y)_{\theta_0}, \\ C_4&=\frac{1}{\pi}(B_y)_{\theta_0}, \quad C_6=-\frac{1}{\pi}(B_x)_{\theta_0} \end{aligned} \tag{9.24}$$

We should note that the rule of signs established in Chapter III for the forces and moments is also valid for the components of the principal vector and the principal moment: for the edge of the shell which, when approached, increases the value of the coordinate, the positive directions of the components coincide with the directions of the respective coordinate axes, while at the other edge they differ in signs. This should be remembered in order to avoid misunderstandings.

Let us return to equation $(9.1)_2$. Since it may be assumed that the function W is known, we replace the function $\widetilde{U}_{,1}$ in the equation by its expression $(W-ic\widetilde{T}_{,1})$. As a result of this we arrive at the basic characteristic equation for the antisymmetric case

$$\begin{aligned} \frac{d^2\widetilde{T}_{,1}}{d\theta^2}+\left[\left(2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{d\widetilde{T}_{,1}}{d\theta}+\frac{R_1}{R_2}\left(1-2\frac{R_1}{R_2}\right)\frac{1}{\sin^2\theta}\widetilde{T}_{,1}+ \\ +i\frac{R_1^2}{cR_2}\widetilde{T}_{,1}=i\frac{R_1^2}{cR_2}F(\theta), \end{aligned} \tag{9.25}$$

where

$$\begin{aligned} F(\theta)&=R_2q_{n,1}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{W}{\sin^2\theta}=R_2q_{n,1}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{R_1R_2\sin^3\theta}\times \\ &\times\left[\frac{C_4}{C_6}+\frac{C_3}{C_5}\cdot\int_{\theta_0}^{\theta}R_1\sin\theta d\theta+\int_{\theta_0}^{\theta}\Phi(\theta)R_1\sin\theta d\theta\right], \\ \Phi(\theta)&=(q_{n,1}\cos\theta-q_{1,1}\sin\theta)R_2^2\sin\theta- \\ &-\int_{\theta_0}^{\theta}(q_{n,1}\sin\theta+q_{1,1}\cos\theta-q_{n,1})R_1R_2\sin\theta d\theta. \end{aligned} \tag{9.26}$$

Using the value of $\widetilde{T}_{,1}$ obtained here, the complex forces, in accordance /134
with equalities (3.4) and (3.5), are determined by means of the following relationships

$$\begin{aligned}
\widetilde{T}_{1,1} &= \frac{W}{R_2 \sin^2\theta} + ic\left\{\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta} - \underline{\frac{1}{R_2\sin^2\theta}\widetilde{T}_{,1}}\right\},\\
\widetilde{T}_{2,1} &= -\frac{W}{R_2\sin^2\theta} + \widetilde{T}_{,1} - ic\left\{\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta} - \underline{\frac{1}{R_2\sin^2\theta}\widetilde{T}_{,1}}\right\},\\
\pm\widetilde{S}_{,1} &= -\frac{1}{R_1\sin\theta}\frac{dW}{d\theta} + ic\left(\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta} - \underline{\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}_{,1}}\right) + \\
&\quad + R_2(q_{n,1}\cos\theta - q_{1,1}\sin\theta)
\end{aligned} \tag{9.27}$$

(The underlined terms are small and may be dropped.)

Let us now proceed with the determination of the displacements. The strain components, according to (3.6), are written in the form

$$\begin{aligned}
\varepsilon_{1,1} &= -\frac{1}{R_1\sin\theta}\frac{du_{z,1}}{d\theta} - \operatorname{ctg}\theta\vartheta_{,1},\\
\varepsilon_{2,1} &= \frac{u_{\rho,1} \pm v_{,1}}{r},\\
\omega_{,1} &= \frac{r}{R_1}\frac{d}{d\theta}\left(\frac{v_{,1}}{r}\right) \mp \frac{\cos\theta u_{\rho,1} - \sin\theta u_{z,1}}{r},\\
\varkappa_{1,1} &= \frac{1}{R_1}\frac{d\vartheta_{,1}}{d\theta},\\
\varkappa_{2,1} &= \frac{\sin\theta(u_{\rho,1} \pm v_{,1}) + \cos\theta(u_{z,1} + \vartheta_{,1})}{r^2},\\
\tau_{,1} &= \pm\frac{1}{R_1}\left[\sin\theta\frac{d}{d\theta}\left(\frac{u_{\rho,1} \pm v_{,1}}{r}\right) + \cos\theta\frac{d}{d\theta}\left(\frac{u_{z,1}}{r}\right)\right].
\end{aligned} \tag{9.28}$$

We introduce three new functions

$$\chi = \vartheta_{,1} + \frac{u_{z,1}}{r}, \quad \beta = \frac{1}{R_1}\frac{u}{d\theta}\left(\frac{u_{z,1}}{r}\right), \quad \gamma = \frac{u_{\rho,1} \pm v_{,1}}{r}. \tag{9.29}$$

By means of these, the strain components are written in the form

$$\varepsilon_{1,1} = -\operatorname{ctg}\theta\,\gamma - R_2\beta, \qquad \varkappa_{1,1} = \frac{1}{R_1}\frac{d\gamma}{d\theta} - \beta,$$

$$\varepsilon_{2,1}=\gamma, \qquad \varkappa_{2,1}=\frac{\cos\theta\chi+\sin\theta\gamma}{R_2\sin\theta},$$
$$\omega_{,1}=\pm\left(\frac{R_2\sin\theta}{R_1}\frac{d\gamma}{d\theta}+R_2\cos\theta\beta+\frac{\chi}{\sin\theta}\right), \quad \tau_{,1}=\pm\left(\frac{\sin\theta}{R_1}\frac{d\chi}{d\theta}+\cos\theta\beta\right). \tag{9.30}$$

Assuming that k = 1 in equalities (3.6) and replacing the components of complex strain by expressions analogous to (9.30), we obtain approximately a compatible system of equations for determining the complex displacements

$$-\operatorname{ctg}\theta\widetilde{\chi}-R_2\widetilde{\beta}=\frac{1}{Eh}\left\{-\nu\widetilde{T}_{,1}+i(1+\nu)c\left(\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{1}{R_2\sin^2\theta}\widetilde{T}_{,1}\right)+\right.$$
$$\left.+(1+\nu)\frac{W}{R_2\sin^2\theta}+\frac{i}{c}M^*_{2,1}\right\},$$
$$\widetilde{\gamma}=\frac{1}{Eh}\left\{\widetilde{T}_{,1}-i(1+\nu)c\left(\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{1}{R_2\sin^2\theta}\widetilde{T}_{,1}\right)-\right.$$
$$\left.-(1+\nu)\frac{W}{R_2\sin^2\theta}+\frac{i}{c}M^*_{1,1}\right\},$$
$$\frac{R_2\sin\theta}{R_1}\frac{d\widetilde{\gamma}}{d\theta}+R_2\cos\theta\widetilde{\beta}+\frac{\widetilde{\chi}}{\sin\theta}=$$
$$=\frac{1}{Eh}\left\{2(1+\nu)\left[ic\left(\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}_{,1}\right)-\right.\right.$$
$$\left.\left.-\frac{1}{R_1\sin\theta}\frac{dW}{d\theta}+R_2(q_{n,1}\cos\theta-q_{1,1}\sin\theta)\right]\mp\frac{i}{c}2H^*_{,1}\right\}, \tag{9.31}$$
$$\frac{1}{R_1}\frac{d\widetilde{\chi}}{d\theta}-\widetilde{\beta}=\frac{i}{Ehc}\left\{\widetilde{T}_{,1}-ic\left(\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{1}{R_2\sin^2\theta}\widetilde{T}_{,1}\right)-\right.$$
$$\left.-\left(\frac{W}{R_2\sin^2\theta}+T^*_{2,1}\right)\right\},$$
$$\frac{\cos\theta\widetilde{\chi}+\sin\theta\widetilde{\gamma}}{R_2\sin\theta}=\frac{i}{Ehc}\left\{ic\left(\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{1}{R_2\sin^2\theta}\widetilde{T}_{,1}\right)+\left(\frac{W}{R_2\sin^2\theta}-T^*_{1,1}\right)\right\},$$
$$\frac{\sin\theta}{R_1}\frac{d\widetilde{\chi}}{d\theta}+\cos\theta\widetilde{\beta}=-\frac{i}{Ehc}\left\{ic\left(\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}_{,1}\right)-\right.$$
$$\left.-\left(\frac{1}{R_1\sin\theta}\frac{dW}{d\theta}\pm S^*_{,1}\right)+R_2(q_{n,1}\cos\theta-q_{1,1}\sin\theta)\right\}.$$

(The quantities $\widetilde{\chi}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are analogous to χ, β, γ, but are constructed from the complex displacements so that ($\chi = \operatorname{Re}\widetilde{\chi}$, $\beta = \operatorname{Re}\widetilde{\beta}$, $\gamma = \operatorname{Re}\widetilde{\gamma}$.)

The function $\widetilde{\gamma}$ is given by the second equation of the system. This equation is also used to obtain the following from the first and third equations

$$\tilde{\chi} = \frac{1}{Eh}\left\{-\frac{R_2^2 \sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{T}_{,1}}{R_2\sin\theta}\right)+(1+\nu)\left[-\frac{1}{R_1R_2\sin^2\theta}\frac{dR_2\sin\theta W}{d\theta}+\right.\right.$$
$$\left.+R_2(q_{n,1}\operatorname{ctg}\theta-2q_{1,1})\right]+ \qquad (9.32)$$
$$\left.+\frac{i}{c}\frac{1}{\sin\theta}\left(\mp 2H^*_{,1}-\frac{R_2\sin\theta}{R_1}\frac{dM^*_{1,1}}{d\theta}+\cos\theta M^*_{2,1}\right)\right\}.$$

In the antisymmetric case the function $\tilde{\chi}$ plays the same role as the function $\tilde{\vartheta}$ in the symmetric case. Its real part χ represents an elastic rotation of the tangent towards the meridian in the section $\varphi = \begin{cases}0\\ \pi/2\end{cases}$. Indeed, it /136 is equal to the total rotation of the tangent towards the meridian $\vartheta_{,1}$ beyond the amount subtracted for the rotation of the section as a rigid whole by an angle $\left(-\frac{u_{z,1}}{r}\right)$ (fig. 20).

The last function $(\tilde{\beta})$ is obtained from the first equation, in which $\tilde{\chi}$ is expressed by (9.32). Dropping small terms we obtain

$$\tilde{\beta} = \frac{1}{Eh}\left\{\frac{\operatorname{ctg}\theta}{R_1}\frac{d\tilde{T}_{,1}}{d\theta}+\right.$$
$$+\left(\frac{\nu}{R_2}-\frac{\cos^2\theta}{R_2\sin^2\theta}\right)\tilde{T}_{,1}-$$
$$-(1+\nu)\left[-\frac{\cos\theta}{R_1R_2^2\sin^4\theta}\times\right.$$
$$\times\frac{dR_2\sin\theta W}{d\theta}+\frac{W}{R_2^2\sin^3\theta}+ \qquad (9.33)$$
$$\left.+\operatorname{ctg}\theta(q_{n,1}\operatorname{ctg}\theta-2q_{1,1})\right]-$$
$$-\frac{i}{c}\frac{\cos\theta}{R_2\sin^2\theta}\left(\mp 2H^*_{,1}-\right.$$
$$\left.\left.-\frac{R_2\sin\theta}{R_1}\frac{dM^*_{1,1}}{d\theta}+\frac{M^*_{2,1}}{\cos\theta}\right)\right\}.$$

Now it is easy to determine the displacement. From $(9.29)_2$ we find

$$\tilde{u}_{z,1} = r\left\{\int_{\theta_0}^{\theta}\tilde{\beta}R_1 d\theta+\frac{\tilde{a}_3}{a_5}\right\}. \qquad (9.34)$$

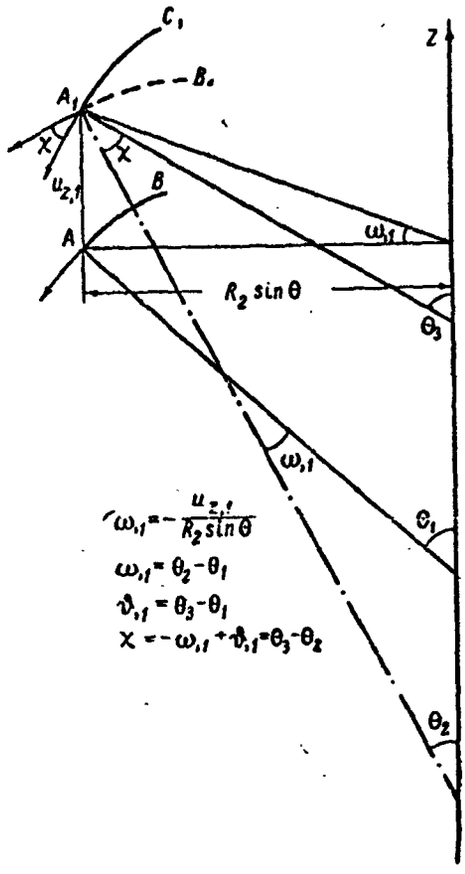

Figure 20

Further, equality $(9.29)_1$ gives us

$$\tilde{\vartheta}_1 = \tilde{\chi} - \frac{\tilde{u}_{z,1}}{r}. \tag{9.35}$$

From the identity

$$\frac{du_{\rho,1}}{d\theta} = R_1\left(\cos\theta\,\tilde{\varepsilon}_{1,1} - \sin\theta\,\tilde{\vartheta}_{,1}\right),$$

which, in accordance with $(9.30)_1$ and (9.35), may be written in the form

$$\frac{d\tilde{u}_{\rho,1}}{d\theta} = R_1\left(-\frac{\tilde{\chi}}{\sin\theta} - R_2\cos\theta\,\tilde{\beta} + \frac{\tilde{u}_{z,1}}{R_2}\right),$$

we obtain

$$\widetilde{u}_{\rho,1} = \frac{\widetilde{a}_4}{a_6} + \int_{\theta_0}^{\theta} \left(-\frac{\widetilde{\chi}}{\sin\theta} - R_2 \cos\theta \widetilde{\beta} + \frac{\widetilde{u}_{z,1}}{R_2} \right) R_1 d\theta. \tag{9.36}$$

Finally, from $(9.29)_3$ we obtain

$$\pm \widetilde{v}_1 = R_2 \sin\theta \widetilde{\gamma} - \widetilde{u}_{\rho,1}. \tag{9.37}$$

To find the displacements it is necessary to separate the real parts in the resulting expressions. Let us clarify the geometric meaning of the constants /137

$$a_3 = \operatorname{Re}\widetilde{a}_3, \quad a_4 = \operatorname{Re}\widetilde{a}_4, \quad a_5 = \operatorname{Re}\widetilde{a}_5 \quad a_6 = \operatorname{Re}\widetilde{a}_6.$$

According to expressions (9.36) and (9.37) the constants a_4 and a_5 pertain to the displacements

$$u_\rho = \begin{cases} a_4 \cos\varphi \\ a_6 \sin\varphi \end{cases}, \quad v = \begin{cases} -a_4 \sin\varphi \\ a_6 \cos\varphi \end{cases}.$$

Using these to construct the projections of the vector of displacements along directions ox and oy (fig. 19)

$$u_x = u_\rho \cos\varphi - v \sin\varphi, \quad u_y = u_\rho \sin\varphi + v \cos\varphi,$$

we obtain for these the following expressions

$$u_x = a_4, \quad u_y = a_6. \tag{9.38}$$

These displacements do not depend on the position of the point on the middle surface and consequently are linear displacements of the shell considered as a rigid whole. In exactly the same manner we can show that

$$a_3 = -\omega_y^0, \quad a_5 = \omega_x^0, \tag{9.39}$$

where ω_x^0 and ω_y^0 are the angles of rigid shell rotation around the ox and oy axes, respectively.

We now consider the conditions for the complex joining of two shells (of equal thickness and with the same elastic constants), which are joined along a common parallel ring. Above we have shown that

$$\widetilde{M}_{1,1} + r\widetilde{Q}_{z,1} = M_{1,1} + rQ_{z,1} = M^*_{1,1} + rQ^*_{z,1}, \tag{9.16}$$

$$\widetilde{Q}_{\rho,1} \mp \widetilde{Q}_{12,1} = Q_{\rho,1} \mp Q_{12,1} = Q^*_{\rho,1} \mp Q^*_{12,1}. \tag{9.19}$$

From these relationships and also from equalities (9.22) it follows that the static system of functions must be selected in such a way that the shear force which corresponds to it and the tumbling moment coincide with those for the sought solution. If this requirement is satisfied, two conditions at the joints will be satisfied automatically

$$\begin{aligned} \{M^*_{1,1} + rQ^*_{z,1}\}^{\mathrm{I}} &= \{\ldots\}^{\mathrm{II}}, \\ \{Q^*_{\rho,1} \mp Q^*_{12,1}\}^{\mathrm{I}} &= \{\ldots\}^{\mathrm{II}}. \end{aligned} \tag{9.40}$$

In addition, relationships (9.16 and (9.19) show that of the four complex quantities (9.10) two, for example $\widetilde{Q}_{\rho,1}$ and $\widetilde{M}_{1,1}$ are substantially different. Thus, to determine the complex constants, as in the symmetric case, we have two complex equalities /138

$$\{\widetilde{Q}_{\rho,1}\}^{\mathrm{I}} = \{\widetilde{Q}_{\rho,1}\}^{\mathrm{II}}, \quad \{\widetilde{M}_{1,1}\}^{\mathrm{I}} = \{\widetilde{M}_{1,1}\}^{\mathrm{II}}. \tag{9.41}$$

By using equality (9.12) these relationships can be written in the form

$$\begin{aligned} \left\{\cos\theta\,\widetilde{T}_{1,1} + ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}_{.1}}{d\theta}\right\}^{\mathrm{I}} &= \{\ldots\}^{\mathrm{II}}, \\ \{\widetilde{T}_{.1} - (1+\nu)\widetilde{T}_{1,1}\}^{\mathrm{I}} &= \{\ldots\}^{\mathrm{II}} \\ \left(\{\widetilde{T}_{2,1} - \nu\widetilde{T}_{1,1}\}^{\mathrm{I}} = \{\ldots\}^{\mathrm{II}}\right). \end{aligned} \tag{9.42}$$

Taking into account equality (9.27), we can impart a more convenient form to these relationships

$$\left\{ic\left(\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta}-\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}_{,1}\right)+\frac{\operatorname{ctg}\theta}{R_2\sin\theta}W\right\}^{I}=\{\ldots\}^{II},$$
$$\left\{\widetilde{T}_{,1}-(1+\nu)\left(\frac{W}{R_2\sin^2\theta}+ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right)\right\}^{I}=\{\ldots\}^{II}. \tag{9.43}$$

The continuity of the combinations which have been written out is equivalent (for shells of equal thickness and with the same elastic constants) to the requirement of continuity (see (9.10)) along the line which joins the four static quantities: Q_ρ, 1, Q_z, 1, Q_{12}, 1, $M_{1,1}$ and two geometric quantities:

the elastic rotation χ and the relative elongation γ of the parallel ring.

In the case when the joining of the shells is smooth ($\theta_0^{I}=\theta_0^{II}=\theta_0$),

the conditions at the joints (9.42)-(9.43) are substantially simplified and take the form

$$\{\widetilde{T}_{,1}\}^{I}=\{\widetilde{T}_{,1}\}^{II},\quad\left\{\frac{d\widetilde{T}_{,1}}{ds_1}\right\}^{I}=\left\{\frac{d\widetilde{T}_{,1}}{ds_1}\right\}^{II}\quad(ds_1=R_1d\theta). \tag{9.44}$$

10. Solution of the Basic Equation. Compliance Coefficients

Let us return to the examination of the basic characteristic equation (9.25)

$$\frac{d^2\widetilde{T}_{,1}}{d\theta^2}+\left[\left(2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{d\widetilde{T}_{,1}}{d\theta}+\frac{R_1}{R_2}\left(1-\frac{R_1}{R_2}\right)\frac{1}{\sin^2\theta}\widetilde{T}_{,1}+$$
$$+i\frac{R_1^2}{cR_2}\widetilde{T}_{,1}=i\frac{R_1^2}{cR_2}F(\theta), \tag{10.1}$$

where

$$F(\theta)=R_2q_{n,1}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{W}{\sin^2\theta}. \tag{10.2}$$

As in the symmetric case when the radii of curvature and the surface loading are smooth, the particular solution may be taken in the form /139

$$\widetilde{T}_{,1}^{part}=F(\theta). \tag{10.3}$$

In this case its error may be estimated by considering the value of the ratio

$$\left|\frac{\frac{d^2F}{d\theta^2}+\left[\left(2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{dF}{d\theta}+\frac{R_1}{R_2}\left(1-2\frac{R_1}{R_2}\right)\frac{1}{\sin^2\theta}F}{\frac{R_1^2}{cR_2}F}\right|. \tag{10.4}$$

We adopt the solution given by the membrane theory as the static system. Setting c = 0 in equation $(9.1)_2$ (which, as we have already stated, implies a transition to the membrane theory), we obtain

$$T^*_{,1}=R_2q_{n,1}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}U^*_{,1}. \tag{10.5}$$

If we take into acccunt the fact that according to (9.15)

$$U^*_{,1}=W^*=W \tag{10.6}$$

and compare the right side of relationship (10.5) with expression (10.2), we arrive at the following equality

$$F(\theta)=T^*_{,1}. \tag{10.7}$$

This equality means that the particular solution is taken from the membrane theory. It follows from $(9.27)_1$ that

$$T^*_{1,1}=\frac{W}{R_2\sin^2\theta}. \tag{10.8}$$

Now the general solution of equation (10.1) may be represented in the form

$$\tilde{T}_{,1}=T^*_{,1}+\tilde{T}^0_{,1}, \tag{10.9}$$

where $\tilde{T}^0_{,1}$ is the general solution of the corresponding homogeneous equation

$$\frac{d^2\widetilde{T}^0_{,1}}{d\theta^2}+\left[\left(2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{d\widetilde{T}^0_{,1}}{d\theta}+$$
$$+\left[\frac{R_1}{R_2}\left(1-2\frac{R_1}{R_2}\right)\frac{1}{\sin^2\theta}+\frac{iR_1^2}{cR_2}\right]\widetilde{T}^0_{,1}=0. \tag{10.10}$$

Let us substitute expression (10.9) into relationship (9.27). By using equality (10.8) and remembering that the functions of the static system are smooth, we obtain, after conventional simplifications, the following expressions for the complex forces /140

$$\begin{aligned}
\widetilde{T}_{1,1}&=T^*_{1,1}+ic\left\{\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0_{,1}}{d\theta}-\frac{1}{R_2\sin^2\theta}\widetilde{T}^0_{,1}\right\},\\
\widetilde{T}_{2,1}&=T^*_{2,1}+\widetilde{T}^0_{,1}-ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0_{,1}}{d\theta},\\
\pm\widetilde{S}_{,1}&=\pm S^*_{,1}+ic\left\{\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}^0_{,1}}{d\theta}-\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}^0_{,1}\right\}\\
&\left(T^*_{2,1}=R_2q_{n,1}-\frac{W}{R_1\sin^2\theta},\ T^*_{1,1}=\frac{W}{R_2\sin^2\theta},\right.\\
\pm S^*_{,1}&=\left.-\frac{1}{R_1\sin\theta}\frac{dW}{d\theta}+R_2(q_{n,1}\cos\theta-q_{1,1}\sin\theta)\right).
\end{aligned} \tag{10.11}$$

Furthermore, from the relationships (9.32), $(9.31)_2$ and (9.33) we find (see (10.8))

$$\begin{aligned}
\widetilde{\chi}&=\frac{1}{Eh}\left\{-\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\left(-\frac{\widetilde{T}^0_{,1}}{R_2\sin\theta}\right)-\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{T^*_{,1}}{R_2\sin\theta}\right)+\right.\\
&\left.+(1+\nu)\left[-\frac{1}{R_1R_2\sin^3\theta}\frac{dR_2\sin\theta W}{d\theta}+R_2(q_{n,1}\operatorname{ctg}\theta-2q_{1,1})\right]\right\},\\
\widetilde{\gamma}&=\frac{1}{Eh}\left\{T^*_{2,1}-\nu T^*_1+\widetilde{T}^0_{,1}-i(1+\nu)c\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0_{,1}}{d\theta}\right\},\\
\widetilde{\beta}&=\frac{1}{Eh}\left\{\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}^0_{,1}}{d\theta}+\left(\frac{\nu}{R_2}-\frac{\cos^2\theta}{R_2\sin^2\theta}\right)\widetilde{T}^0_{,1}-\right.\\
&-(1+\nu)\left[-\frac{\cos\theta}{R_1R_2^2\sin^4\theta}\frac{dR_2\sin\theta W}{d\theta}+\frac{W}{R_1R_2\sin^2\theta}+\right.\\
&\left.\left.+q_{n,1}(\operatorname{ctg}^2\theta-1)-2q_{1,1}\operatorname{ctg}\theta\right]+\frac{\operatorname{ctg}\theta}{R_1}\frac{dT^*_{,1}}{d\theta}-\frac{T^*_{,1}}{R_2\sin^2\theta}\right\}.
\end{aligned} \tag{10.12}$$

Comparing the homogeneous equation (10.10) with the corresponding equation for the symmetric case (7.2), we note that they differ by terms which are not taken into account during asymptotic integration. Therefore, from analogy with (7.21) we can immediately write the expression for the solution of the homogeneous equation

$$\widetilde{T}^0_{\cdot,1} = \frac{1}{R_2^{3/4}\sin^{1/2}\theta}\left\{\widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}\right\}. \tag{10.13}$$

In practice, as in the symmetric case, a cruder approximation

$$\widetilde{T}^0_{\cdot,1} = \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}, \qquad \beta_1 = \int_{\theta_1}^{\theta} \frac{R_1 d\theta}{\sqrt{2cR_2}}, \quad \beta_2 = \int_{\theta_1}^{\theta} \frac{R_1 d\theta}{\sqrt{2cR_2}}, \tag{10.14}$$

is used, which is based on the neglect of terms of order $\sqrt{\frac{c}{R_0}}$ compared with unity. Retaining this accuracy successively, we obtain the following expressions for the quantities of interest to us /141

$$\widetilde{T}_{1,1} = T^*_{1,1}, \quad \widetilde{S}_{\cdot,1} = S^*_{\cdot,1} \tag{10.15}$$

$$\widetilde{T}_{2,1} = T^*_{2,1} + \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2};$$

$$\widetilde{\chi} = \chi^* - \frac{R_2}{EhR_1}\frac{d\widetilde{T}^0_{\cdot,1}}{d\theta},$$

$$\widetilde{\gamma} = \gamma^* + \frac{1}{Eh}\widetilde{T}^0_{\cdot,1},$$

$$\widetilde{\beta} = \beta^* + \frac{\operatorname{ctg}\theta}{EhR_1}\frac{\partial\widetilde{T}^0_{\cdot,1}}{d\theta}, \tag{10.16}$$

where

$$Eh\chi^* = -\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{T^*_{\cdot,1}}{R_2\sin\theta}\right) + (1+\nu)\left[-\frac{1}{R_1R_2\sin^3\theta}\frac{dR_2\sin\theta W}{d\theta} + R_2(q_{n,1}\operatorname{ctg}\theta - 2q_{1,1})\right],$$

$$Eh\gamma^* = T^*_{2,1} - \nu T_{1,1} = Ehe^*_{2,1},$$

$$Eh\beta^* = \frac{\operatorname{ctg}\theta}{R_1}\frac{dT^*_{\cdot,1}}{d\theta} - \frac{T^*_{\cdot,1}}{R_2\sin^2\theta} - (1+\nu)\left[-\frac{\cos\theta}{R_1R_2^2\sin^4\theta}\frac{dR_2\sin\theta W}{d\theta} + \frac{W}{R_1R_2\sin^2\theta} + q_{n,1}(\operatorname{ctg}^2\theta - 1) - 2q_{1,1}\operatorname{ctg}\theta\right]. \tag{10.17}$$

By means of the last equalities we obtain the following relationships, from (9.34)-(9.37)

$$\widetilde{u}_{z,1}=u^*_{z,1}+\frac{R_2\sin\theta}{Eh}\left(\int_{\theta_0}^{\theta}\operatorname{ctg}\theta\,\frac{d\widetilde{T}^0_{,1}}{d\theta}\,d\theta\right)=u^*_{z,1}+\frac{R_2\cos\theta}{Eh}\,\widetilde{T}^0_{,1},$$
$$\widetilde{\vartheta}_{,1}=\vartheta^*_{,1}-\frac{R_2}{EhR_1}\frac{d\widetilde{T}^0_{,1}}{d\theta},\quad \widetilde{u}_{\rho,1}=u^*_{\rho,1}+\frac{R_2\sin\theta}{Eh}\,\widetilde{T}^0_{,1},\tag{10.18}$$
$$\widetilde{v}_{,1}=R_2\sin\theta\,\widetilde{\gamma}-\widetilde{u}_{\rho,1}=R_2\sin\theta\,\gamma^*-u^*_{\rho,1},$$

where

$$u^*_{z,1}=\frac{R_2\sin\theta}{Eh}\int_{\theta_0}^{\theta}\left\{\frac{\operatorname{ctg}\theta}{R_1}\frac{dT^*_{,1}}{d\theta}-\frac{T^*_{,1}}{R_2\sin^2\theta}-\right.$$
$$-(1+\nu)\left[-\frac{\cos\theta}{R_1R_2^2\sin^4\theta}\frac{dR_2\sin\theta\,W}{d\theta}+\frac{W}{R_1R_2\sin^2\theta}+\right.$$
$$\left.\left.+q_{n,1}(\operatorname{ctg}^2\theta-1)-2q_{1,1}\operatorname{ctg}\theta\right]\right\}R_1d\theta+\frac{a_3}{a_6}R_2\sin\theta,$$
$$\vartheta^*_{,1}=\chi^*-\frac{u^*_{z,1}}{R_2\sin\theta},$$
$$u^*_{\rho,1}=\frac{a_4}{a_6}+\int_{\theta_0}^{\theta}\left\{-\frac{\chi^*}{\sin\theta}-R_2\cos\theta\,\beta^*+\frac{u^*_{z,1}}{R_2}\right\}R_1d\theta=\tag{10.19}$$
$$=\frac{a_4}{a_6}+\frac{1}{Eh}\int_{\theta_0}^{\theta}\left\{\frac{R_2\sin\theta}{R_1}\frac{dT^*_{,1}}{d\theta}+(1+\nu)\left[\frac{1}{R_1R_2\sin^2\theta}\frac{dR_2\sin\theta\,W}{d\theta}+\right.\right.$$
$$\left.\left.+\frac{\cos\theta}{R_1\sin^2\theta}W-2R_2(q_{n,1}\cos\theta-q_{1,1}\sin\theta)\right]\right\}R_1d\theta+$$
$$+\int_{\theta_0}^{\theta}\frac{u^*_{z,1}}{R_2}R_1d\theta.$$

Finally, from $(9.12)_3$ and (10.15) we find /142

$$\widetilde{Q}_{\rho,1}=Q^*_{\rho,1}+ic\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}^0_{,1}}{d\theta}\quad(Q^*_{\rho,1}=T^*_{1,1}\cos\theta).\tag{10.20}$$

Separating the real and imaginary parts in these equalities we arrive at the following expressions ($\widetilde{C}_1 = A_1 + iB_1$, $\widetilde{C}_2 = A_2 + iB_2$)

$$T_{2,1}=T^*_{2,1}+[(A_1\cos\beta_1-B_1\sin\beta_1)e^{-\beta_1}+(A_2\cos\beta_2+B_2\sin\beta_2)e^{\beta_2}],$$
$$M_{1,1}=-c[(B_1\cos\beta_1+A_1\sin\beta_1)e^{-\beta_1}+(B_2\cos\beta_2-A_2\sin\beta_2)e^{\beta_2}], \qquad (10.21)$$
$$M_{2,1}=\nu M_{1,1};\quad T_{1,1}=T^*_{1,1},\quad S_{,1}=S^*_{,1};$$

$$\chi=\chi^*+\frac{1}{Eh}\sqrt{\frac{R_2}{2c}}\{[(A_1+B_1)\cos\beta_1+(A_1-B_1)\sin\beta_1]e^{-\beta_1}+$$
$$+[-(A_2+B_2)\cos\beta_2+(A_2-B_2)\sin\beta_2]e^{\beta_2}\},$$
$$\gamma=\gamma^*+\frac{1}{Eh}[(A_1\cos\beta_1-B_1\sin\beta_1)e^{-\beta_1}+(A_2\cos\beta_2+B_2\sin\beta_2)e^{\beta_2}], \qquad (10.22)$$

$$Q_{\rho,1}=Q^*_{\rho,1}+\sqrt{\frac{c}{2R_2}}\frac{1}{\sin\theta}\{[(-A_1+B_1)\cos\beta_1+$$
$$+(A_1+B_1)\sin\beta_1]e^{-\beta_1}+[(A_2-B_2)\cos\beta_2+$$
$$+(A_2+B_2)\sin\beta_2]e^{\beta_2}\};$$
$$u_{z,1}=u^*_{z,1}+\frac{R_2\cos\theta}{Eh}\{(A_1\cos\beta_1-B_1\sin\beta_1)e^{-\beta_1}+$$
$$+(A_2\cos\beta_2+B_2\sin\beta_2)e^{\beta_2}\},$$
$$u_{\rho,1}=u^*_{\rho,1}+\frac{R_2\sin\theta}{Eh}\{(A_1\cos\beta_1-B_1\sin\beta_1)e^{-\beta_1}+ \qquad (10.23)$$
$$+(A_2\cos\beta_2+B_2\sin\beta_2)e^{\beta_2}\},$$
$$\vartheta_{,1}=\vartheta^*_{,1}+\frac{1}{Eh}\sqrt{\frac{R_2}{2c}}\{[(A_1+B_1)\cos\beta_1+(A_1-B_1)\sin\beta_1]e^{-\beta_1}+$$
$$+[-(A_2+B_2)\cos\beta_2+(A_2-B_2)\sin\beta_2]e^{\beta_2}\},$$
$$\pm v_{,1}=R_2\sin\theta\,\gamma^*-u^*_{\rho,1}.$$

By considering the expressions for $M_{1,1}$ $Q_{\rho,1}$, χ, $r\gamma$ we see that they are, respectively, like the symmetric quantities M_1, Q_ρ, ϑ, u_ρ. Therefore, without repeating the steps which have already been covered, we make use of the results presented in section 8. Thus the following relationships take place at the edge of the shell /143

$$u_0=u^*_0+\alpha_{11}(Q_0-Q^*_0)+\alpha_{12}M_0,$$
$$\vartheta_0=\vartheta^*_0+\alpha_{12}(Q_0-Q^*_0)+\alpha_{22}M_0, \qquad (10.24)$$

where, as in the symmetric case

$$\alpha_{11}=\mu 2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)^{3/2}\sqrt{\sin\theta_0}\,\frac{1}{E},$$
$$\alpha_{12}=\alpha_{21}=-2\sqrt{3(1-\nu^2)}\left(\frac{r_0}{h}\right)\frac{1}{Eh}, \qquad (10.25)$$
$$\alpha_{22}=\mu 4[3(1-\nu)^2]^{3/4}\sqrt{\frac{r_0}{h}}\frac{1}{\sqrt{\sin\theta_0}}\frac{1}{Eh^2},$$

but

$$u_0 = (r\gamma)_{\theta_0}, \quad \vartheta_0 = \chi(\theta_0),$$
$$M_0 = M_{1,1}(\theta_0), \quad Q_0 = Q_{p,1}(\theta_0), \quad Q_0^* = Q_{p,1}^*(\theta_0) = (T_{1,1}^* \cos\theta)_{\theta_0},$$
$$u_0^* = (r_1)_{\theta_0} = \frac{1}{Eh}\left\{r\left(T_{2,1}^* - \nu T_{1,1}^*\right)\right\}_{\theta_0},$$
$$\vartheta_0^* = \chi^*(\theta_0) = \frac{1}{Eh}\left\{-\frac{R_2^2 \sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{T_{1,1}^*}{R_2 \sin\theta}\right) + \right. \tag{10.26}$$
$$\left. + (1+\nu)\left[-\frac{1}{R_1 R_2 \sin^2\theta}\frac{dR_2 \sin\theta\, W}{d\theta} + R_2\left(q_{n,1}\operatorname{ctg}\theta - 2q_{1,1}\right)\right]\right\}.$$

Moreover, if θ_0 is understood to be θ_1, then $\mu = -1$; $\mu = +1$ when $\theta = \theta_2$. /144

As in the symmetric case, it is easy to realize that the quantity ϑ_0^* may be dropped in relationships (10.24).

Relationships (10.24) play the same role as equalities (8.15) in the symmetric case. The substantial difference is that they do not contain the total rotation of the tangent towards the meridian and the horizontal displacement, but rather quantities which do not contain rigid displacements: elastic rotation and the relative elongation of the edge. If we remember that in the symmetric case the quantities u_ρ and ϑ coincide with the strain components ϵ_2 and $\varkappa_2$ with an accuracy determined by the multiplier, it becomes clear that, in essence, relationships (8.15) and (10.24) are the deformation boundary conditions of a special form.

Assuming furthermore, that $\beta = \beta_1$ when considering the edge $\theta = \theta_1$ and that $\beta = -\beta$ for the edge $\theta = \theta_2$ and introducing the quantities

$$\sigma_{M_0} = \frac{6M_0}{h^2}, \quad \sigma_k = 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\sqrt{\sin\theta_0}\,\frac{Q_0 - Q_0^*}{h}, \tag{10.27}$$

which have the same meaning as in the symmetric case, we arrive at the following expressions for the stresses

$$\sigma_{M_{1,1}} = \sigma_{M_0}(\cos\beta + \sin\beta)e^{-\beta} - \mu\left(\frac{3}{\sqrt{3(1-\nu^2)}}\right)\sigma_k \sin\beta e^{-\beta},$$
$$\sigma_{T_{2,1}} = \sigma_{T_{2,1}^*} + \mu\sigma_k \cos\beta e^{-\beta} - \left(\frac{\sqrt{3(1-\nu^2)}}{3}\right)\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta}, \tag{10.28}$$
$$\sigma_{M_{2,1}} = \nu\sigma_{M_{1,1}}, \quad \sigma_{T_{1,1}} = \sigma_{T_{1,1}^*}, \quad \sigma_{S,1} = \sigma_{S_{,1}^*}.$$

On the other hand, the displacements and angle of rotation are determined by means of the equalities

$$u_{\rho,1}=u^{*}_{\rho,1}+$$
$$+\frac{R_2\sin\theta}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta-\sin\beta)e^{-\beta}+\mu\sigma_k\cos\beta e^{-\beta}\right\},$$
$$u_{z,1}=u^{*}_{z,1}+$$
$$+\frac{R_2\cos\theta}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta-\sin\beta)e^{-\beta}+\mu\sigma_k\cos\beta e^{-\beta}\right\}, \qquad (10.29)$$
$$\vartheta_{,1}=\vartheta^{*}_{,1}+\frac{\sqrt[4]{3(1-\nu^2)}}{E}\sqrt{\frac{R_2}{h}}\left\{\mu\frac{2\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}\cos\beta e^{-\beta}-\right.$$
$$\left.-\sigma_k(\cos\beta+\sin\beta)e^{-\beta}\right\},$$
$$\pm v_{,1}=\frac{R_2\sin\theta}{Eh}(T^{*}_{2,1}-\nu T^{*}_{1,1})-u^{*}_{1,1}.$$

11. Circular Cylindrical Shell

A circular cylindrical shell (closed with respect to φ) is a rather special case of the general cylindrical shells considered in the next chapter. At the same time it can be looked upon as a shell of revolution for which /145

$$\theta=\frac{\pi}{2},\quad R_2=r=r_0=\text{const},\quad \frac{1}{R_1}=0. \qquad (11.1)$$

This makes it possible for us to use the results of the preceding sections quite simply. In this connection we shall limit ourselves at this time to a short presentation of the method for separating variables. Other questions applicable to cylindrical shells of the general type will be considered in the following chapter. In the relationships of the preceding sections, with the exception of (11.1), we must set

$$R_1d\theta=ds_1=r_0d\xi. \qquad (11.2)$$

This introduces a new coordinate $\xi=\frac{s_1}{r_0}$ --the dimensionless distance along the generatrix (fig. 21). We assign a value $\xi=\xi_1$ to the upper edge of the shell and the value $\xi=\xi_2$ to the lower edge.

From equality (1.10) we obtain

$$u_\rho=w,\quad u_z=-u. \qquad (11.3)$$

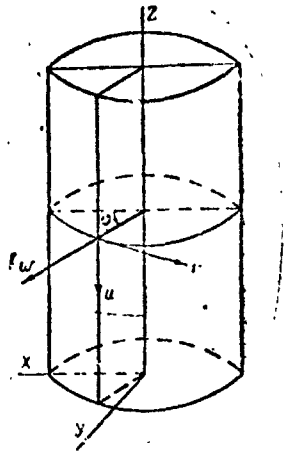

Figure 21

Retaining the variation in the unknown and assigned quantities as a function of φ in form (3.2), we arrive at the following basic system (3.3)

$$\begin{aligned} \Delta_k \widetilde{U}_{,k} + k^2 r_0 \left(1 + i\frac{c}{r_0}\right)\widetilde{T}_{,k} &= r_0^2\left(\pm kq_{2,k} - \frac{dq_{1,k}}{d\xi}\right), \\ -i\frac{c}{r_0}\Delta_k \widetilde{T}_{,k} + \widetilde{T}_{,k} - \frac{1}{r_0}\widetilde{U}_{,k} &= r_0 q_{n,k}. \end{aligned} \tag{11.4}$$

Here

$$\Delta_k = \frac{d^2}{d\xi^2} - k^2.$$

By using the functions $\widetilde{T}_{,k}$ and $\widetilde{U}_{,k}$, which were obtained from the basic system, we determine the complex forces by means of the following relationships (3.4)

$$\begin{aligned} \widetilde{T}_{1,k} &= \frac{1}{r_0}\widetilde{U}_{,k}, \quad \widetilde{T}_{2,k} = \widetilde{T}_{,k} - \frac{1}{r_0}\widetilde{U}_{,k}, \\ \pm k\widetilde{S}_{,k} &= -\frac{1}{r_0}\frac{d\widetilde{U}_{,k}}{d\xi} - r_0 q_{1,k}. \end{aligned} \tag{11.5}$$

For determining the complex displacements we have the following system of equations (3.6), which is approximately compatible /146

$$
\begin{aligned}
&(r_0\tilde{\varepsilon}_{1,k}) \quad \frac{d\tilde{u}_{,k}}{d\xi} = \frac{r_0}{Eh}\left\{\tilde{T}_{1,k} - \nu\tilde{T}_{2,k} + \frac{i}{c}M^*_{2,k}\right\},\\
&(r_0\tilde{\varepsilon}_{2,k}) \quad \pm k\tilde{v}_{,k} + \tilde{w}_{,k} = \frac{r_0}{Eh}\left\{\tilde{T}_{2,k} - \nu\tilde{T}_{1,k} + \frac{i}{c}M^*_{1,k}\right\},\\
&(r_0\tilde{\omega}_{,k}) \quad \frac{d\tilde{v}_{,k}}{d\xi} \mp k\tilde{u}_{,k} = \frac{r_0}{Eh}\left\{2(1+\nu)\tilde{S}_{,k} - \frac{i}{c}2H^*_{,k}\right\},\\
&(r_0^2\tilde{\varkappa}_{1,k}) \quad -\frac{d^2\tilde{w}_{,k}}{d\xi^2} = i\frac{r_0^2}{Ehc}\left\{\tilde{T}_{2,k} - T^*_{2,k}\right\},\\
&(r_0^2\tilde{\varkappa}_{2,k}) \quad k^2\tilde{w}_{,k} \pm k\tilde{v}_{,k} = i\frac{r_0^2}{Ehc}\left\{\tilde{T}_{1,k} - T^*_{1,k}\right\},\\
&(r_0^2\tilde{\tau}_{,k}) \quad \pm k\frac{d\tilde{w}_{,k}}{d\xi} + \frac{d\tilde{v}_{,k}}{d\xi} = -i\frac{r_0^2}{Ehc}\left\{\tilde{S}_{,k} - S^*_{,k}\right\}.
\end{aligned}
\tag{11.6}
$$

The boundary quantities (3.7) take the form

$$
\begin{aligned}
&\tilde{Q}_{\rho,k} = Q_{\rho,k} - iEhc\varkappa_{\rho,k} = Q^*_{\rho,k} - iEhc\tilde{\varkappa}_{\rho,k},\\
&\tilde{Q}_{z,k} = Q_{z,k} + iEhc\varkappa_{z,k} = Q^*_{z,k} + iEhc\tilde{\varkappa}_{z,k},\\
&\tilde{Q}_{12,k} = Q_{12,k} + iEhc\varkappa_{21,k} = Q^*_{12,k} + iEhc\tilde{\varkappa}_{21,k},\\
&\tilde{M}_{1,k} = M_{1,k} + iEhc\varepsilon_{2,k} = M^*_{1,k} + iEhc\tilde{\varepsilon}_{2,k},
\end{aligned}
\tag{11.7}
$$

where

$$
\begin{aligned}
&\varkappa_{\rho,k} = -\frac{1}{r_0^2}\left(\frac{dw_{,k}}{d\xi} + k^2u_{,k}\right),\\
&\varkappa_{21,k} = \pm\frac{k}{r_0^2}\left(\frac{dv_{,k}}{d\xi} + u_{,k}\right),\\
&\varkappa_{z,k} = -\frac{1}{r_0^2}(\pm kv_{,k} + k^2w_{,k}),\\
&\varepsilon_{2,k} = \frac{1}{r_0}(\pm kv_{,k} + w_{,k});
\end{aligned}
\tag{11.8}
$$

$$
\begin{aligned}
&Q_{12,k} = S_{,k} + \frac{2H_{,k}}{r_0},\\
&Q_{\rho,k} = Q_{1n,k} = \frac{1}{r_0}\left(\frac{dM_{1,k}}{d\xi} \mp kH_{,k}\right),\\
&Q_{z,k} = T_{1,k}.
\end{aligned}
\tag{11.9}
$$

Moreover, the following approximate expressions are valid for the complex boundary quantities

$$\widetilde{Q}_{\rho,k}=\widetilde{Q}_{1n,k}=i\frac{c}{r_0}\frac{d\widetilde{T}_{,k}}{d\xi},\quad \widetilde{Q}_{z,k}=\widetilde{T}_{1,k},\quad \widetilde{Q}_{12,k}=\widetilde{S}_{,k},$$
$$\widetilde{M}_{1,k}=ic\left(\widetilde{T}_{,k}-(1+\nu)\widetilde{T}_{1,k}\right). \tag{11.10}$$

If, for the static system, we adopt the membrane solution, then the auxiliary function $U^*_{,k}$ will be determined from equation (see (4.15)) /147

$$\frac{d^2U^*_{,k}}{d\xi^2}=r_0^2\left\{-\frac{dq_{1,k}}{d\xi}-k^2q_{n,k}\pm kq_{2,k}\right\}. \tag{11.11}$$

In this case

$$T^*_{1,k}=\frac{1}{r_0}U^*_{,k},\quad T^*_{2,k}=r_0q_{n,k},$$
$$\pm kS^*_{,k}=-\frac{1}{r_0}\frac{dU^*_{,k}}{d\xi}-r_0q_{1,k} \tag{11.12}$$

Thus, for a circular cylindrical shell, the determination of forces is reduced to the problem of finding two quadratures. This is true in regard to any shell of zero Gaussian curvature. Using the forces which have been established. the displacements are determined (again by means of two quadratures) from a system of equations

$$\frac{du^*_{,k}}{d\xi}=\frac{r_0}{Eh}\left(T^*_{1,k}-\nu T^*_{2,k}\right),$$
$$\pm kv^*_{,k}+w^*_{,k}=\frac{r_0}{Eh}\left(T^*_{2,k}-\nu T^*_{1,k}\right), \tag{11.13}$$
$$\frac{dv^*_{,k}}{d\xi}\mp ku^*_{,k}=\frac{2(1+\nu)r_0}{Eh}S^*_{,k}.$$

The Second Symmetric Case

In this case

$$S_{,0}=\frac{C_2}{r_0^2}-r_0\int_{\xi_1}^{\xi}q_{2,0}\,d\xi\quad\left(C_2=\frac{1}{2\pi}B_z(\xi_1)\right), \tag{11.14}$$

$$v_{,0} = r_0\left\{\frac{2(1+\nu)}{Eh}\int_{\xi_1}^{\xi_2} S_{,0}\,d\xi + a_2\right\} \tag{11.15}$$

(a_2 is the rotation of the shell as a rigid whole around the axis of rotation).

In this case the shear force and displacement coincide with those established in the membrane theory.

The First Symmetric Case

For the first symmetric case ("0" is dropped)

$$U = U^* = C_1 - r_0^2\int_{\xi_1}^{\xi_2} q_{1,1}\,d\xi \quad \left(C_1 = -\frac{1}{2\pi}\,\bar{r}_z(\xi_1)\right). \tag{11.16}$$

It is interesting to note that the asymptotic solution of a homogeneous equation, even in the first approximation, /148

$$\widetilde{T}^0 = \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}, \qquad \begin{aligned}\beta_1 &= \sqrt{\frac{r_0}{2c}}\,(\xi - \xi_1),\\ \beta_2 &= \sqrt{\frac{r_0}{2c}}\,(\xi - \xi_2),\end{aligned} \tag{11.17}$$

is exact. This is easy to verify by direct substitution.

Furthermore, when the surface load is smooth (all points of the shell are ordinary), it is legitimate to borrow the particular solution from the membrane theory. Therefore we shall immediately make use of relationships from section 8.

First of all we have

$$\widetilde{T}_1 = T_1^*, \quad \widetilde{T}_2 = T_2^* + \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2} \qquad \left(T_1^* = \frac{1}{r_0}\,U, \quad T_2^* = r_0 q_n\right). \tag{11.18}$$

Furthermore,

$$\begin{aligned} u_0 &= u_0^* + \alpha_{11}Q_0 + \alpha_{12}M_0,\\ \vartheta_0 &= \vartheta_0^* + \alpha_{21}Q_0 + \alpha_{22}M_0,\end{aligned} \tag{11.19}$$

where

$$\alpha_{11}=\mu 2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)^{3/2}\frac{1}{E},$$
$$\alpha_{12}=\alpha_{21}=-2\sqrt{3(1-\nu^2)}\left(\frac{r_0}{h}\right)\frac{1}{Eh}, \qquad (11.20)$$
$$\alpha_{22}=\mu 4\,[3(1-\nu^2)]^{3/4}\sqrt{\frac{r_0}{h}}\frac{1}{Eh^2}.$$

Here

$$u_0=w(\xi_1),\quad \vartheta_0=\vartheta(\xi_1),\quad Q_0=Q_{1n}(\xi_1),\quad M_0=M_1(\xi_1),$$
$$u_0^*=w^*(\xi_1),\quad \vartheta_0^*=\vartheta^*(\xi_1),\quad \mu=-1$$

when we consider the edge $\xi = \xi_1$ and

$$u_0=w(\xi_2),\quad \vartheta_0=\vartheta(\xi_2),\quad Q_0=Q_{1n}(\xi_2),\quad M_0=M_1(\xi_2),$$
$$u_0^*=w^*(\xi_2),\quad \vartheta_0^*=\vartheta^*(\xi_2),\quad \mu=+1$$

at the other edge.

We recall that within the framework of accuracy associated with the utilized solution, the quantity ϑ_0^* in equalities (11.19) and (11.29) may be dropped. The stresses in the shell are determined by relationships

$$\sigma_{M_1}=\sigma_{M_0}(\cos\beta+\sin\beta)e^{-\beta}-\mu\left(\frac{3}{\sqrt{3(1-\nu^2)}}\right)\sigma_k\sin\beta e^{-\beta},$$
$$\sigma_{T_2}=\sigma_{T_2}^*+\mu\sigma_k\cos\beta e^{-\beta}-\left(\frac{\sqrt{3(1-\nu^2)}}{3}\right)\sigma_{M_0}(\cos\beta-\sin\beta)e^{-\beta}, \qquad (11.21)$$
$$\sigma_{M_2}=\nu\sigma_{M_1},\quad \sigma_{T_1}=\sigma_{T_1}^*.$$

In these expressions

$$\sigma_{M_0}=\frac{6M_0}{h_2},\quad \sigma_k=2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\frac{Q_0}{h},$$
$$T_1^*=\frac{U}{r_0},\quad T_2^*=r_0q_n, \qquad (11.22)$$
$$\varepsilon^*=-\frac{1}{Eh}\frac{dT^*}{d\xi},$$

$$w^* = \frac{r_0}{Eh}\left(T_2^* - \nu T_1^*\right),$$

$$u^* = a_1 + \frac{r_0}{Eh}\int_{\xi_1}^{\xi}\left(T_1^* - \nu T_2^*\right)d\xi.$$

Finally, the displacements and the angles of rotation are equal to /149

$$\begin{aligned} w &= w^* + \frac{r_0}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \mu\sigma_k\cos\beta e^{-\beta}\right\}, \\ \vartheta &= \vartheta^* + \frac{\sqrt[4]{3(1-\nu^2)}}{E}\sqrt{\frac{r_0}{h}}\left\{\mu\frac{2\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}\cos\beta e^{-\beta} - \right. \\ &\qquad \left. - \sigma_k(\cos\beta + \sin\beta)e^{-\beta}\right\}, \\ u &= u^*. \end{aligned} \tag{11.23}$$

The conditions for the complex joining with a shell of rotation which has a general form, are written in the following form (in accordance with (6.17) and (6.18))

$$\begin{aligned} \left\{\widetilde{T} - (1+\nu)\frac{U}{r_0}\right\}^{\text{cyl}} &= \left\{\widetilde{T} - (1+\nu)\widetilde{T}_1\right\}^{\text{II}}, \\ \left\{i\frac{c}{r_0}\frac{d\widetilde{T}}{d\xi}\right\}^{\text{cyl}} &= \left\{\cos\theta\widetilde{T}_1 + ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}}{d\theta}\right\}^{\text{II}}. \end{aligned} \tag{11.24}$$

In the case when junction takes place without a break in the middle surface, the conditions are substantially simplified

$$\left\{\widetilde{T}\right\}^{\text{cyl}} = \left\{\widetilde{T}\right\}^{\text{II}}, \quad \left\{\frac{1}{r_0}\frac{d\widetilde{T}}{d\xi}\right\}^{\text{cyl}} = \left\{\frac{1}{R_1}\frac{d\widetilde{T}}{d\theta}\right\}^{\text{II}} \tag{11.25}$$

The Antisymmetric Case

In antisymmetric cases (9.8)

$$W = \frac{1}{r_0}\left\{\frac{C_4}{C_6} + \frac{C_3}{C_5}r_0(\xi - \xi_0) + r_0\int_{\xi_0}^{\xi}\Phi(\xi)\,d\xi\right\}, \tag{11.26}$$

where

$$\Phi(\xi)=r_0^2\left[-q_{1,1}-\int_{\xi_0}^{\xi}(q_{n,1}-q_{2,1})\,d\xi\right];$$
$$\begin{matrix}C_3\\C_5\end{matrix}=\left|r_0\left(Q_{1n,1}\mp Q_{12,1}\right)\right|_{\xi=\xi_0}=\frac{1}{\pi}\begin{matrix}F_x(\xi_0)\\F_y(\xi_0)\end{matrix};\qquad(11.27)$$
$$\begin{matrix}C_4\\C_6\end{matrix}=\left|r_0(M_{1,1}+r_0T_{1,1})\right|_{\xi=\xi_0}=\frac{1}{\pi}\begin{matrix}B_y(\xi_0)\\-B_x(\xi_0)\end{matrix}.$$

Adopting again the asymptotic solution in its simplified form, and borrowing the particular solution from the membrane theory, we arrive at the following relationships /150

$$\widetilde{T}_{2,1}=T^*_{2,1}+\widetilde{C}_1e^{-(1-i)\beta_1}+\widetilde{C}_2e^{(1-i)\beta_2},\qquad\beta_1=\sqrt{\frac{r_0}{2c}}(\xi-\xi_1),\qquad\beta_2=\sqrt{\frac{r_0}{2c}}(\xi-\xi_2),\qquad(11.28)$$

$$\widetilde{T}_{1,1}=T^*_{1,1},\quad\widetilde{S}_{,1}=S^*_{,1};$$
$$u_0=u_0^*+\alpha_{11}Q_0+\alpha_{12}M_0,$$
$$\vartheta_0=\vartheta_0^*+\alpha_{21}Q_0+\alpha_{22}M_0,\qquad(11.29)$$

where

$$u_0=r_0\varepsilon_{2,1}(\xi_0)=r_0\gamma(\xi_0),\quad\vartheta_0=\left(\vartheta_{,1}+\frac{w}{r_0}\right)_{\xi=\xi_0}=\chi(\xi_0);$$
$$Q_0=Q_{1n,1}(\xi_0),\quad M_0=M_{1,1}(\xi_0);\qquad(11.30)$$
$$u_0^*=\frac{r_0}{Eh}\left(T^*_{2,1}-\nu T^*_{1,1}\right)_{\xi=\xi_0}=w^*_{,1}(\xi_0),\quad\vartheta_0^*\approx0.$$

Here the interpretation of ξ_0 depends on the edge which is being considered, either ξ_1 ($\mu=-1$) or ξ_2 ($\mu=+1$).

Furthermore, according to (10.19), $(10.18)_4$ and $(10.17)_1$,

$$u^*_{,1}=-u^*_{z,1}=\frac{r_0}{Eh}\int_{\xi_1}^{\xi}\left\{T^*_{,1}-(1+\nu)r_0q_{n,1}\right\}d\xi+r_0\frac{a_3}{a_5},\qquad(11.31)$$
$$w^*_{,1}=u^*_{\rho,1}=\frac{r_0}{Eh}\int_{\xi_0}^{\xi}\left\{\frac{dT^*_{,1}}{d\xi}+(1+\nu)\left[\frac{1}{r_0}\frac{dW}{d\xi}+2r_0q_{1,1}\right]\right\}d\xi-$$

$$-\int_{\xi_1}^{\xi} u^*_{,1}d\xi + a_4;$$

$$\vartheta^*_{,1} = -\frac{1}{Eh}\left\{\frac{dT^*_{,1}}{d\xi} + (1+\nu)\left(\frac{1}{r_0}\frac{dW}{d\xi} - 2r_0 q_{1,1}\right)\right\} + \frac{u^*_{,1}}{r_0},$$

$$\pm v^*_{,1} = \frac{r_0}{Eh}\left(T^*_{2,1} - \nu T^*_{1,1}\right) - \omega^*_{,1}.$$

Finally, the displacements and the angle of rotation (10.23) are given by the relationships /151

$$\begin{aligned}
w_{,1} &= w^*_{,1} + \frac{r_0}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \right.\\
&\qquad \left. + \mu\sigma_k \cos\beta e^{-\beta}\right\},\\
\vartheta_{,1} &= \vartheta^*_{,1} + \frac{\sqrt[4]{3(1-\nu^2)}}{E}\sqrt{\frac{r_0}{h}}\left\{\mu\frac{2\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}\cos\beta e^{-\beta} - \right.\\
&\qquad \left. - \sigma_k(\cos\beta + \sin\beta)e^{-\beta}\right\},\\
u_{,1} &= u^*_{,1}, \quad v_{,1} = v^*_{,1}.
\end{aligned} \tag{11.32}$$

The membrane forces which enter into these equalities are determined by means of the following relationships (9.27)

$$T^*_{1,1} = \frac{1}{r_0}W, \quad T^*_{2,1} = r_0 q_{n,1}, \quad S^*_{,1} = -\frac{1}{r_0}\frac{dW}{d\xi} - r_0 q_{1,1}. \tag{11.33}$$

The stresses are determined from the following equations (10.28)

$$\begin{aligned}
\sigma_{M_{1,1}} &= \sigma_{M_0}(\cos\beta + \sin\beta)e^{-\beta} - \mu\left(\frac{3}{\sqrt{3(1-\nu^2)}}\right)\sigma_k \sin\beta e^{-\beta},\\
\sigma_{T_{2,1}} &= \sigma_{T^*_{2,1}} + \mu\sigma_k\cos\beta e^{-\beta} - \left(\frac{\sqrt{3(1-\nu^2)}}{3}\right)\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta},\\
\sigma_{T_{1,1}} &= \sigma_{T^*_{1,1}}, \quad \sigma_{S,1} = \sigma_{S^*_{,1}}, \quad \sigma_{M_{2,1}} = \nu\sigma_{M_{1,1}}, \quad \sigma_{H,1} = 0,
\end{aligned} \tag{11.34}$$

$$\sigma_{M_0} = \frac{6M_0}{h^2}; \quad \sigma_k = 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\frac{Q_0}{h}.$$

Finally, the conditions at the joints when a cylindrical shell is joined with a shell of general form, are written in the form (9.42)

$$\left\{\widetilde{T}_{,1}-(1+\nu)\frac{W}{r_0}\right\}^{\text{cyl}}=\left\{\widetilde{T}_{,1}-(1+\nu)\widetilde{T}_{1,1}\right\}^{\text{II}},$$
$$\left\{i\frac{c}{r_0}\frac{d\widetilde{T}_{,1}}{d\xi}\right\}^{\text{cyl}}=\left\{\cos\theta\widetilde{T}_{1,1}+ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right\}^{\text{II}}. \tag{11.35}$$

In the case of smooth joining they acquire a simpler form

$$\{\widetilde{T}_{,1}\}^{\text{cyl}}=\{\widetilde{T}_{,1}\}^{\text{II}},\quad\left\{\frac{1}{r_0}\frac{d\widetilde{T}_{,1}}{d\xi}\right\}^{\text{cyl}}=\left\{\frac{1}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right\}^{\text{II}}. \tag{11.36}$$

12. Circular Conic Shell

A circular conic shell (fig. 22) is referred to coordinates

$$\alpha_1=s,\quad\alpha_2=\varphi, \tag{12.1}$$

and in this case

$$A_1=1,\quad A_2=r=\cos\theta_0 s,\quad R_2=\frac{r}{\sin\theta_0}=\operatorname{ctg}\theta_0 s. \tag{12.2}$$

For a truncated cone (fig. 23), the values of the coordinate s corresponding to the upper and the lower edges are determined by means of the following equalities /152

$$s=s_1=\frac{r_1}{\cos\theta_0},\quad s=s_2=\frac{r_2}{\cos\theta_0}, \tag{12.3}$$

while the length of the generatrix is equal to

$$s_2-s_1=\frac{r_2-r_1}{\cos\theta_0}. \tag{12.4}$$

We note that the quantity $\alpha=\frac{\pi}{2}-\theta_0$ is usually called the angle of conicity.

As in the preceding section, we shall present only the basic information required for practical applications, assuming that the reader, as the need arises, will be able to obtain the additional relationships by himself, by making the following substitutions in the equalities of sections 1-10

$$\theta=\theta_0,\quad R_2=\operatorname{ctg}\theta_0\, s,\quad \frac{1}{R_1}=0,\quad \frac{d}{R_1 d\theta}=\frac{d}{ds_1}=\frac{d}{ds}. \tag{12.5}$$

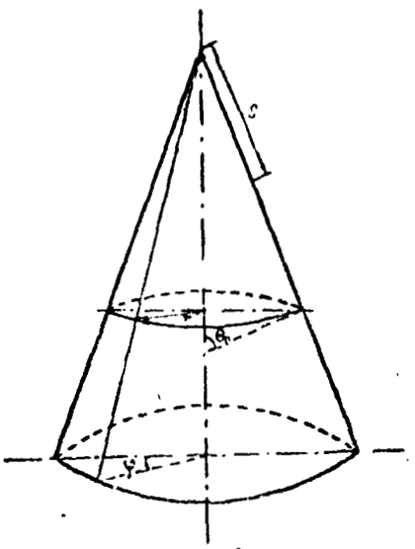

Figure 22

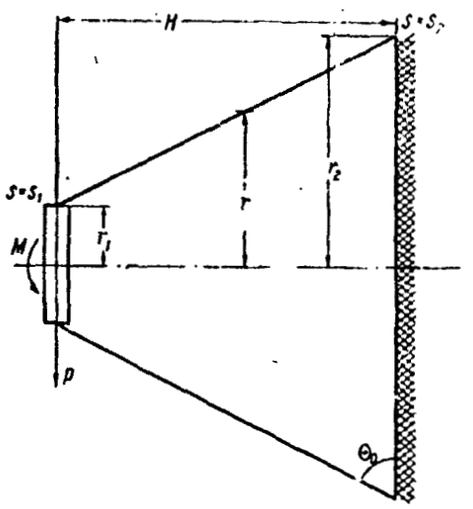

Figure 23

We note the following situation. As we can see from (12.3), $R_2 \underset{s\to 0}{\to} 0$ and the ratio h/R_2 become infinitely large as we approach the apex of the cone. Thus, near the apex the cone ceases to be thin and a three-dimensional state of strain takes place in it. Therefore, in the future we shall consider only truncated cones for which s is not too small. In addition, we shall assume that the angle of conicity α is substantially less than $\pi/2$, which eliminates the consideration of a shallow cone close to a plate. For the reader who finds these limitations unacceptable, we recommend references 129-148. /153

Below we shall consider the first symmetric case and the antisymmetric case.

The First Symmetric Case

For this case

$$U=U^*=C_1+\cos\theta_0\int_{s_0}^{s}(q_n\cos\theta_0-q_1\sin\theta_0)s'\,ds'. \tag{12.6}$$

The asymptotic solution in the first approximation (8.1) takes on the form

$$\widetilde{T}^0 = \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}, \qquad \begin{aligned} \beta_1 &= \sqrt{\frac{2\operatorname{tg}\theta_0}{c}}(\sqrt{s} - \sqrt{s_1}), \\ \beta_2 &= -\sqrt{\frac{2\operatorname{tg}\theta_0}{c}}(\sqrt{s_2} - \sqrt{s}). \end{aligned} \tag{12.7}$$

It is associated with the complex forces

$$\widetilde{T}_1 = T_1^*, \quad \widetilde{T}_2 = T_2^* + \widetilde{C}_1 e^{-(1-i)\beta_1} + \widetilde{C}_2 e^{(1-i)\beta_2}. \tag{12.8}$$

where

$$T_1^* = \frac{1}{\sin\theta_0 \cos\theta_0 s} U, \quad T_2^* = \operatorname{ctg}\theta_0 s q_n. \tag{12.9}$$

Furthermore

$$\begin{aligned} u_0 &= u_0^* + \alpha_{11}(Q_0 - Q_0^*) + \alpha_{12} M_0, \\ \vartheta_0 &= \vartheta_0^* + \alpha_{12}(Q_0 - Q_0^*) + \alpha_{22} M_0, \end{aligned} \tag{12.10}$$

where

$$\begin{aligned} \alpha_{11} &= \mu \cdot 2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)^{3/2}\sqrt{\sin\theta_0}\,\frac{1}{E}, \\ \alpha_{12} &= -2\sqrt{3(1-\nu^2)}\left(\frac{r_0}{h}\right)\frac{1}{Eh}, \\ \alpha_{22} &= \mu 4\,[3(1-\nu^2)]^{3/4}\sqrt{\frac{r_0}{h}}\,\frac{1}{\sqrt{\sin\theta_0}}\,\frac{1}{Eh^2}. \end{aligned} \tag{12.11}$$

Here

$$\begin{gathered} r_0 = r(s_1),\ u_0 = u_\rho(s_1),\ \vartheta_0 = \vartheta(s_1),\ u_0^* = u_\rho(s_1),\ \vartheta^* = \vartheta^*(s_1), \\ Q_0 = Q_\rho(s_1),\quad M_0 = M_1(s_1),\quad Q_0^* = Q_\rho^*(s_1),\quad \mu = -1,\ \beta = \beta_1 \end{gathered}$$

when we consider the edge $s = s_1$ and

$$r_0 = r(s_2),\quad u_0 = u_\rho(s_2),\ \ldots;\quad \mu = +1,\quad \beta = -\beta_2$$

at the other edge.

We recall that within the frame ork of accuracy associated with the solution which we are using the term ϑ^* may be dropped in equality (12.10). The stresses, the angle of rotation and displacement are determined by relationships

$$\begin{aligned}
&\sigma_{M_1} = \sigma_{M_0}(\cos\beta + \sin\beta)e^{-\beta} - \mu\left(\frac{3}{\sqrt{3(1-\nu^2)}}\right)\sigma_k \sin\beta\, e^{-\beta},\\
&\sigma_{T_1} = \sigma_{T_2}^* + \mu\sigma_k \cos\beta\, e^{-\beta} - \left(\frac{\sqrt{3(1-\nu^2)}}{3}\right)\sigma_{M_0}(\cos\beta - \sin\beta)\, e^{-\beta},\\
&\sigma_{M_2} = \nu\sigma_{M_1}, \quad \sigma_{T_2} = \sigma_{T_1}^*;
\end{aligned} \tag{12.12}$$

$$\begin{aligned}
\vartheta &= \vartheta^* + \frac{\sqrt[4]{3(1-\nu^2)}}{E}\sqrt{\frac{s\cos\theta_0}{h}}\left\{\mu\frac{2\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}\cos\beta e^{-\beta} - \right.\\
&\qquad \left. - \sigma_k(\cos\beta + \sin\beta)e^{-\beta}\right\},\\
u_\rho &= u_\rho^* + \frac{s\cos\theta_0}{E}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \right.\\
&\qquad \left. + \mu\sigma_k\cos\beta\, e^{-\beta}\right\},\\
u_z &= u_z^* + \frac{s\cos^2\theta_0}{E\sin\theta_0}\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \right.\\
&\qquad \left. + \mu\sigma_k\cos\beta\, e^{-\beta}\right\}.
\end{aligned} \tag{12.13}$$

In these expressions

$$\begin{aligned}
&\sigma_{M_0} = \frac{6M_0}{h^2} \qquad \sigma_k = 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\sqrt{\sin\theta_0}\,\frac{Q_0 - Q_0'}{h},\\
&u_\rho^* = \frac{\cos\theta_0 s}{Eh}(T_2^* - \nu T_1^*), \quad \vartheta^* = -\frac{\operatorname{ctg}\theta_0}{Eh}s\frac{dT^*}{ds},\\
&u_z^* = a_1 + \frac{1}{Eh}\int_{s_1}^{s}\left[\frac{\cos^2\theta_0 s}{\sin\theta_0}\frac{dT^*}{ds} - \sin\theta_0(T_1^* - \nu T_2^*)\right]ds.
\end{aligned} \tag{12.14}$$

Finally, the conditions for complex junction with a shell of rotation of the general type assume the following form

$$\begin{aligned}
&\left\{\widetilde{T} - \frac{1+\nu}{\sin\theta_0\cos\theta_0 s}U\right\}^{\mathrm{con}} = \left\{\widetilde{T} - (1+\nu)\widetilde{T}_1\right\}^{\mathrm{II}},\\
&\left\{\frac{1}{s\sin\theta_0}U + ic\sin\theta_0\frac{d\widetilde{T}}{ds}\right\}^{\mathrm{con}} = \left\{\cos\theta_0\widetilde{T}_1 + ic\frac{\sin\theta_0}{R_1}\frac{d\widetilde{T}}{d\theta}\right\}^{\mathrm{II}},
\end{aligned} \tag{12.15}$$

which is substantially simplified when there is smooth junction

$$\{\widetilde{T}\}^{\text{con}} = \{\widetilde{T}\}^{\text{II}}, \quad \left\{\frac{d\widetilde{T}}{ds}\right\}^{\text{con}} = \left\{\frac{1}{R_1}\frac{d\widetilde{T}}{d\theta}\right\}^{\text{II}}. \tag{12.16}$$

Antisymmetric Case

In these cases

$$W = W^* = \\ = \frac{1}{\cos\theta_0 s}\left\{\begin{matrix} C_4 \\ C_6 \end{matrix} + \begin{matrix} C_3 \\ C_5 \end{matrix}\sin\theta_0 (s - s_0) + \sin\theta_0 \int_{s_0}^{s} \Phi(s')\, ds'\right\}, \tag{12.17}$$

where

$$\Phi(s) = \left(q_{n,1}\cos\theta_0 - q_{1,1}\sin\theta_0\right)\frac{\cos^2\theta_0}{\sin\theta_0} s^2 - \\ - \cos\theta_0 \int_{s_0}^{s} \left(q_{n,1}\sin\theta_0 + q_{1,1}\cos\theta_0 - q_{2,1}\right) s'\, ds', \\ \begin{matrix} C_3 \\ C_5 \end{matrix} = \frac{1}{\pi}\begin{matrix} F_x(s_0) \\ F_y(s_0) \end{matrix}; \quad \begin{matrix} C_4 \\ C_6 \end{matrix} = \frac{1}{\pi}\begin{matrix} B_y(s_0) \\ -B_x(s_0) \end{matrix}. \tag{12.18}$$

As in the symmetric case we adopt the membrane solution for the particular solution. For the general solution of the nonhomogeneous equation we adopt the simplified asymptotic solution. We obtain /155

$$\widetilde{T}_{2,1} = T^*_{2,1} + \widetilde{C}_1 e^{-(1-i)\beta_1 s} + \widetilde{C}_2 e^{(1-i)\beta_2 s}, \quad \begin{matrix} \beta_1 = \sqrt{\frac{2\operatorname{tg}\theta_0}{c}}\left(\sqrt{s} - \sqrt{s_1}\right), \\ \beta_2 = -\sqrt{\frac{2\operatorname{tg}\theta_0}{c}}\left(\sqrt{s_2} - \sqrt{s}\right), \end{matrix} \tag{12.19}$$

$$\widetilde{T}_{1,1} = T^*_{1,1}, \quad \widetilde{S}_{,1} = S^*_{,1}.$$

Relationships (12.10) remain valid and in these we now have

$$u_0 = r_0 \varepsilon_{2,1}(s_0) = (r\gamma)_{s=s_0}, \quad \vartheta_0 = \left(\vartheta_{,1} + \frac{u_{z,1}}{r}\right)_{s=s_0} = \chi(s_0), \\ Q_0 = Q_{p,1}(s_0), \quad M_0 = M_{1,1}(s_0), \\ Q^*_0 = Q^*_{p,1}(s_0) = \left(T^*_{1,1}\cos\theta_0\right)_{s=s_0}, \\ u^*_0 = (r\gamma^*)_{s=s_0} = \frac{1}{Eh}\left\{r\left(T^*_{2,1} - \nu T^*_{1,1}\right)\right\}_{s=s_0}, \quad \vartheta^*_0 \approx 0. \tag{12.20}$$

In this case, at the edge $s_0 = s_1$ we should take $\mu = -1$; $\beta = \beta_1$; for $s_0 = s_2$ $\mu = +1$, $\beta = -\beta_2$.

Furthermore

$$\begin{aligned}
\sigma_{M_{1,1}} &= \sigma_{M_0}(\cos\beta + \sin\beta)e^{-\beta} - \mu\left(\frac{3}{\sqrt{3(1-\nu^2)}}\right)\sigma_k \sin\beta e^{-\beta},\\
\sigma_{T_{2,1}} &= \sigma_{T^*_{2,1}} + \mu\sigma_k \cos\beta e^{-\beta} - \left(\frac{\sqrt{3(1-\nu^2)}}{3}\right)\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta},\\
\sigma_{M_{2,1}} &= \nu\sigma_{M_{1,1}},\quad \sigma_{T_{1,1}} = \sigma_{T^*_{1,1}},\quad \sigma_{S_{,1}} = \sigma_{S^*_{,1}};
\end{aligned} \tag{12.21}$$

$$\begin{aligned}
u_{\rho,1} &= u^*_{\rho,1} + \frac{s\cos\theta_0}{E}\times\\
&\times\left\{-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \mu\sigma_k\cos\beta e^{-\beta}\right\},\\
u_{z,1} &= u^*_{z,1} + \frac{s\cos^2\theta_0}{\sin\theta_0 E}\times\\
&\times\left\{-\frac{\nu\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta - \sin\beta)e^{-\beta} + \mu\sigma_k\cos\beta e^{-\beta}\right\},\\
\pm v_{,1} &= \frac{s\cos\theta_0}{Eh}\left(T^*_{2,1} - \nu T^*_{1,1}\right) - u^*_{\rho,1},\\
\vartheta_{,1} &= \vartheta^*_{,1} + \frac{\sqrt[4]{3(1-\nu^2)}}{E}\sqrt{\frac{s\cos\theta_0}{h}}\times\\
&\times\left\{\mu\frac{2\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}\cos\beta e^{-\beta} - \sigma_k(\cos\beta + \sin\beta)e^{-\beta}\right\}.
\end{aligned} \tag{12.22}$$

In these equalities

$$\begin{aligned}
\sigma_{M_0} &= \frac{6M_0}{h^2},\quad \sigma_k = 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\sqrt{\sin\theta_0}\,\frac{Q_0 - Q^*_0}{h},\\
\chi^* &= \frac{1}{Eh}\left\{-\frac{s^2\cos\theta_0}{\sin\theta_0}\frac{d}{ds}\left(\frac{T^*_{,1}}{s}\right) + (1+\nu)\left[-\frac{1}{s\sin^2\theta_0}\frac{ds\,W}{ds} + \right.\right.\\
&\left.\left. + s\operatorname{ctg}\theta_0(q_{n,1}\operatorname{ctg}\theta_0 - 2q_{1,1})\right]\right\},\\
u^*_{z,1} &= \frac{s\cos\theta_0}{Eh}\int_{s_1}^{s}\left\{\operatorname{ctg}\theta_0\frac{dT^*_{,1}}{ds'} - \frac{1}{s'\sin\theta_0\cos\theta_0}T^*_{,1} - \right.\\
&\left. - (1+\nu)\left[-\frac{1}{s^2\sin^2\theta_0}\int ds' W + q_{n,1}(\operatorname{ctg}^2\theta_0 - 1) - 2q_{1,1}\operatorname{ctg}\theta_0\right]\right\}\times\\
&\times ds' + \frac{a_3}{a_5}s\cos\theta_0,\\
\vartheta^*_{,1} &= \chi^* - \frac{u^*_{z,1}}{s\cos\theta_0},\\
u^*_{\rho,1} &= \frac{a_4}{a_6} + \frac{1}{Eh}\int_{s_1}^{s}\left\{s'\cos\theta_0\frac{dT^*_{,1}}{ds'} + (1+\nu)\left[\frac{1}{s'\sin\theta_0}\frac{ds'W}{ds'} - \right.\right.
\end{aligned} \tag{12.23}$$

$$-2\cos\theta_0\, s(\operatorname{ctg}\theta_0 q_{n,1} - q_{1,1})\Big]\Big\}\, ds' + \operatorname{tg}\theta_0 \int_{s_1}^{s} \frac{u_{z,1}}{s'}\, ds',$$

$$T^*_{1,1} = \frac{W}{s\sin\theta_0\cos\theta_0}, \quad T^*_{2,1} = s\operatorname{ctg}\theta_0 q_{n,1},$$

$$\pm S^*_{,1} = -\frac{1}{\sin\theta_0}\frac{dW}{ds} + s\operatorname{ctg}\theta_0(q_{n,1}\cos\theta_0 - q_{1,1}\sin\theta_0).$$

Finally, we present conditions for the compatible junction of a conic shell with a general shell of rotation /156

$$\begin{aligned}\left\{\widetilde{T}_{,1} - \frac{(1+\nu)}{\sin\theta_0\cos\theta_0 s} W\right\}^{\text{con}} &= \left\{\widetilde{T}_{,1} - (1+\nu)\widetilde{T}_{1,1}\right\}^{\text{II}},\\ \left\{\frac{1}{\sin\theta_0} W + ic\sin\theta_0 \frac{d\widetilde{T}_{,1}}{ds}\right\}^{\text{con}} &= \left\{\cos\theta\,\widetilde{T}_{1,1} + ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right\}^{\text{II}},\end{aligned} \tag{12.24}$$

In the case of smooth junction we adopt a simpler form

$$\{\widetilde{T}_{,1}\}^{\text{con}} = \{\widetilde{T}_{,1}\}^{\text{II}}, \quad \left\{\frac{d\widetilde{T}_{,1}}{ds}\right\}^{\text{con}} = \left\{\frac{1}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta}\right\}^{\text{II}}. \tag{12.25}$$

13. Round Plate

A round plate may be looked upon as a unique shell of revolution which is obtained by rotating a section of a straight line around an axis perpendicular to the line. Many of the relationships obtained in sections 1-10 are applicable to the plate, if we make the following substitutions in these relationships

$$\begin{gathered}\frac{1}{R_1} = \frac{1}{R_2} = 0, \quad \ell = 0, \quad R_2\sin\theta = r = r_0\rho,\\ \frac{1}{R_1}\frac{\partial}{\partial\vartheta} = \frac{\partial}{\partial s_1} = \frac{1}{r_0}\frac{\partial}{\partial\rho}.\end{gathered} \tag{13.1}$$

When the plate is prescribed in this manner, it is defined by the conventional dimensionless polar coordinates

$$\alpha_1 = \frac{r}{r_0} = \rho, \quad \alpha_2 = \varphi \quad (A_1 = r_0,\ A_2 = r_0\rho). \tag{13.2}$$

In this coordinate system the system of equations containing the complex forces (1.14) takes the form

$$\frac{\partial_2\widetilde{T}_1}{\partial\rho} + \frac{\partial\widetilde{S}}{\partial\varphi} - \widetilde{T}_2 = -r_0\rho q_1, \tag{13.3}$$

$$\frac{\partial \widetilde{T}_1}{\partial \varphi} + \frac{1}{\rho}\frac{\partial \rho^2 \widetilde{S}}{\partial \rho} = -r_0 \rho q_2,$$
$$\Delta \widetilde{T} = \frac{i}{c} q_n,$$

where we now have

$$\Delta = \frac{1}{r_0^2}\left\{\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right\}. \tag{13.4}$$

Replacing $\widetilde{T}_2$ in the above system with expression $(\widetilde{T}\text{-}\widetilde{T}_1)$ and eliminating $\widetilde{S}$, we arrive at a system of two equations

$$\frac{\partial}{\partial \rho}\rho\frac{\partial \rho^2 \widetilde{T}_1}{\partial \rho} + \rho\frac{\partial^2 \widetilde{T}_1}{\partial \varphi^2} - \left[\frac{\partial}{\partial \rho}\rho^2 \widetilde{T} + \rho\frac{\partial^2 \widetilde{T}}{\partial \varphi^2}\right] = r_0\left\{\rho^2\frac{\partial q_2}{\partial \varphi} - \frac{\partial \rho^3 q_1}{\partial \rho}\right\}, \tag{13.5}$$
$$\Delta \widetilde{T} = \frac{i}{c} q_n.$$

As in the preceding sections we shall consider in detail the method of separating the variables, retaining the previous relationship between the quantities and the angle φ. Then the basic complex system (13.5) is transformed into /158

$$\frac{d}{d\rho}\rho\frac{d\rho^2 \widetilde{T}_{1,k}}{d\rho} - k^2\rho\widetilde{T}_{1,k} - \left[\frac{d\rho^2 \widetilde{T}_{,k}}{d\rho} - k^2\rho\widetilde{T}_{,k}\right] = r_0\left\{\pm k\rho^2 q_{2,k} - \frac{d\rho^3 q_{1,k}}{d\rho}\right\}, \tag{13.6}$$
$$\Delta_k \widetilde{T} = \frac{i}{c} q_{n,k},$$

where

$$\Delta_k = \frac{1}{r_0^2}\left\{\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{k^2}{\rho^2}\right\}. \tag{13.7}$$

By using the values $\widetilde{T}_{,k}$ and $\widetilde{T}_{1,k}$ obtained from this system, the remaining complex forces are determined, in accordance with $(13.3)_1$ and $(13.3)_2$ by means of relationships

$$\widetilde{T}_{2,k}=\widetilde{T}_{,k}-\widetilde{T}_{1,k},$$
$$\pm k\widetilde{S}_{,k}=-\frac{d\rho\widetilde{T}_{1,k}}{d\rho}+\widetilde{T}_{2,k}-r_0\rho q_{1,k} \tag{13.8}$$
$$\left(\widetilde{S}_{,0}=\frac{1}{\rho^2}\left\{\widetilde{C}_2-r_0\int_{\rho_0}^{\rho}\rho'^2q_{2,0}d\rho'\right\}\right).$$

Furthermore, the complex displacements are obtained from the system

$$\begin{aligned}
&\left(r_0\widetilde{\varepsilon}_{1,k}\right)\quad \frac{d\widetilde{u}_{,k}}{d\rho}=\frac{r_0}{Eh}\left\{\widetilde{T}_{1,k}-\nu\widetilde{T}_{2,k}+\frac{i}{c}M^*_{2,k}\right\},\\
&\left(r_0\widetilde{\varepsilon}_{2,k}\right)\quad \frac{1}{\rho}\left(\pm k\widetilde{v}_{,k}+\widetilde{u}_{,k}\right)=\frac{r_0}{Eh}\left\{\widetilde{T}_{2,k}-\nu\widetilde{T}_{1,k}+\frac{i}{c}M^*_{1,k}\right\},\\
&\left(r_0\widetilde{\omega}_{,k}\right)\quad \rho\frac{d}{d\rho}\left(\frac{\widetilde{v}_{,k}}{\rho}\right)\mp\frac{k\widetilde{u}_{,k}}{\rho}=\frac{r_0}{Eh}\left\{2(1+\nu)\widetilde{S}_{,k}-\frac{i}{c}2H^*_{,k}\right\},\\
&\left(r_0^2\widetilde{\varkappa}_{1,k}\right)\quad -\frac{d^2\widetilde{w}_{,k}}{d\rho^2}=i\frac{r_0^2}{Ehc}\left\{\widetilde{T}_{2,k}-T^*_{2,k}\right\},\\
&\left(r_0^2\widetilde{\varkappa}_{2,k}\right)\quad -\frac{1}{\rho}\frac{d\widetilde{w}_{,k}}{d\rho}+\frac{k^2}{\rho^2}\widetilde{w}_{,k}=i\frac{r_0^2}{Ehc}\left\{\widetilde{T}_{1,k}-T^*_{1,k}\right\},\\
&\left(r_0^2\widetilde{\tau}_{,k}\right)\quad \pm k\left(\frac{1}{\rho}\frac{d\widetilde{w}_{,k}}{d\rho}-\frac{1}{\rho^2}\widetilde{w}_{,k}\right)=-i\frac{r_0^2}{Ehc}\left\{\widetilde{S}_{,k}-S^*_{,k}\right\},
\end{aligned} \tag{13.9}$$

On the other hand, the complex boundary quantities take on the form

$$\begin{aligned}
&\widetilde{Q}_{\rho,k}=Q_{\rho,k}-iEhc\varkappa_{\rho,k}=Q^*_{\rho,k}-iEhc\widetilde{\varkappa}_{\rho,k},\\
&\widetilde{Q}_{z,k}=Q_{z,k}+iEhc\varkappa_{z,k}=Q^*_{z,k}+iEhc\widetilde{\varkappa}_{z,k},\\
&\widetilde{Q}_{12,k}=Q_{12,k}+iEhc\varkappa_{21,k}=Q^*_{12,k}+iEhc\widetilde{\varkappa}_{2,k},\\
&\widetilde{M}_{1,k}=M_{1,k}+iEhc\varepsilon_{2,k}=M^*_{1,k}+iEhc\widetilde{\varepsilon}_{2,k}.
\end{aligned} \tag{13.10}$$

Here

$$\begin{aligned}
&\left(\vartheta_{,k}=-\frac{1}{r_0}\frac{dw_{,k}}{d\rho}\right)\\
&\varkappa_{\rho,k}=\frac{1}{r_0\rho}\left\{\vartheta_{,k}+\frac{k^2}{r_0\rho}w_{,k}\right\},\quad \varkappa_{21,k}=\frac{1}{r_0\rho}\left\{\mp k\left(\vartheta_{,k}+\frac{w_{,k}}{r_0\rho}\right)\right\},\\
&\varkappa_{z,k}=\frac{1}{r_0\rho}\left\{\mp kv_{,k}-k^2u_{,k}\right\},\quad \varepsilon_{2,k}=\frac{1}{r_0\rho}\left\{\pm kv_{,k}+u_{,k}\right\};
\end{aligned} \tag{13.11}$$

$$Q_{12,k}=S_{,k},\quad Q_{\rho,k}=T_{1,k},$$
$$-Q_{z,k}=Q_{1n,k}=\frac{1}{r_0\rho}\left\{\frac{d\rho M_{1,k}}{d\rho}\pm k2H_{,k}-M_{2,k}\right\}. \tag{13.12}$$

The following approximate expressions are valid for the quantities (13.10) /159

$$\widetilde{Q}_{\rho,k}=\widetilde{T}_{1,k},\quad -\widetilde{Q}_{z,k}=\widetilde{Q}_{1n,k}=i\frac{c}{r_0}\frac{d\widetilde{T}_{,k}}{d\rho},$$
$$\widetilde{Q}_{12,k}=\widetilde{S}_{,k},\quad \widetilde{M}_{1,k}=ic\left\{\widetilde{T}_{2,k}-\nu\widetilde{T}_{1,k}\right\}. \tag{13.13}$$

Let us consider in detail the two most interesting cases--the symmetric case and the antisymmetric case.

The First Symmetric Case

For this case it is more convenient to proceed directly from system (13.3), which assumes the following form because all of the quantities are independent of φ and because $\widetilde{S}$ is equal to 0

$$\frac{1}{\rho}\frac{d\rho^2\widetilde{T}_1}{d\rho}-\widetilde{T}=-r_0\rho q_1,$$
$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\widetilde{T}}{d\rho}=i\frac{r_0^2}{c}q_n.$$

From the second equation it follows that

$$\frac{d\widetilde{T}}{d\rho}=\frac{1}{\rho}\left\{i\frac{r_0^2}{c}\int_{\rho_0}^{\rho}\rho' q_n d\rho'+\widetilde{A}\right\}. \tag{*}$$

The constant $\widetilde{A}$ is easily associated with the axial force. Indeed, first of all, in accordance with equalities (13.10) and (13.11) the quantity $\widetilde{Q}_z$ is real. Therefore it follows from relationship $(13.13)_2$ that

$$\widetilde{Q}_z(\rho_1)=-\frac{F_z^0}{2\pi r_0\rho_1}=-i\frac{c}{r_0\rho_1}\widetilde{A},\quad \widetilde{A}=-\frac{i}{c}\frac{F_z^0}{2\pi}.$$

Taking this into account and integrating (*) once again, we obtain

$$\widetilde{T}=i\frac{r_0^2}{c}\left\{\int_{\rho_1}^{\rho}\frac{d\rho'}{\rho'}\int_{\rho_1}^{\rho'}\rho''q_n d\rho''-\frac{F_z^0}{2\pi r_0^2}\ln\rho\right\}+\widetilde{C}_1. \qquad (13.14)_1$$

We substitute the expression which we have obtained into the first equation and integrate it. As a result we obtain an expression for $\widetilde{T}_1$ /160

$$\widetilde{T}_1=\frac{1}{\rho^2}\left\{i\frac{r_0^2}{c}\int_{\rho_1}^{\rho}\rho'd\rho'\int_{\rho_1}^{\rho'}\frac{d\rho''}{\rho''}\int_{\rho_1}^{\rho''}\rho'''q_n d\rho'''-r_0\int_{\rho_1}^{\rho}\rho'^2q_1d\rho'-\right.$$
$$\left.-\frac{i}{c}\frac{F_z^0}{2\pi}\left(\frac{\rho^2}{2}\ln\rho-\frac{\rho^2}{4}\right)+\widetilde{C}_1\frac{\rho^2}{2}+\widetilde{C}_2\right\}. \qquad (13.14)_2$$

In the case of a solid plate ($\rho_1 = 0$), when there is no concentrated force at the center, we should let

$$F_z^0=0,\quad \widetilde{C}_2=0.$$

From the established $\widetilde{T}$ and $\widetilde{T}_1$ we determine

$$\widetilde{T}_2=\widetilde{T}-\widetilde{T}_1,\quad T_1=\operatorname{Re}\widetilde{T}_1\quad T_2=\operatorname{Re}\widetilde{T}_2,$$
$$M_1=-c\operatorname{Im}\left\{\widetilde{T}_2+\nu\widetilde{T}_1\right\},\quad M_2=-c\operatorname{Im}\left\{\widetilde{T}_1+\nu\widetilde{T}_2\right\}, \qquad (13.15)$$

and then from equality (13.9)

$$\frac{u}{\rho}=\frac{r_0}{Eh}\operatorname{Re}\left\{\widetilde{T}_2-\nu\widetilde{T}_1\right\},$$
$$\frac{1}{\rho}\frac{dw}{d\rho}=\frac{r_0^2}{Ehc}\operatorname{Im}\widetilde{T}_1 \qquad (13.16)$$

the displacements are found.

Finally, the conditions for junction with shells of revolution of the general type are written in the following form

$$\left\{\widetilde{T}_1\right\}^{\text{plate}} = \left\{\frac{\operatorname{ctg}\theta}{R_2 \sin\theta} U + i \frac{c}{R_1 \sin\theta} \frac{d\widetilde{T}}{d\theta}\right\}^{\text{II}},$$
$$\left\{\widetilde{T} - (1+\nu)\widetilde{T}_1\right\}^{\text{plate}} = \left\{\widetilde{T} - ic(1+\nu)\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}}{d\theta} - (1+\nu)\frac{U}{R_2 \sin^2\theta}\right\}^{\text{II}}. \tag{13.17}$$

The Antisymmetric Case

For these cases system (13.6) is transformed into

$$\frac{d}{d\rho}\rho\frac{d\rho^2\widetilde{T}_{1,1}}{d\rho} - \rho\widetilde{T}_{1,1} - \left[\frac{d\rho^2\widetilde{T}_{,1}}{d\rho} - \rho^2\widetilde{T}_{,1}\right] = r_0\left\{\pm\rho^2 q_{2,1} - \frac{d\rho^3 q_{1,1}}{d\rho}\right\},$$
$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\widetilde{T}_{,1}}{d\rho} - \frac{\widetilde{T}_{,1}}{\rho^2} = i\frac{r_0^2}{c}q_{n,1}. \tag{13.18}$$

It is more convenient to write it in a slightly different form

$$\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d\rho^2\widetilde{T}_{1,1}}{d\rho} - \rho\frac{d\rho\widetilde{T}_{,1}}{d\rho} = r_0\left\{\pm\rho^2 q_{2,1} - \frac{d\rho^3 q_{1,1}}{d\rho}\right\},$$
$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d\rho\widetilde{T}_{,1}}{d\rho} = i\frac{r_0^2}{c}q_{n,1}. \tag{13.19}$$

From this we can see that, as in the symmetric case, the solution of the principal complex system is reduced to a successive determination of four quadratures.[1] From the established $\widetilde{T}_{,1}$ and $\widetilde{T}_{1,1}$ we determine /161

$$\widetilde{T}_{2,1} = \widetilde{T}_{,1} - \widetilde{T}_{1,1}, \quad \pm\widetilde{S}_{,1} = -\frac{d\rho\widetilde{T}_{1,1}}{d\rho} + \widetilde{T}_{2,1} - \rho r_0 q_{1,1},$$
$$T_{1,1} = \operatorname{Re}\widetilde{T}_{1,1}, \quad T_{2,1} = \operatorname{Re}\widetilde{T}_{2,1}, \quad S_{,1} = \operatorname{Re}\widetilde{S}_{,1},$$
$$M_{1,1} = -c\operatorname{im}\left\{\widetilde{T}_{2,1} + \nu\widetilde{T}_{1,1}\right\}, \quad M_{2,1} = -c\operatorname{Im}\left\{\widetilde{T}_{1,1} + \nu\widetilde{T}_{2,1}\right\}, \tag{13.20}$$
$$H_{,1} = (1-\nu)c\operatorname{Im}\widetilde{S}.$$

Furthermore, the displacements are determined from equalities (13.9)

[1] As in the symmetric case we can show that two of the four complex constants obtained during integration are purely imaginary. It is easy to associate them with the corresponding components of the principal vector and the principal moment.

$$\frac{du_{,1}}{d\rho} = \frac{r_0}{Eh}\operatorname{Re}\left\{\widetilde{T}_{1,1} - \nu\widetilde{T}_{2,1}\right\},$$
$$\frac{\pm v_{,1} + u_{,1}}{\rho} = \frac{r_0}{Eh}\operatorname{Re}\left\{\widetilde{T}_{2,1} - \nu\widetilde{T}_{1,1}\right\}, \qquad (13.21)$$
$$\frac{1}{\rho^2}\frac{d\rho w_{,1}}{d\rho} = \frac{r_0^2}{Ehc}\operatorname{Im}\widetilde{T}_{1,1}$$

The complex conditions at the joints take on the form

$$\left\{\widetilde{T}_{1,1}\right\}^{\text{plate}} = \left\{ic\left(\frac{1}{R_1\sin\theta}\frac{d\widetilde{T}_{,1}}{d\theta} - \frac{\operatorname{ctg}\theta}{R_2\sin\theta}\widetilde{T}_{,1}\right) + \frac{\operatorname{ctg}\theta}{R_2\sin\theta}W\right\}^{\text{II}},$$
$$\left\{\widetilde{T}_{,1} - (1+\nu)\widetilde{T}_{1,1}\right\}^{\text{plate}} = \left\{\widetilde{T}_{,1} - (1+\nu)\left(ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}_{,1}}{d\theta} + \frac{W}{R_2\sin^2\theta}\right)\right\}^{\text{II}}. \qquad (13.22)$$

The relationships which have been obtained in this section should be utilized in cases when the plate is a composite part of a shell of revolution (for example, when it represents a flat end plate having the same thickness as the walls of the vessel).

In looking over the relationships obtained above it is easy to note that for the plate the overall stress-strain state breaks down into a purely flexural state and a membrane state. This is explained by the fact that for small flexures the distortion of the middle surface does not produce a tangential strain, and vice versa. As a result the analysis of the plate breaks down into two independent parts:

(a) the problem of transverse bending produced by an external load perpendicular to the middle surface;

(b) the problem of the tangential deformation of a plate by a tangential external load. The second state is usually called the generalized two-dimensional state of stress. In the subsequent sections we shall consider both problems briefly. /162

14. The Transverse Flexure of a Plate

By separating the real and imaginary parts in the relationships of the preceding section, it is easy to obtain

$$M_{1,k} = -c\operatorname{Im}\left\{\widetilde{T}_{2,k} + \nu\widetilde{T}_{1,k}\right\} = D(\varkappa_{1,k} + \nu\varkappa_{2,k}),$$
$$M_{2,k} = -c\operatorname{Im}\left\{\widetilde{T}_{1,k} + \nu\widetilde{T}_{2,k}\right\} = D(\varkappa_{2,k} + \nu\varkappa_{1,k}), \qquad (14.1)$$
$$H_{,k} = (1-\nu)c\operatorname{Im}\widetilde{S}_{,k} = D(1-\nu)\tau_{,k} \quad \left(D = Ehc^2 = \frac{Eh^3}{12(1-\nu^2)}\right).$$

where

$$r_0^2 \varkappa_{1,k} = -\frac{d^2 w_{,k}}{d\rho^2},$$
$$r_0^2 \varkappa_{2,k} = -\frac{1}{\rho}\frac{dw_{,k}}{d\rho} + \frac{k^2}{\rho^2} w_{,k}, \qquad (14.2)$$
$$r_0^2 \tau_{,k} = \pm k\left(\frac{1}{\rho}\frac{dw_{,k}}{d\rho} - \frac{w_{,k}}{\rho^2}\right).$$

Furthermore we have

$$\operatorname{Im} T_{,k} = -Ehc(\varkappa_{1,k} + \varkappa_{2,k}) =$$
$$= \frac{Ehc}{r_0^2}\left(\frac{dw_{,k}}{d\rho^2} + \frac{1}{\rho}\frac{dw_{,k}}{d\rho} - \frac{k^2}{\rho^2} w_{,k}\right) = Ehc\,\Delta_k(w_{,k}). \qquad (14.3)$$

If the last expression is substituted into the imaginary part of the third equation in (13.6), we obtain a well-known (ref. 41) equation which determines the flexure of a plate

$$\Delta_k \Delta_k w_{,k} = \frac{q_{n,k}}{D}. \qquad (14.4)$$

Finally, from relationships (13.11) and (13.12), with the aid of equality (14.1), we obtain

$$Q_{z,k} = \frac{D}{r_0^3}\left\{\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{k^2}{\rho^2}\right) w_{,k}\right\} = \frac{D}{r_0}\frac{d\Delta_k w_{,k}}{\rho d\rho},$$
$$\varkappa_{\rho,k} = \frac{1}{r_0^2}\left\{-\frac{1}{\rho}\frac{dw_{,k}}{d\rho} + \frac{k^2}{\rho^2} w_{,k}\right\}, \qquad (14.5)$$
$$\varkappa_{21,k} = \frac{1}{r_0^2}\left\{\pm k\left(\frac{1}{\rho}\frac{dw_{,k}}{d\rho} - \frac{1}{\rho^2} w_{,k}\right)\right\}, \quad \vartheta_{,k} = -\frac{1}{r_0}\frac{dw_{,k}}{d\rho}.$$

It is interesting to note that by determining $Q_{z,k}$ from the simplified expression $(13.13)_2$ we arrive at the same quantity.

Let us return to equation (14.4) and rewrite it in an expanded form

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{k^2}{\rho^2}\right)\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{k^2}{\rho^2}\right) w_{,k} = \frac{r_0^4 q_{n,k}}{D}. \qquad (14.6)$$

We shall seek a particular solution of the corresponding homogeneous equation

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{k^2}{\rho^2}\right)\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{k^2}{\rho^2}\right)w^0_{\cdot,k}=0 \tag{14.7}$$

in the form

$$w^0_{\cdot,k}=\rho^s \tag{14.8}$$

Substituting this expression into the equation, we arrive at the following characteristic equation

$$(s^2-k^2)\left[(s-2)^2-k^2\right]=0. \tag{14.9}$$

From this equation we find

$$s_1=k,\quad s_2=-k,\quad s_3=2+k,\quad s_4=2-k. \tag{14.10}$$

Thus, when $k \geq 2$, the general solution of the homogeneous equation is given by the following expression

$$w^0_{\cdot,k}=\frac{r_0^4}{D}\left(A_k\rho^k+B_k\rho^{-k}+C_k\rho^{k+2}+D_k\rho^{-k+2}\right). \tag{14.11}$$

The case $k = 0$ and $k = 1$ (the symmetric case and the antisymmetric case) pertaining to multiple roots will be considered specially below.

Let us return to the nonhomogeneous equation (14.6). All forms of surface loading which are of practical interest are assigned either by polynomials in ρ or are represented by series of the form

$$q_{n,k}=\sum_{l=-1}^{\infty}G_k^{(l)}\rho^l. \tag{14.12}$$

Therefore, it is necessary, first of all, to find the particular solution of the equation

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{k^2}{\rho^2}\right)\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{k^2}{\rho^2}\right)X_k^{(l)}(\rho)=\rho^l. \tag{14.13}$$

We seek this solution in the form

$$X_k^{(l)}(\rho) = F_k^{(l)} \rho^s. \tag{14.14}$$

Substituting the last expression into (14.13) we obtain (when $l \neq k-4$ and $l \neq k-2$)

$$s = l+4, \quad F_k^l = \frac{1}{[(l+4)^2 - k^2][(l+2)^2 - k^2]}. \tag{14.15}$$

If, on the other hand, $l = k-4$ or $l = k-2$, then, by seeking the solution in the form

$$X_k(\rho) = F_k^{(l)} \rho^s \ln \rho, \tag{14.16}$$

we arrive at the equality

$$F_k^{(l)} = [2(s-2)(s^2-k^2) + 2s[(s-2)^2 - k^2]] \rho^{s-4} = \rho^l.$$

From this equality it follows that

$$F_k^{(k-4)} = -\frac{1}{8k(k-1)} \quad \begin{matrix} s = l+4, \\ \text{when} \quad l = k-4 \quad (s=k), \end{matrix} \tag{14.17}$$

$$F_k^{(k-4)} = \frac{1}{8k(k+1)} \quad \text{when} \quad l = k-2 \quad (s = k+2). \tag{14.18}$$

Taking into account the relationships which we have obtained, the particular solution of the nonhomogeneous equation may be represented in the form /164

$$w_{*,k}^{*} = \frac{r_0^4}{D}\left\{\sum_{l=-1}^{\infty}{}' G_k^{(l)} \frac{1}{[(l+4)^2 - k^2][(l+2)^2 - k^2]} \rho^{l+4} - \frac{G_k^{(l)}}{8k(k-1)} \rho^k \ln \rho \delta_l^{2k-4} + \frac{G_k^{(l)}}{8k(k+1)} \rho^{k+2} \ln \rho \delta_l^{2k-2}\right\} \tag{14.19}$$

$$\left(\delta_j^i = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}\right).$$

The symbol " means that the terms corresponding to $i = k-4$ and $i = k-2$ are dropped in the sum. They are written out separately. We note that the particular solution which has been obtained is also valid when $k = 0$ and $k = 1$.

It is not difficult to find the particular solution also for the case when the surface loading is of the general form. Indeed, it is easy to become convinced that equation (14.6) may be written in the following form

$$\left\{\rho^{k-1}\frac{d}{d\rho}\rho^{-2k+1}\frac{d}{d\rho}\rho\right\}\left\{\rho^{k-1}\frac{d}{d\rho}\rho^{-2k+1}\frac{d}{d\rho}\rho\right\}w_{n,k}=\frac{r_0^4 q_{n,k}}{D}.$$

From this we can see that the search for the particular solution in the general form is reduced to a successive determination of four quadratures.

Symmetric Case

In the symmetric case the basic characteristic equation assumes the following form

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dw}{d\rho}=\frac{r_0^4}{D}q_n. \tag{14.20}$$

By integrating successively we obtain

$$w=w^*+w^0, \tag{14.21}$$

where

$$w^*=\frac{r_0^4}{D}\left\{\int_{\rho_1}^{\rho}\frac{d\rho'}{\rho'}\int_{\rho_1}^{\rho'}\rho''d\rho''\int_{\rho_1}^{\rho''}\frac{d\rho'''}{\rho'''}\int_{\rho_1}^{\rho'''}\rho^{(IV)}q_n(\rho^{(IV)})\,d\rho^{(IV)}\right\} \tag{14.22}$$

is a particular solution

$$w^0=\frac{r_0^4}{D}\{A_0+B_0\rho^2+C_0\ln\rho+D_0\rho^2\ln\rho\}$$

is a general solution of the homogeneous equation. In particular for a uniform pressure $q_n = p = \text{const}$

$$w^*=\frac{r_0^4 p}{64D}\rho^4. \tag{14.23}$$

By using the relationships obtained above we find

$$\vartheta = \vartheta^* - \frac{r_0^3}{D}\left\{2B_0\rho + C_0\frac{1}{\rho} + D_0(2\rho\ln\rho + \rho)\right\}, \tag{14.24}$$

where

$$\vartheta^* = -\frac{1}{r_0}\frac{dw^*}{d\rho}, \tag{14.25}$$

$$\begin{aligned} M_1 &= M_1^* - r_0^2\left\{2(1+\nu)B_0 - (1-\nu)C_0\frac{1}{\rho^2} + \right. \\ &\left. + D_0[2(1+\nu)\ln\rho + 3 + \nu]\right\}, \\ M_2 &= M_2^* - r_0^2\left\{2(1+\nu)B_0 + (1-\nu)C_0\frac{1}{\rho^2} + \right. \\ &\left. + D_0[2(1+\nu)\ln\rho + 1 + 3\nu]\right\}. \end{aligned} \tag{14.26}$$

Here

$$\begin{aligned} M_1^* &= -\frac{D}{r_0^2}\left\{\frac{d^2w^*}{d\rho^2} + \nu\frac{1}{\rho}\frac{dw^*}{d\rho}\right\}, \\ M_2^* &= -\frac{D}{r_0^2}\left\{\nu\frac{d^2w^*}{d\rho^2} + \frac{1}{\rho}\frac{dw^*}{d\rho}\right\}, \end{aligned} \tag{14.27}$$

$$Q_z = Q_z^* + r_0\left\{\frac{4D_0}{\rho}\right\}, \tag{14.28}$$

where

$$Q_z^* = -\frac{1}{r_0\rho}\left(\frac{d\rho M_1^*}{d\rho} - M_2^*\right). \tag{14.29}$$

If the surface load $q_n(\rho)$ does not have singularities when $\rho = 0$, /165

which is what we shall assume in the future, all quantities with the sign * will be finite. (It is easy to see this if we let $\rho_1 = \rho_1' = \ldots = \rho_1^{(IV)} = 0$ in expression (14.22) and expand $q_n(\rho)$ in the series $q_n(\rho) = q_n(0) + \frac{q_n'(0)}{1}\rho + \ldots$.)

It follows from this that for a solid circular plate the conditions that the solution be finite are reduced to the requirement $C_0 = D_0 = 0$. In this case

$$M_1 = M_1^* - r_0^2 [2(1+\nu) B_0], \quad w = w^* + \frac{r_0^4}{D} [A_0 + B_0 \rho^2],$$
$$M_2 = M_2^* - r_0^2 [2(1+\nu) B_0], \quad \vartheta = \vartheta^* - \frac{r_0^3}{D} [2B_0 \rho]. \tag{14.30}$$

(r_0 is the radius of the plate). Let

$$M_0 = M_1(1), \quad M_0^* = M_1^*(1), \quad \vartheta_0 = \vartheta(1), \quad \vartheta_0^* = \vartheta^*(1).$$

Then it follows from (14.30) that

$$B_0 = -\frac{M_0 - M_0^*}{2(1+\nu) r_0^2}.$$

Substituting B_0 into expression $(14.30)_4$ and assuming that in this expression $\rho = 1$, we obtain /166

$$\vartheta_0 = \vartheta_0^* + \alpha_{22} (M_0 - M_0^*), \tag{14.31}$$

where the flexural compliance coefficient of the edge α_{22} is equal to

$$\alpha_{22} = \frac{12(1-\nu)}{Eh^2} \left(\frac{r_0}{h}\right). \tag{14.32}$$

This expression can be used in joining a plate with the general shell of rotation. Substituting the determined value of B_0 into the remaining quantities

and stipulating that w (1) = 0 for fixing of the rigid axial displacement of the plate, we obtain

$$M_1 = M_1^* + (M_0 - M_0^*), \quad M_2 = M_2^* + (M_0 - M_0^*),$$
$$w = w^* - w^*(1) + \frac{r_0^2}{2(1+\nu) D} (M_0 - M_0^*)(1 - \rho^2), \tag{14.33}$$
$$\vartheta = \vartheta^* + \frac{r_0}{2(1+\nu) D} (M_0 - M_0^*) \rho.$$

Let us return briefly to the ring plate and compute the value of the vertical force acting on the internal contour ($\rho = \rho_1$). According to (14.28)

$$-F_z(\rho_1) = 2\pi r_0 \rho_1 Q_z^* + 2\pi r_0^2 4 \cdot D_0.$$

Now, by constricting the contour and remembering that Q_z^* is a finite quantity, we obtain

$$F_z^0 = \lim_{\rho_1 \to 0} F_z(\rho_1) = -8\pi r_0^2 D_0. \tag{14.34}$$

This relationship associates the constant D_0 with the normal concentrated force which is applied at the center of the plate. The corresponding moments and displacements are equal to

$$\begin{aligned} M_1 &= [2(1+\nu)\ln\rho + 3 + \nu]\frac{F_z^0}{8\pi}, \quad w = -\frac{r_0^2 F_z^0}{8\pi D}[\rho^2 \ln\rho], \\ M_2 &= [2(1+\nu)\ln\rho + 1 + 3\nu]\frac{F_z^0}{8\pi}, \quad \vartheta = \frac{r_0 F_z^0}{8\pi D}[2\rho\ln\rho + \rho]. \end{aligned} \tag{14.35}$$

In the second part of the book we shall consider in detail the question concerning the nature of singularities which occur in the neighborhood of the place where concentrated forces are applied. We shall consider this question as it applies to the shell of the general type. Here, on the other hand, we shall limit ourselves to the remarks that the moments have a uniform logarithmic singularity.

Let us consider in more detail a circular plate loaded with uniform pressure. For this plate /167

$$\begin{aligned} w^* &= \frac{r_0^4 p}{64D}\rho^4, \quad w^*(1) = \frac{r_0^4 p}{64D}, \quad \vartheta^* = -\frac{r_0^3 p}{16D}\rho^3, \\ \vartheta_0^* &= -\frac{r_0^3 p}{16D}, \quad M_1^* = -\frac{r_0^2 p}{16}(3+\nu)\rho^2, \quad M_2^* = -\frac{r_0^2 p}{16}(1+3\nu)\rho^2, \\ M_0^* &= -\frac{r_0^2 p}{16}(1+3\nu). \end{aligned} \tag{14.36}$$

Let us pause to consider two most important versions of the boundary conditions.

A Freely Supported Plate

$$w(1) = M_0 = 0$$

The first of these conditions is satisfied automatically. The second determines the sole arbitrary constant M_0 which is contained in expressions (14.33). Substituting it into expression (14.33) we obtain

$$w = \frac{r_0^4 p}{64D}(1-\rho^2)\left(\frac{5+\nu}{1+\nu}-\rho^2\right). \tag{14.37}$$

The maximum flexure, as was to be expected, takes place at the center of the plate

$$w_{\max} = w(0) = \frac{(5+\nu)\,pr_0^4}{64(1+\nu)D}. \tag{14.38}$$

We obtain the following expression for the angle of rotation

$$\vartheta = \frac{r_0^3 p}{16D}\rho\left(\frac{3+\nu}{1+\nu}-\rho^2\right), \tag{14.39}$$

in which case

$$\vartheta_{\max} = \vartheta(1) = \frac{r_0^3 p}{8(1+\nu)D}.$$

Finally

$$\begin{aligned} M_1 &= \frac{r_0^2 p}{16}(3+\nu)(1-\rho^2), \\ M_2 &= \frac{r_0^2 p}{16}[3+\nu-(1+3\nu)\rho^2]. \end{aligned} \tag{14.40}$$

Both of the moments achieve their maximum value at the center of the plate

$$M_{\max} = M_1(0) = M_2(0) = \frac{3+\nu}{16} r_0^2 p. \tag{14.41}$$

By introducing the maximum flexural stress

$$\sigma_0 = \frac{6M_{max}}{h^2} = \frac{3(3+\nu)}{8}\left(\frac{r_0}{h}\right)^2 p. \tag{14.42}$$

We obtain from (14.40)

$$\sigma_{M_1} = \sigma_0(1-\rho^2), \quad \sigma_{M} = \sigma_0\left(1 - \frac{1+3\nu}{3+\nu}\rho^2\right). \tag{14.43}$$

The variation in the stresses along the radius of the plate is shown in figure 24. It is assumed that $\nu = 0.3$. /168

A Rigidly Fixed Edge

$$w(1) = \vartheta_0 = 0$$

The second of these conditions makes it possible to determine the value of the constant M_0 from equality $(14.33)_4$. In this case we obtain the following expression for the flexure

$$w = \frac{r_0^4 p}{64D}(1-\rho^2)^2, \tag{14.44}$$

which assumes a maximum value at the center of the plate

$$w_{max} = \frac{r_0^4 p}{64D}. \tag{14.45}$$

Furthermore,

$$\vartheta = \frac{r_0^3 p}{16D}\rho(1-\rho^2),$$
$$M_1 = \frac{r_0^2 p}{16}[1+\nu-(3+\nu)\rho^2], \quad M_2 = \frac{r_0^2 p}{16}[1+\nu-(1+3\nu)\rho^2]. \tag{14.46}$$

The maximum moment occurs at the edge of the plate. If we introduce the corresponding flexural stress

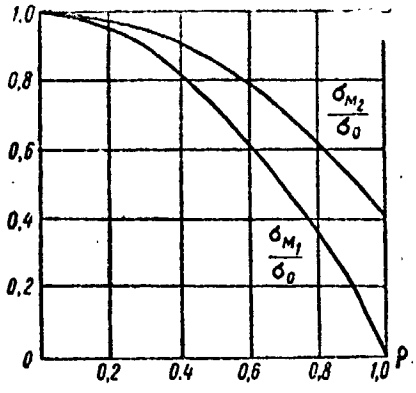

Figure 24

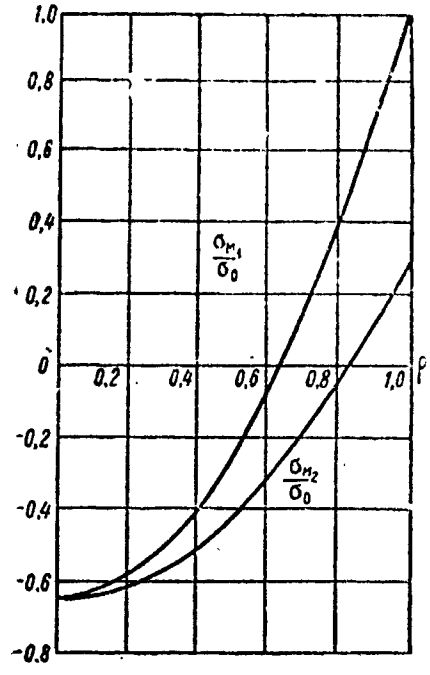

Figure 25

$$\sigma_0 = -\frac{6M_1(1)}{h^2} = \frac{3}{4}\left(\frac{r_0}{h}\right)^2 p, \tag{14.47}$$

we obtain the following expressions for the flexural stresses

$$\sigma_{M_1} = \sigma_0\left\{\frac{3+\nu}{2}\rho^2 - \frac{1+\nu}{2}\right\}, \quad \sigma_{M_2} = \sigma_0\left\{\frac{1+3\nu}{2}\rho^2 - \frac{1+\nu}{2}\right\}. \tag{14.48}$$

The distribution of stresses along the radius is shown in figure 25.

Comparing the corresponding relationships obtained for both cases, /169 we can see that when $\nu = 0.3$ the maximum stresses in a freely supported plate are greater by a factor of 1.65 than those in a clamped plate. Flexure, on the other hand, is approximately four times greater. In actual structures the conditions which fix the plate are intermediate, usually closer to the conditions of free support.

Antisymmetric Case

Inasmuch as

$$\Delta_1 = \frac{1}{r_0^2}\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho, \tag{14.49}$$

the basic equation (14.4) is written in the form

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d\rho w_{\cdot,1}}{d\rho} = \frac{r_0^4 q_{n,1}}{D}. \tag{14.50}$$

Integrating this equation we obtain

$$w_{\cdot,1} = w^*_{\cdot,1} + w^0_{\cdot,1}, \tag{14.51}$$

where

$$\begin{aligned} w^*_{\cdot,1} &= \frac{r_0^4}{D}\left\{\frac{1}{\rho}\int_{\rho_1}^{\rho}\rho'\,d\rho'\int_{\rho_1}^{\rho'}\frac{d\rho''}{\rho''}\int_{\rho_1}^{\rho''}\rho'''\,d\rho'''\int_{\rho_1}^{\rho'''}q_{n,1}(\rho^{IV})\,d\rho^{IV}\right\},\\ w^0_{\cdot,1} &= \frac{r_0^4}{D}\left\{A_1\rho + B_1\rho^3 + C_1\frac{1}{\rho} + D_1\rho\ln\rho\right\}. \end{aligned} \tag{14.52}$$

The following correspond to the established solution

$$\begin{aligned} \vartheta_{\cdot,1} &= \vartheta^*_{\cdot,1} - \frac{r_0^3}{D}\left\{A_1 + 3B_1\rho^2 - C_1\frac{1}{\rho^2} + D_1(\ln\rho + 1)\right\},\\ M_{1,1} &= M^*_{1,1} - r_0^2\left\{2(3+\nu)B_1\rho + 2(1-\nu)C_1\frac{1}{\rho^3} + (1+\nu)D_1\frac{1}{\rho}\right\},\\ M_{2,1} &= M^*_{2,1} - r_0^2\left\{2(1+3\nu)B_1\rho - 2(1-\nu)C_1\frac{1}{\rho^3} + (1+\nu)D_1\frac{1}{\rho}\right\},\\ H_{\cdot,1} &= H^*_{\cdot,1} \pm r_0^2(1-\nu)\left\{2B_1\rho - 2C_1\frac{1}{\rho^3} + D_1\frac{1}{\rho}\right\}, \end{aligned} \tag{14.53}$$

where

$$\begin{aligned} \vartheta^*_{\cdot,1} &= -\frac{1}{r_0}\frac{dw^*_{\cdot,1}}{d\rho},\\ M^*_{1,1} &= -\frac{D}{r_0^2}\left\{\frac{d^2w^*_{\cdot,1}}{d\rho^2} + \nu\left(\frac{1}{\rho}\frac{dw^*_{\cdot,1}}{d\rho} - \frac{1}{\rho^2}w^*_{\cdot,1}\right)\right\}. \end{aligned} \tag{14.54}$$

$$M^*_{2,1} = -\frac{D}{r_0^2}\left\{\nu\frac{d^2w^*_{,1}}{d\rho^2} + \left(\frac{1}{\rho}\frac{dw^*_{,1}}{d\rho} - \frac{1}{\rho^2}w^*_{,1}\right)\right\},$$

$$H^*_{,1} = \pm\frac{(1-\nu)D}{r_0^2}\left\{\frac{1}{\rho}\frac{dw^*_{,1}}{d\rho} - \frac{1}{\rho^2}w^*_{,1}\right\}.$$

Finally

$$Q_{z,1} = Q^*_{z,1} + r_0\left\{2(3+\nu)B_1 + 2(1-\nu)C_1\frac{1}{\rho^4} - (3-\nu)\frac{D_1}{\rho^2}\right\} \qquad (14.55)$$

Here

$$Q^*_{z,1} = -\frac{1}{r_0\rho}\left\{\frac{d\rho M^*_{1,1}}{d\rho} \pm 2H^*_{,1} - M^*_{2,1}\right\}. \qquad (14.56)$$

If we assume that $q_n(\rho)$ has no singularities when $\rho = 0$, all of the /170 quantities with the sign * will be finite. In this case, for a solid plate we should let

$$\rho_1 = \rho_1 = \rho_1^* = \rho_1^{III} = \rho_1^{IV} = 0.$$

It follows from this that for a solid plate the conditions for a finite solution are reduced to the requirement $C_1 = D_1 = 0$. Therefore

$$\begin{aligned}
M_{1,1} &= M^*_{1,1} - r_0^2\{2(3+\nu)B_1\rho\},\\
M_{2,1} &= M^*_{2,1} - r_0^2\{2(3+\nu)B_1\rho\},\\
H_{,1} &= H^*_{,1} \pm r_0^2\{2(1-\nu)B_1\rho\},\\
w_{,1} &= w^*_{,1} + \frac{r_0^4}{D}\{A_1\rho + B_1\rho^3\},\\
\vartheta_{,1} &= \vartheta^*_{,1} - \frac{r_0^3}{D}\{A_1 + 3B_1\rho^2\}.
\end{aligned} \qquad (14.57)$$

From the last two expressions we obtain the elastic rotation (see section 9)

$$\chi = \vartheta_{,1} + \frac{w_{,1}}{r_0\rho} = \chi^* + \frac{r_0^3}{D}\{-2B_1\rho^3\} \quad \left(\chi^* = \vartheta^*_{,1} + \frac{w^*_{,1}}{r_0\rho}\right). \qquad (14.58)$$

Now, by introducing the designation

$$M_0 = M_{1,1}(1), \quad M_0^* = M_{1,1}^*(1), \quad \vartheta_0 = \chi(1), \quad \vartheta_0^* = \chi^*(1).$$

we obtain

$$B_1 = -\frac{M_0 - M_0^*}{2(3+\nu)r_0^2}.$$

If we substitute B_1 into relationship (14.58) and let $\rho = 1$, we obtain the equality

$$\vartheta_0 = \vartheta_0^* + \alpha_{22}(M_0 - M_0^*). \tag{14.59}$$

where

$$\alpha_{22} = \frac{12(1-\nu^2)}{(3+\nu)Eh^2}\left(\frac{r_0}{h}\right). \tag{14.60}$$

Furthermore, for the established value of the constant B_1, equalities (14.57) take the form

$$\begin{aligned}
M_{1,1} &= M_{1,1}^* + (M_0 - M_0^*)\rho, \\
M_{2,1} &= M_{2,1}^* + \frac{1+3\nu}{3+\nu}(M_0 - M_0^*)\rho, \\
H_{,1} &= H_{,1}^* \pm \frac{1-\nu}{3+\nu}(M_0 - M_0^*)\rho, \\
w_{,1} &= w_{,1}^* - \frac{(M_0 - M_0^*)r_0^2}{2(3+\nu)D}\rho^3 \mp r_0\frac{(\omega_y)_0}{(\omega_x)_0}\cdot\rho, \\
\vartheta_{,1} &= \vartheta_{,1}^* + \frac{3(M_0 - M_0^*)r_0^2}{2(3+\nu)D}\rho^2 \pm \frac{(\omega_y)_0}{(\omega_x)_0}.
\end{aligned} \tag{14.61}$$

Here $(w_x)_0$, $(w_y)_0$ are the angles of rotation of the plate as a whole rigid /171 body around the X and Y axes, respectively.

Let us return to the general case of a ring plate. By using equalities (9.22) we compute the tumbling moment acting on the internal contour

$$\pm \begin{matrix} B_y \\ B_x \end{matrix} = \pi r_0 \rho_1 (M_{1,1} + r_0 \rho_1 Q_{z,1}) = \pm \begin{matrix} B_y^* \\ B_x^* \end{matrix} - \pi r_0^3 4 D_1. \tag{14.62}$$

where

$$\begin{matrix} B_y^* \\ B_x^* \end{matrix} = \pi r_0 \rho_1 (M_{1,1}^* + r_0 \rho_1 Q_{z,1}^*).$$

If we let the radius of the internal contour approach 0, then in view of the fact pointed out above that all quantities with the sign * are finite, we arrive at an equality which associates the constant D_1 with the concentrated tumbling moment applied at the center of the plate

$$D_1 = \mp \frac{1}{4\pi r_0^3} \frac{(B_y)_0}{(B_x)_0}.$$

This concentrated action is associated with the following

$$\begin{aligned} M_{1,1} &= \pm \frac{1+\nu}{4\pi r_0} \frac{(B_y)_0}{(B_x)_0} \frac{1}{\rho}, \\ M_{2,1} &= \pm \frac{1+\nu}{4\pi r_0} \frac{(B_y)_0}{(B_x)_0} \frac{1}{\rho}, \\ H_{,1} &= -\frac{1-\nu}{4\pi r_0} \frac{(B_y)_0}{(B_x)_0} \frac{1}{\rho}, \\ w_{,1} &= \mp \frac{r_0}{4\pi D} \frac{(B_y)_0}{(B_x)_0} \rho \ln \rho, \\ \vartheta_{,1} &= \pm \frac{1}{4\pi D} \frac{(B_y)_0}{(B_x)_0} (\ln \rho + 1). \end{aligned} \tag{14.63}$$

15. Circular Plate, Reinforced With Radial Ribs

"Structurally anisotropic" plates play a large role in engineering. This is the name normally given to plates in which the variation in elastic properties as a function of direction is explained not by the properties of the material, but rather by structural peculiarities. Thus, in order to increase the rigidity of shells, they are frequently reinforced with ribs. The placement of ribs substantially increases the rigidity of the plate at the expense of a slight increase in its weight. /172

In the case when the ribs are quite frequent, the calculation of reinforced ribs follows the scheme of "structural anisotropy." The essence of this widely used method is that the rigidity of the ribs during flexure is uniformly distributed (spread out) over the entire shell. In this case a shell

with ribs is replaced with some equivalent plate with different elastic properties in different directions. This actually is what explains the term "structurally anisotropic."

We should note that when using the method of "structural anisotropy" the following question usually arises: how frequent must the ribs be so that the solution which is obtained gives stresses and strains which are close to those actually occurring in the shell? Comparison with existing experimental data shows that the method is highly effective, even when the distribution of reinforcements is sparse. The same conclusion is reached by considering the few existing "exact" solutions of certain specific problems in this class. Below we present one of such solutions (ref. 190).

Let us consider a circular plate fortified by m-equally spaced ribs of constant cross section. As we know, the equation for the bent axis of the rib (beam) has the form

$$\frac{d^4w^b}{d\rho^4}=\frac{r_0^4}{EI}q^b(\rho). \tag{15.1}$$

Here I is the moment of inertia of the rib with respect to the neutral line of cross section, while $q^b(\rho)$ is the linear load distribution.

We specify the elastic line of the rib by means of the following expression

$$w^b=\sum_{s=0}^{N}\Phi_s\rho^s, \tag{15.2}$$

where Φ_s is, at this time, an undetermined constant. Substituting this expression into (15.1), we find that the linear load distribution necessary for bending the rib in a manner shown by (15.2) has the form

$$q^b(\rho)=\frac{EI}{r_0^4}\sum_{s=4}^{N}\Phi_s p_s\rho^{s-4}, \tag{15.3}$$

where

$$p_s=s(s-1)(s-2)(s-3). \tag{15.4}$$

For simplicity we shall assume that the axis for the beam lies in the middle plane of the plate. Compatibility in the operation of the plate and of the rib is reduced to the fact that the following coincide:

(a) the flexures of the ribs and of the plates at the points where they join /173

$$w^{pl} = w^{b}, \tag{15.5}$$

(b) the loading of the plate from the side of the rib and loading on the rib from the side of the plate

$$q^{pl} = -q^{b} \tag{15.6}$$

(The minus sign in the last equality is explained by the fact that the positive directions of the normals in the plane and in the ribs coincide.)

Assuming that the width of a rib is small, we represent the action of the rib on the plate by a linear load acting along the radius which coincides with the axis of the rib. In this case, according to (15.3) and (15.6) the intensity of this load is equal to

$$q^{pl} = -\frac{EI}{r_0^4}\sum_{s=4}^{N}\Phi_{s}\mu_{s}\rho^{s-4}.$$

The load acting on the plate from all ribs represents m-equally separated loads (fig. 26). The following Fourier expansion is valid for a load of this type

$$q(\rho, \varphi) = -\frac{mEI}{2\pi r_0^5}\left(\sum_{s=4}^{N}\Phi_{s}\mu_{s}\rho^{s-5}\right)\left(1 + 2\sum_{k=1}^{\infty}\cos km\varphi\right).$$

Let us assume that the plate reinforced with the ribs is loaded with a uniform pressure p. Then the total load on the plate is equal to

$$q_n(\rho, \varphi) = p + q(\rho, \varphi) = p + \sum_{k=0}^{\infty} q_{n,km}\cos km\varphi. \tag{15.7}$$

where

$$q_{n,\,km}=-\frac{mEI}{2\pi r_0^5}2\left(\sum_{s=4}^{N}\Phi_s\mu_s\rho^{s-5}\right)(k\geqslant 1),$$
$$q_{n,\,0}=-\frac{mEI}{2\pi r_0^5}\left(\sum_{s=4}^{N}\Phi_s\mu_s\rho^{s-5}\right). \tag{15.8}$$

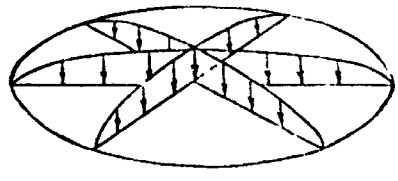

Figure 26

The uniform pressure is associated with normal flexure

$$w^0=\frac{r_0^4}{D}\left\{\frac{p}{64}\rho^4+A_0+B_0\rho^2\right\}; \tag{15.9}$$

In this case by selecting the constants A_0 and B_0 in a suitable manner /174
we can satisfy the boundary conditions at the edge of the plate.

Now let us consider the terms which take into account the action of the ribs. Comparing (14.12) with (15.7) we see that, in the considered problem,

$$G_{km}^{(s-5)}=-2\gamma_k\frac{mEI}{2\pi r_0^5}\mu_s\Phi_s, \tag{15.10}$$

where

$$\gamma_k=\begin{cases}1 & k\geqslant 1,\\ 1/2 & k=0.\end{cases}$$

Therefore, the additional flexure may be written in the form

$$w^{\mathrm{d}}=-2\lambda\sum_{k=0}^{\infty}\left\{\sum_{s=4}^{N}\gamma_k\left[\frac{\mu_s}{[(s-1)^2-(km)^2]\,[(s-3)^2-(km)^2]}\rho^{s-1}+\right.\right.$$
$$+A_{km}^s\rho^{km}+B_{km}^s\rho^{km+2}\Big]\Phi_s+$$
$$+\left[-\frac{\mu_s}{8km\,(km-1)}\rho^{km}\ln\rho+A_{km}^s\rho^{km}+B_{km}^s\rho^{km+2}\right]\Phi_s\delta_s^{km+1}+$$
$$\left.+\left[\frac{\mu_s}{8km\,(km+1)}\rho^{km+2}\ln\rho+A_{km}^s\rho^{km}+B_{km}^s\rho^{km+2}\right]\Phi_s\delta^{km+3}\right\}\cos km\varphi.$$

Here $\lambda = \frac{mEI}{2\pi r_0 D}$ is the ratio of the overall rigidity of the ribs to the flexural rigidity of the plate's edge. It is appropriate to call this quantity the relative rigidity of the ribs. Combining this solution with (15.9), we obtain

$$w = w^0 - \lambda \sum_{s=4}^{N} \Phi_s w^{(s)}. \tag{15.11}$$

Here

$$\begin{aligned} w^{(s)} = 2\Bigg\{ \sum_{k=0}^{\infty}{}' \gamma_k \Bigg[& \frac{\mu_s}{[(s-1)^2-(km)^2][(s-3)^2-(km)^2]} \rho^{s-1} + A^s_{km}\rho^{km} + \\ & + B^s_{km}\rho^{km+2}\Bigg] \cos km\varphi + \\ & + \Bigg[-\frac{\mu_s}{8km(km-1)} \rho^{km} \ln \rho + A^s_{km}\rho^{km} + B^s_{km}\rho^{k\cdot} \Bigg] \cos km\varphi\, \delta_s^{km+1} + \\ & + \Bigg[\frac{\mu_s}{8k\,n\,(km+1)} \rho^{km+2} \ln \rho + A^s_{km}\rho^{km} + B^s_{km}\rho^{km+2} \Bigg] \cos km\varphi \delta_s^{km+3} \Bigg\}. \end{aligned} \tag{15.12}$$

The constants A^s_{km} and B^s_{km} contained in these equations are selected from the condition that the expressions in each square bracket in (15.12) satisfy the respective homogeneous boundary conditions of the problem. Then the expression for the total flexure (15.11) for any values of λ and Φ_s satisfies the boundary conditions of the problem.

Of the conditions formulated above for the compatible operation of the plate and of the ribs, the second is satisfied automatically because we have used it in deriving relationships (15.7). In regard to the first one (compatibility of flexure), it leads to the conditions for determining the constants Φ_s. The flexure of the plate under the rib (since the distribution of the ribs is symmetric, it is sufficient to consider the first rib for which $\varphi = 0$) is equal to /175

$$w(\rho, 0) = w^0(\rho) - \lambda \sum_{s=4}^{N} \Phi_s w^{(s)}(\rho, 0), \tag{15.13}$$

where

$$w^{(s)}(\rho, 0) = 2\Bigg\{ \sum_{k=3}^{\infty}{}' \gamma_k \Bigg[\frac{\mu_s}{[(s-1)^2-(km)^2][(s-3)^2-(km)^2]} \rho^{s-1} + \tag{15.14}$$

$$+A^s_{km}\rho^{km}+B^s_{km}\rho^{km+2}\Big]+$$
$$+\Big[-\frac{\mu_s}{8km(km-1)}\rho^{km}\ln\rho+A^s_{km}\rho^{km}+B^s_{km}\rho^{km+2}\Big]\delta_s^{km+1}+$$
$$+\Big[\frac{\mu_s}{8km(km+1)}\rho^{km+2}\ln\rho+A^s_{km}\rho^{km}+B^s_{km}\rho^{km+2}\Big]\delta_s^{km+3}\Big\}.$$

As we can see from (15.13) the constants Φ_0, Φ_1, Φ_2, Φ_3 which determine the flexure of the rib have no direct effect on the magnitude of the plate flexure. They are determined from the boundary conditions for the rib.

Let us assume, for example, that the rib is rigidly fixed along the contour of the plate, i.e.,

$$w^b(1)=\left(\frac{dw^b}{d\rho}\right)_{\rho=1}=0. \tag{15.15}$$

We obtain two other conditions, assuming that at the other end of the rib (when $\rho = 0$) the angle of rotation and the cutting force are absent

$$\left(\frac{dw^b}{d\rho}\right)_{\rho=0}=\left(\frac{d^3w^b}{d\rho^3}\right)_{\rho=0}=0. \tag{15.16}$$

Substituting the expression for w^b into conditions (15.15) and (15.16) we find

$$\Phi_1=\Phi_3=0,\ \Phi_0+\Phi_2+\sum_{s=4}^{N}\Phi_s=0,$$
$$2\Phi_2+\sum_{s=4}^{N}s\Phi_s=0,$$

from which it follows that

$$\Phi_0=\sum_{s=4}^{N}\left(\frac{s}{2}-1\right)\Phi_s,\quad \Phi_2=-\frac{1}{2}\sum_{s=4}^{N}s\Phi_s.$$

Thus, for a beam with a fixed end, the expression for flexure has the following form /176

$$w^b = \sum_{s=4}^{N} \Phi_s \left\{ \frac{s}{2} - 1 - \frac{s}{2}\rho^2 + \rho^s \right\}. \qquad (15.17)$$

When the number of parameters Φ_s is finite, it is of course impossible to achieve an exact equivalence in the flexures of the plate and of the rib at their junction. We can only speak of the approximate coincidence in some sense or other. For example, we may require that the flexures coincide at N-3 points of the radius (according to the number of undetermined parameters Φ_s). In this case, if the expressions for the flexures are smooth, there is good coincidence at the intermediate points.

As an example, we should consider a plate reinforced with six ribs and clamped along the edges. The solution for $w^{(0)}$, when the edge is clamped, is presented in section 14. As far as the expressions for $w^{(s)}$ are concerned, for them the boundary condition pertaining to a clamped plate $w'^{(s)}(1) = \left(\frac{dw^{(s)}}{d\rho}\right)_{\rho=1} = 0$ will be satisfied if we let

$$\left.\begin{aligned} A^s_{km} &= \frac{\mu_s}{2[(s-1)^2-(km)^2][s-3+km]}, \\ B^s_{km} &= -\frac{\mu_s}{2[s-1+km][(s-3)^2-(kn)^2]} \end{aligned}\right\} \text{ when } \begin{matrix} s \neq 6k+1 \\ s \neq 6k+3, \end{matrix} \qquad (15.18)$$

$$A^s_{km} = B^s_{km} = \frac{\mu_s}{16\,km\,(km-1)} \text{ when } s = 6k+1,$$

$$A^s_{km} = -B^s_{km} = \frac{\mu_s}{16\,km\,(km+1)} \text{ when } s = 6k+3.$$

The following values were assumed for the relative rigidity in the numeric calculations carried out in reference 190: $\lambda = 0;\ 0.1;\ 0.3;\ 0.5$. Three approximations were computed for each of the values specified for λ.

First Approximation. One arbitrary constant remained in expression (15.13)

$$w(\rho, 0) = w^0(\rho) - \lambda\Phi_4 w^{(4)}(\rho, 0).$$

This constant was determined from the condition that the flexures of the plate end of the rib coincide when $\rho = 0$

$$w\ (0.0) = w^b\ (0).$$

Second Approximation. Two constants were retained

$$w(\rho, 0) = w^0(\rho) - \lambda\left[\Phi_4 w^{(4)}(\rho, 0) + \Phi_5 w^{(5)}(\rho, 0)\right].$$

In this case the constants were determined from the system

$$w(0.0) = w^b(0),\ w(0.5;\ 0) = w^b(0.5). \qquad \text{/177}$$

Third Approximation. Three constants were retained

$$w(\rho, 0) = w^0(\rho) - \lambda\left\{\Phi_4 w^{(4)}(\rho, 0) + \Phi_5 w^{(5)}(\rho, 0) + \Phi_6 w^{(6)}(\rho, 0)\right\}.$$

The constants were determined by means of the following equations

$$w(0;\ 0) = w^b(0),\ w(0.25;\ 0) = w^b(0.25),\ w(0.625;\ 0) = w^b(0.625).$$

The flexures, angles of rotation and stresses were computed by using the established constants. The results of these simple, although somewhat cumbersome computations, were compared. A comparison was also made with the solution obtained by the method of "structural anisotropy."

The results of the calculations showed that the method which was utilized has good convergence. For the practical application of this method it is sufficient to use three approximations. Figures 27 and 28 compare stresses $\sigma_r^0 = \sigma_{M_r}\left/\frac{3p}{32}\left(\frac{r_0}{h}\right)^2\right.$ and $\sigma_\theta^0 = \sigma_{M_\theta}\left/\frac{3p}{32}\left(\frac{r_0}{h}\right)^2\right.$, computed by means of the third approximation with analogous computations carried out by using the "structural anisotropy" method (see ref. 88). In these figures the solid lines designate the stresses under the ribs, the dot and dash lines designate the stresses between the ribs and the broken lines designate the stresses in the "anisotropic" plate.

Figures 29 and 30 demonstrate the effect of relative rib rigidity on the value of stresses in the plate. We note that when $\lambda = 0.5$ the distributions of the angles of rotation computed by the two methods practically coincide. In this case the maximum angle of rotation in the reinforced plate decreases by 40 percent, compared with its value for the unreinforced plate.

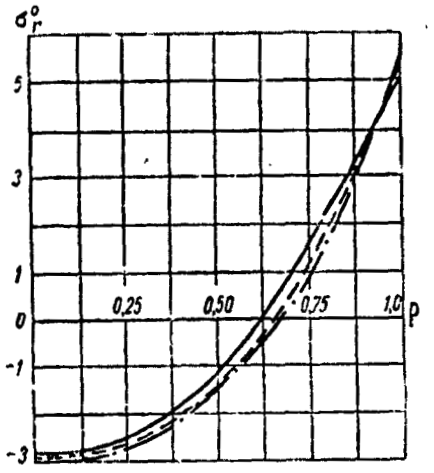

Figure 27

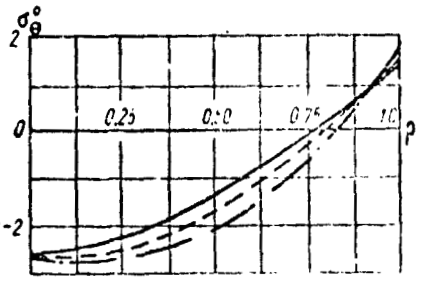

Figure 28

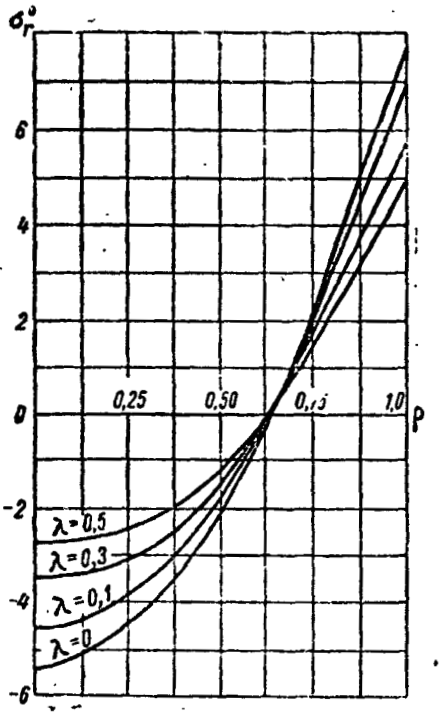

Figure 29

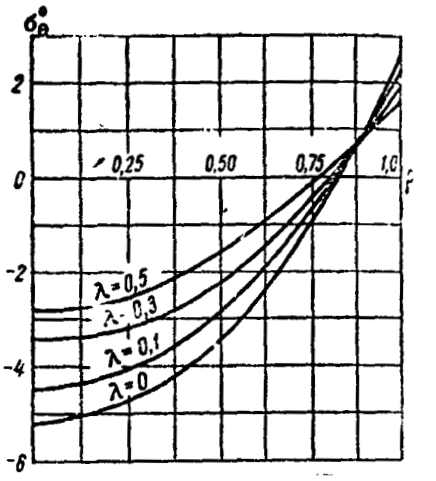

Figure 30

The solution which we have obtained for this particular problem makes it possible for us to draw conclusions of a general nature:

(a) the installment of ribs with small or medium relative rigidity ($\lambda \leq 0.5$) substantially reduces the flexure, the angle of rotation and stresses and is therefore expedient;

(b) for ribs with medium relative rigidity the method of "structural anisotropy" produces useful results even when the spacing of the ribs is sparse.

16. Generalized Plane State of Stress

By utilizing the real and imaginary parts of relationships in section 13, we can consider yet another case of plate deformation when the plate is deformed in its own plane under the action of external tangential forces. As above, we limit ourselves to the method involving the separation of variables and obtain

$$T_{1,k}=T^{*}_{1,k}-\frac{Ehc}{r_0^2}\left\{\frac{1}{\rho}\frac{d\bar{w}_{,k}}{d\rho}-\frac{k^2}{\rho^2}\bar{w}_{,k}\right\},$$
$$T_{2,k}=T^{*}_{2,k}-\frac{Ehc}{r_0^2}\left\{\frac{d^2\bar{w}_{,k}}{d\rho^2}\right\},$$
$$S_{,k}=S^{*}_{,k}\mp\frac{Ehc}{r_0^2}\left\{\frac{k}{\rho}\left(\frac{d\bar{w}_{,k}}{d\rho}-\frac{\bar{w}_{,k}}{\rho}\right)\right\}; \tag{16.1}$$
$$(r_0\varepsilon_{1,k})\frac{du_{,k}}{d\rho}=\frac{r_0}{Eh}(T^{*}_{1,k}-\nu T^{*}_{2,k})-\frac{c}{r_0}\left\{\frac{1}{\rho}\frac{d\bar{w}_{,k}}{d\rho}-\frac{k^2}{\rho^2}\bar{w}_{,k}-\nu\frac{d^2\bar{w}_{,k}}{d\rho^2}\right\},$$

$$(r_0\varepsilon_{2,k})\ \frac{1}{\rho}(\pm kv_{,k}+u_{,k})=\frac{r_0}{Eh}(T^{*}_{2,k}-\nu T^{*}_{1,k})-\frac{c}{r_0}\left\{\frac{d^2\bar{w}_{,k}}{d\rho^2}-\right.$$
$$\left.-\nu\left(\frac{1}{\rho}\frac{d\bar{w}_{,k}}{d\rho}-\frac{k^2}{\rho^2}\bar{w}_{,k}\right)\right\}, \tag{16.2}$$
$$(r_0\omega_{,k})\rho\frac{d}{d\rho}\left(\frac{v_{,k}}{\rho}\right)\mp k\frac{u_{,k}}{\rho}=\frac{2(1+\nu)S^{*}_{,k}}{Eh}\mp 2(1+\nu)\frac{c}{r_0}\frac{k}{\rho}\left(\frac{d\bar{w}_{,k}}{d\rho}-\frac{\bar{w}_{,k}}{\rho}\right);$$

$$Q_{\rho,k}=T_{1,k},\quad Q_{12,k}=S_{,k},$$
$$\varkappa_{z,k}=\frac{1}{r_0\rho}(\mp kv_{,k}-k^2u_{,k}),\quad \varepsilon_{2,k}=\frac{1}{r_0\rho}(\pm kv_{,k}+u_{,k}). \tag{16.3}$$

In this case the quantities $T^{*}_{1,k}$, $T^{*}_{2,k}$, $S^{*}_{,k}$ are obtained from the system

$$\frac{d\rho T^{*}_{1,k}}{d\rho}\pm kS^{*}_{,k}-T^{*}_{2,k}=-r_0\rho q_{1,k},$$
$$\frac{1}{\rho}\frac{d\rho^2S^{*}_{,k}}{d\rho}\mp kT^{*}_{2,k}=-r_0\rho q_{2,k}, \tag{16.4}$$

while the stress function $\bar{w}_{,k}$ is a solution of the equation

$$\Delta_k \Delta_k \bar{w}_{,k} = \frac{1}{Ehc}\left\{\Delta_k T^*_{,k} + \frac{1+\nu}{r_0 \rho}\left(\frac{d}{d\rho} q_{1,k} \pm k q_{2,k}\right)\right\} \qquad (16.5)$$
$$\left(\Delta_k = \frac{1}{r_0^2}\left\{\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{k^2}{\rho^2}\right\}\right).$$

With three of the unknown functions at our disposal, it is easy to satisfy the two equations (16.4). In regard to equation (16.5), a solution for an equation of the same type was considered in section 14. Bearing this in mind we shall consider in detail only the symmetric cases and antisymmetric case whose solution is associated with certain difficulties.

The Second Symmetric Case

From (16.1), (16.4), we find

$$S_{,0} = S^*_{,0} = \frac{1}{\rho^2}\left\{-r_0 \int_{\rho_1}^{\rho} \rho' q_{2,k} d\rho' + \frac{C_2}{r_0^2}\right\}. \qquad (16.6)$$

From this and from $(16.2)_3$ it follows that

$$v_{,0} = \rho\left\{a_2 r_0 + \frac{2(1+\nu)}{Eh}\int_{\rho_1}^{\rho} \frac{S^*_{,0}}{\rho'} d\rho'\right\}. \qquad (16.7)$$

In the case of a circular plate (without a hole) we should let $C_2 = 0$. As in section 14, we can show that the retention of this constant signifies that a concentrated twisting moment is applied at the center of the plate.

$$(B_z)_0 = 2\pi C_2. \qquad (16.8)$$

The following correspond to this twisting moment /180

$$S^*_{,0} = \frac{(B_z)_0}{2\pi r_0^2}\frac{1}{\rho^2}, \quad v_{,0} = -\frac{(1+\nu)(B_z)_0}{2\pi r_0 Eh}\frac{1}{\rho}. \qquad (16.9)$$

The First Symmetric Case

The basic equation takes the form

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\bar{w}}{d\rho}=\frac{r_0^2}{\bar{E}}\left\{\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dT^*}{d\rho}+\frac{(1+\nu)r_0}{\rho}\frac{d\rho q_1}{d\rho}\right\}.$$

Integrating it once, we obtain

$$\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\bar{w}}{d\rho}=\frac{r_0^2}{Ehc}\left\{\rho\frac{dT^*}{d\rho}+(1+\nu)\rho r_0 q_1+D_0\right\}. \tag{16.10}$$

We shall show that the constant D_0 contained in the above equation should be set to zero. By utilizing equalities (16.1), the relationships of Hooke's law and the first of the equilibrium equations (16.4), we first obtain

$$r_0\left\{\frac{d\rho\varepsilon_2}{d\rho}-\varepsilon_1\right\}=\frac{r_0}{Eh}\left\{\rho\frac{dT^*}{d\rho}+(1+\nu)\rho r_0 q_1\right\}-\frac{c}{r_0}\left\{\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\bar{w}}{d\rho}\right\}.$$

A comparison of this equation with (16.10) shows that

$$-\frac{r_0}{c}\left\{\frac{d\rho\varepsilon_2}{d\rho}-\varepsilon_1\right\}=D_0. \tag{16.11}$$

In view of the first two equations from (16.2), the left part of the last equality and consequently the constant D_0 are equal to zero.

It is also easy to establish the geometric meaning when the constant $D_0 = 0$. For this purpose we must first remember that in accordance with (1.15), (1.17), in the case considered,

$$\varkappa_z=\varkappa_{2\alpha}=-\frac{1}{r_0\rho}\left(\frac{d\rho\varepsilon_2}{d\rho}-\varepsilon_1\right)=\frac{c}{r_0^2\rho}D_0.$$

Furthermore, in section 6 it was established that the equality $\varkappa_z = 0$ provides a single value for the displacement v. Therefore the equality $D_0 = 0$ is one of the conditions providing for single-valued displacements.

Dropping D_0 in (16.10) and continuing the integration we obtain

$$\bar{w} = \bar{w}^{**} + \bar{w}^{0}, \tag{16.12}$$

where

$$\bar{w}^{**} = \frac{r_0^2}{Ehc}\left\{\int_{\rho_1}^{\rho}\frac{d\rho'}{\rho'}\int_{\rho_1'}^{\rho'}\rho''T^{*}d\rho'' + (1+\nu)\int_{\rho_1}^{\rho}\frac{d\rho'}{\rho'}\int_{\rho_1'}^{\rho'}\rho''d\rho''\int_{\rho_1''}^{\rho''}q_1 d\rho'''\right\}, \tag{16.13}$$

$$\bar{w}^{0} = \frac{r_0^2}{Ehc}(A_0 + B_0\rho^2 + C_0\ln\rho).$$

The established stress functions are associated with the following /181

$$\begin{aligned} T_1 &= T_1^{*} - \frac{Ehc}{r_0^2}\frac{1}{\rho}\frac{d\bar{w}^{**}}{d\rho} - \left\{2B_0 + \frac{C_0}{\rho^2}\right\},\\ T_2 &= T_2^{*} - \frac{Ehc}{r_0^2}\frac{d^2\bar{w}^{**}}{d\rho^2} - \left\{2B_0 - \frac{C_0}{\rho^2}\right\},\\ u &= \frac{r_0\rho}{Eh}(T_2^{*} - \nu T_1^{*}) - \frac{c}{r_0}\left(\rho\frac{d^2\bar{w}^{**}}{d\rho^2} - \nu\frac{d\bar{w}^{**}}{d\rho}\right) - \\ &\quad - \frac{r_0}{Eh}\left(2(1-\nu)B_0\rho - (1+\nu)\frac{C_0}{\rho}\right). \end{aligned} \tag{16.14}$$

In the case of a solid circular plate we should let $C_0 = 0$.

$$\rho_1 = \rho_1' = \rho_1'' = \rho_1''' = 0.$$

We introduce the usual designations

$$\begin{aligned} Q_0 &= T_1(1),\ u_0 = u(1),\ Q_0^{*} = \left\{T_1^{*} - \frac{Ehc}{r_0^2}\frac{d\bar{w}^{**}}{d\rho}\right\}_{\rho=1},\\ u_c^{*} &= \left\{\frac{r_0}{Eh}(T_2^{*} - \nu T_1^{*}) - \frac{c}{r_0}\left(\frac{d^2\bar{w}^{**}}{d\rho^2} - \nu\frac{d\bar{w}^{**}}{d\rho}\right)\right\}_{\rho=1}. \end{aligned} \tag{16.15}$$

Then from $(16.14)_1$ we find

$$B_0 = -\frac{Q_0 - Q_0^{*}}{2}.$$

Substituting B_0 into relationship $(16.14)_3$ and assuming that $\rho = 1$, we obtain

$$u_0 = u_1^* + a_{11}(Q_0 - Q_0^*), \tag{16.16}$$

where

$$a_{11} = \frac{1-\nu}{E}\left(\frac{r_0}{h}\right). \tag{16.17}$$

Finally, in accordance with $(16.14)_{1,2}$ we have

$$\begin{aligned} \sigma_{r_1} &= \frac{Q_0 - Q_0^*}{h} + \frac{1}{h}\left(T_1^* - \frac{Ehc}{r_0^2}\frac{1}{\rho}\frac{d\bar{w}^{**}}{d\rho}\right), \\ \sigma_{r_2} &= \frac{Q_0 - Q_0^*}{h} + \frac{1}{h}\left(T_2^* - \frac{Ehc}{r_0^2}\frac{d^2\bar{w}^{**}}{d\rho^2}\right). \end{aligned} \tag{16.18}$$

The most interesting form of surface loading in the case considered by us is a centrifugal force, which is the design load for a disk rotating with a constant angular velocity ω_0. In this case

$$q_1 = Br_0\rho. \tag{16.19}$$

Here $B = m\omega_0^2 h$, $m = \frac{\gamma}{g}$ is the density of the disk material. For this loading T_1^* and T_2^* are determined from the equation

$$\frac{d_\rho T_1^*}{d\rho} - T_2^* = -r_0^2 B\rho^2.$$

One of the unknown functions may be assigned arbitrarily. From the general considerations presented above, it follows that the arbitrary manner in which the functions of the static system are selected should not be reflected in the magnitude of the unknown forces and displacements. In the relatively simple case which is being considered, this can be understood directly. In this connection we select two versions of solutions for the equations /182

(a) $T_1^* = 0$, $T_2^* = r_0^2 B\rho^2$;

(b) $T_1^* = -\frac{1}{3}r_0^2 B\rho^3$, $T_2^* = 0$.

Substituting them into the relationships written above, we become convinced that both versions lead to the same expressions

$$Q_0^* = -\frac{3+\nu}{8} r_0^2 B, \qquad M_0^* = -\frac{1-\nu^2}{8} \frac{r_0^2 B}{Eh},$$
$$\sigma_{T_1} = \frac{Q_0}{h} + \frac{3+\nu}{8}(1-\rho^2)\frac{r_0^2 B}{h},$$
$$\sigma_{T_2} = \frac{Q_0}{h} + \left(\frac{3+\nu}{8} - \frac{1+3\nu}{8}\rho^2\right)\frac{r_0^2 B}{h}, \tag{16.20}$$
$$u = \frac{r_0^2 B}{Eh}\left\{\frac{1-\nu}{8}\rho\left[3+\nu-(1+\nu)\rho^2\right]\right\} + (1-\nu)\frac{r_0\rho}{Eh} Q_0.$$

For a disk with a free edge it is necessary to let $Q_0 = 0$. Equations (16.20) show that in this case these stresses, which are tensile everywhere, achieve their maximum value at the center of the disk.

The Antisymmetric Case

If we have in mind the application of the theory of plates to the computation of complex shells of rotation which are composed of circular plates, the homogeneous problem is of the greatest interest in antisymmetric cases. Therefore, even though the consideration of the general case is not very difficult, we shall limit ourselves to the homogeneous problem.

By using equalities (14.49) the basic equation may be presented in the form

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d\rho\bar{w}_{,1}}{d\rho} = \frac{r_0^2}{Ehc}\left\{\frac{d}{d\rho}\frac{1}{\rho}\frac{d\rho T_{,1}^*}{d\rho}\right\}.$$

Integrating this equation successively we find

$$\bar{w}_{,1} = \frac{r_0^2}{Ehc}\left\{\frac{1}{\rho}\int_{\rho_1}^{\rho}\rho'\,d\rho'\int_{\rho_1}^{\rho'} T_{,1}^*\,d\rho'' + A_1\rho + B_1\rho^3 + C_1\frac{1}{\rho} + D_1\rho\ln\rho\right\}. \tag{16.21}$$

According to (16.4), the functions of the static system are determined from the equations /183

$$\frac{d\rho T_{1,1}^*}{d\rho} \pm S_{,1}^* - T_{2,1}^* = 0,$$
$$\frac{1}{\rho}\frac{d\rho S_{,1}^*}{d\rho} \mp T_{2,1}^* = 0. \tag{16.22}$$

Multiplying the second equation by ∓ 1 and adding it to the first, we obtain

$$\frac{d\rho(T^*_{1,1} \mp S^*_{,1})}{d\rho} = 0.$$

Integrating this equality and utilizing the second equation of (16.22), we obtain

$$T^*_{1,1} = \frac{C_3}{C_5} \cdot \frac{1}{r_0\rho} \pm S^*_{,1}, \quad T^*_{2,1} = \pm \frac{1}{\rho} \frac{d_\rho^2 S^*_{,1}}{d\rho}. \tag{16.23}$$

The function $S^*_{,1}$ which is contained in these equations may be selected arbitrarily by setting it, for example, equal to zero. In this case

$$T^*_{1,1} = \frac{C_3}{C_5} \cdot \frac{1}{r_0\rho}, \qquad T^*_{2,1} = S^*_{,1} = 0. \tag{16.24}$$

Substituting the last expressions into (16.21) and computing the forces from the established value $\overline{w}_{,1}$, we find

$$\begin{aligned} T_{1,1} &= \frac{C_3}{C_5} \cdot \frac{1}{r_0\rho} - 2B_1\rho + 2C_1\frac{1}{\rho^3} - \frac{D_1}{\rho}, \\ T_{2,1} &= -6B_1\rho - 2C_1\frac{1}{\rho^3} - \frac{D_1}{\rho}, \\ \pm S_{,1} &= -2B_1\rho + 2C_1\frac{1}{\rho^3} - \frac{D_1}{\rho}. \end{aligned} \tag{16.25}$$

The displacements are determined from the system

$$\begin{aligned} &(r_0\varepsilon_{1,1}) \qquad \frac{du_{,1}}{d\rho} = \frac{r_0}{Eh}(T_{1,1} - \nu T_{2,1}), \\ &(r_0\varepsilon_{2,1}) \qquad u_{,1} \pm v_{,1} = \frac{r_0\rho}{Eh}(T_{2,1} - \nu T_{1,1}), \\ &(\pm r_0\omega_{,1}) \qquad \rho\frac{d}{d\rho}\left(\frac{v_{,1}}{\rho}\right) \mp \frac{u_{,1}}{\rho} = \frac{2(1+\nu)r_0}{Eh} S_{,1}. \end{aligned} \tag{16.26}$$

As in the symmetric cases we can show that the condition for the compatibility of this system is the equality

$$D_1 = \frac{1-\nu}{4r_0} \cdot \frac{C_3}{C_5}, \tag{16.27}$$

which provides a single-value for the displacements in a doubly connected region.

Taking into account equalities (16.27) the following relationships define the displacements

$$\begin{aligned} u_{,1} &= \frac{r_0}{Eh}\left\{\frac{3+2\nu-\nu^2}{4} \cdot \frac{C_3}{C_5}\frac{\ln\rho}{r_0} - \right. \\ &\left. -(1-3\nu)B_1\rho^2 - (1+\nu)C_1\frac{1}{\rho^2}\right\} + \frac{a_3}{a_5}, \\ \pm v_{,1} &= \frac{r_0}{Eh}\left\{-\left[\frac{(1+\nu)^2}{4} + \frac{3+2\nu-\nu^2}{4}\ln\rho\right]\frac{1}{r_0} \cdot \frac{C_3}{C_5} - \right. \\ &\left. -(5+\nu)B_1\rho^2 - (1+\nu)C_1\frac{1}{\rho}\right\} - \frac{a_3}{a_5}. \end{aligned} \tag{16.28}$$

Here a_3 is a rigid displacement of the plate along the X-axis, while a_5 /184 is a displacement along the Y-axis.

The forces, on the other hand, are determined from the equalities

$$\begin{aligned} T_{1,1} &= \frac{3+\nu}{4}\frac{C_3}{C_5} \cdot \frac{1}{r_0\rho} - 2B_1\rho + 2C_1\frac{1}{\rho^2}, \\ T_{2,1} &= -\frac{1-\nu}{4}\frac{C_3}{C_5} \cdot \frac{1}{r_0\rho} - 6B_1\rho - 2C_1\frac{1}{\rho^2}, \\ \pm S_{,1} &= -\frac{1-\nu}{4}\frac{C_3}{C_5} \cdot \frac{1}{r_0\rho} - 2B_1\rho + 2C_1\frac{1}{\rho^2}. \end{aligned} \tag{16.29}$$

For a solid plate the conditions for the finite value of forces and displacements are reduced to the requirement $C_3 = C_5 = C_1 = 0$. In this case

$$\begin{aligned} T_{1,1} &= -2B_1\rho,\ T_{2,1} = -6B_1\rho,\ \pm S_{,1} = -2B_1\rho, \\ u_{,1} &= -(1-3\nu)\frac{r_0}{Eh}B_1\rho^2 + \frac{a_3}{a_5}, \\ \pm v_{,1} &= -(5+\nu)\frac{r_0}{Eh}B_1\rho^2 - \frac{a_3}{a_5}, \\ r_0\rho e_{2,1} &= u_{,1} \pm v_{,1} = -2(3-\nu)\frac{r_0}{Eh}B_1\rho^2. \end{aligned} \tag{16.30}$$

Identifying r_0 with the radius of the plate, we obtain the following at the edge ($\rho = 1$)

$$Q_0 = T_{1,1}(1) = -2B_1,$$
$$u_0 = (r_0 \rho \varepsilon_{2,1})_{\rho=1} = -(3-\nu) 2B_1 \frac{r_0}{Eh}. \tag{16.31}$$

From the first equality we find $B_1 = -\frac{Q_0}{2}$. Substituting B_1 into the second equality, we obtain

$$u_0 = a_{11} Q_0, \tag{16.32}$$

where $a_{11} = \frac{3-\nu}{E}\left(\frac{r_0}{h}\right)$. Furthermore it follows from (16.30) that

$$\sigma_{T_{1,1}} = \frac{Q_0}{h}\rho, \quad \pm \sigma_{s,1} = \frac{Q_0}{h}\rho, \quad \sigma_{T_{2,1}} = \frac{3Q_0}{h}\rho. \tag{16.33}$$

Using an approach analogous to the one considered above, it can be shown that for a solid plate the retention of terms with $\begin{matrix} C_3 \\ C_5 \end{matrix}$ means that there is a concentrated tangential force applied at the center of the plate. In this case

$$(F_x)_0 = \pi C_3, \qquad (F_y)_0 = \pi C_5. \tag{16.34}$$

This concentrated force is associated with the following

$$T_{1,1} = \frac{3+\nu}{4\pi r_0} \cdot \frac{(F_x)_0}{(F_y)_0} \cdot \frac{1}{\rho}, \quad T_{2,1} = -\frac{1-\nu}{4\pi r_0} \cdot \frac{(F_x)_0}{(F_y)_0} \cdot \frac{1}{\rho},$$
$$\pm S_{,1} = -\frac{1-\nu}{4\pi r_0} \cdot \frac{(F_x)_0}{(F_y)_0} \cdot \frac{1}{\rho}, \tag{16.35}$$
$$u_{,1} = \frac{1}{\pi Eh} \frac{(F_x)_0}{(F_y)_0}\left(\frac{3+2\nu-\nu^2}{4} \ln \rho\right),$$
$$\pm v_{,1} = -\frac{1}{\pi Eh} \cdot \frac{(F_x)_0}{(F_y)_0}\left(\frac{3+2\nu-\nu^2}{4} \ln \rho + \frac{(1+\nu)^2}{4}\right).$$

17. The Calculation of Pressure Vessels

Let us apply the general relationships which we have obtained to the calculation of pressure vessels and their end plates. For these vessels the design load is a uniform normal pressure /185

$$q_n = p = \text{const} \qquad (q_1 = 0, \quad q_2 = 0).$$

For this load the integral contained in the expression for U is taken when the form of the meridian is arbitrary. Indeed, in this case

$$U = p \int_{\theta_0}^{\theta} R_1 R_2 \sin\theta \cos\theta \, d\theta + C_1 =$$

$$= p \int_{\theta_0}^{\theta} (R_2 \sin\theta) \, d(R_2 \sin\theta) + C_1 = \frac{p(R_2 \sin\theta)^2}{2} + C.$$

From this equation and from (4.5) it follows that

$$T_1^* = \frac{pR_2}{2} + \frac{C}{R_2 \sin^2\theta}, \qquad (17.1)$$

$$T_2^* = pR_2 - \frac{R_2}{R_1} T_1^* = \frac{pR_2}{2}\left(2 - \frac{R_2}{R_1}\right) - \frac{C}{R_1 \sin^2\theta}.$$

Let us apply the equations, which we have obtained, to specific shells.

For a spherical shell ($R_1 = R_2 = R$)

$$T_1^* = \frac{pR}{2} + \frac{C}{R \sin^2\theta}, \quad T_2^* = \frac{pR}{2} - \frac{C}{R \sin^2\theta}. \qquad (17.2)$$

If the spherical shell contains any of its poles ($\theta = 0$ or $\theta = \pi$) then, because the forces are finite, we should let $C = 0$. In this case

$$T_1^* = T_2^* = \frac{pR}{2}. \qquad (17.3)$$

Expressions (17.3) give the general solution of the problem for a closed spherical shell. We can see from this equation that for a uniform normal pressure the spherical vessel is a shell of uniform resistance. In addition, it is well known that for a given surface area a spherical shell provides a maximum volume. It follows from this that a sphere is an ideal and most /186 economical form for vessels. Therefore it is frequently used for stationary gas storage tanks, reactor vessels and other similar structures. At the same time a whole series of other considerations (dimensions, fabrication techniques, etc.) make it necessary to use vessels of other form.

Most widespread are the cylindrical vessels covered at their ends with end plates of various form. We shall consider several of them in more detail.

For a cylindrical shell

$$\frac{1}{R_1}=0, \quad R_2=r_0, \quad \theta=\frac{\pi}{2} \qquad (\sin\theta=1, \cos\theta=0)$$

and expressions in (17.1) assume the form

$$T_1^*=A, \qquad T_2^*=pr_0.$$

The constant A is determined simply from the condition that the axial forces acting along the line joining the cylindrical shell to its end plate are equal (the latter construction is usually called a boiler). Indeed the axial force produced by the pressure on the end plate is equal to

$$F_z=\int_{\varphi=0}^{2\pi}\int_{0}^{\theta_0} p\cos\theta\,(R_2\sin\theta\,d\varphi)\,(R_1 d\theta)$$

According to (1.4) and $(1.5)_2$

$$R_2\sin\theta=r, \quad \cos\theta\,R_1 d\theta=\cos\theta\,ds_1=dr$$

Therefore

$$F_z=2\pi p\int_0^{r_0} r\,dr=\pi r_0^2 p.$$

Since this force is associated with a reaction

$$A=T_1^*=\frac{\pi r_0^2 p}{2\pi r_0}=\frac{pr_0}{2}.$$

Thus for the boiler

$$T_1^*=\frac{pr_0}{2}, \qquad T_2^*=pr_0. \tag{17.4}$$

Let the end plate of the vessel represent a spherical segment (fig. 31). The geometry of such a vessel is characterized by two (if we do not take into account the wall thickness h) parameters r_0 and θ_0. In this case, as we can easily see from figure 31, radius of the sphere R and the height of the end plate H are equal

$$R=\frac{r_0}{\sin\theta_0}, \quad H=r_0\operatorname{tg}\frac{\theta_0}{2}. \tag{17.5}$$

The membrane forces in the spherical end plate are determined by the expressions

$$T_1^0=T_2^0=\frac{pr_0}{2\sin\theta_0}. \tag{17.6}$$

The meridian force produces a thrust along the edge of the end plate /187

$$Q_\varphi^0=T_1^0\cos\theta_0=\frac{pr_0}{2}\operatorname{ctg}\theta_0. \tag{17.7}$$

This thrust can be taken up by the cylindrical shell only by means of cutting forces. Thus at the line joining the end plate to the boiler one of the significant conditions for the existence of a membrane state of stress is violated: the boundary element of the shell must be balanced in a membrane fashion. The violation of this condition leads to the occurrence of substantial cutting forces and bending moments in the neighborhood of the junction line. To evaluate these we consider an example presented in the monograph of V. V. Novozhilov (ref. 31).

EXAMPLE 1.

Let

$$\frac{H}{r_0}=\frac{1}{2}, \quad \frac{r_0}{h}=100, \quad \nu=0.3.$$

For this ratio of end plate height to shell radius we have

$$\theta_0\approx 53^\circ \quad (\sin\theta_0=0.8, \quad \cos\theta_0=0.6).$$

The solution for the boiler has the form

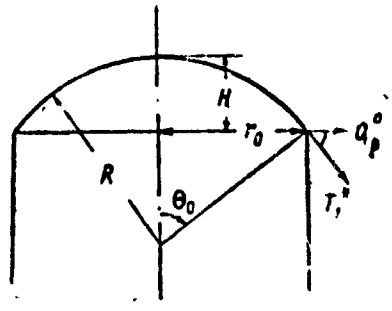

Figure 31

$$T_1^*=\frac{pr_0}{2}, \qquad T_2^*=pr_0, \quad U=\frac{pr_0^2}{2},$$
$$\widetilde{T}_1=pr_0\{0.5\}, \quad \widetilde{T}_2=pr_0\{1+\widetilde{A}e^{-(1-i)\beta_1}\}. \tag{17.8}$$

where

$$\beta_1=\sqrt{\frac{r_0}{2c}}\xi.$$

From this it follows that

$$\widetilde{T}=pr_0\{1.5+\widetilde{A}e^{-(1-i)\beta_1}\},$$
$$\frac{d\widetilde{T}}{d\xi}=pr_0\left\{-(1-i)\sqrt{\frac{r_0}{2c}}\,\widetilde{A}e^{-(1-i)\beta_1}\right\}. \tag{17.9}$$

For a spherical end plate

$$T_1^*=T_2^*=pr_0\frac{1}{2\sin\theta_0}$$
$$\widetilde{T}_1=pr_0\left\{\frac{1}{2\sin\theta_0}\right\}, \quad \widetilde{T}_2=pr_0\left\{\frac{1}{2\sin\theta_0}+\widetilde{B}e^{(1-i)\beta_2}\right\},$$

$$\beta_2=\sqrt{\frac{r_0}{2c\sin\theta_0}}\,(\theta-\theta_0), \tag{17.10}$$
$$\widetilde{T}=pr_0\left\{\frac{1}{\sin\theta_0}+\widetilde{B}e^{(1-i)\beta_2}\right\},$$
$$\frac{d\widetilde{T}}{d\theta}=pr_0\left\{(1-i)\sqrt{\frac{r_0}{2c\sin\theta_0}}\,\widetilde{B}e^{(1-i)\beta_2}\right\}.$$

The conditions for complex junction (6.17) are written in the following manner /188

$$pr_0\left\{1-\frac{\nu}{2}+\widetilde{A}\right\}=pr_0\left\{\frac{1-\nu}{2\sin\theta_0}+\widetilde{B}\right\},$$
$$pr_0\left\{-(1+i)\sqrt{\frac{c}{2r_0}}\widetilde{A}\right\}=pr_0\left\{\frac{\cos\theta_0}{2\sin\theta_0}+(1+i)\frac{1}{\sin\theta_0}\sqrt{\frac{c\sin\theta_0}{2r_0}}\widetilde{B}\right\} \qquad (17.11)$$

or, for the considered parameters of the vessel

$$\widetilde{B}-\widetilde{A}=0{,}413,$$
$$(1+i)\left[0{,}0436\,\widetilde{B}+0{,}0390\,\widetilde{A}\right]=-0{,}375.$$

This system is satisfied by

$$\widetilde{A}=-2{,}49+i\,2{,}27,\quad \widetilde{B}=-2{,}08+i\,2{,}27.$$

Thus for the cylinder

$$\widetilde{T}_1=pr_0\{0{,}5\},\quad \widetilde{T}_2=pr_0\{1+(-2{,}49+i\,2{,}27)\,e^{-(1-i)\beta_1}\}, \qquad (17.12)$$

where

$$\beta_1 = 12.9\ \xi,$$

and for the sphere

$$\widetilde{T}_1=pr_0\{0{,}625\},\quad \widetilde{T}_2=pr_0\{0{,}625+(-2{,}08+i\,2{,}27)\,e^{-(1-i)\beta_2}\}, \qquad (17.13)$$

where

$$\beta_2 = 14.4\ (\theta-\theta_0).$$

Remembering that

$$e^{-(1-i)\beta}=\cos\beta e^{-\beta}+i\sin\beta e^{-\beta},$$

it is easy to use table I and construct the picture of stress distribution along the meridian. To do this we must compute the quantities

$$\sigma_{T_1}=\frac{\operatorname{Re}\widetilde{T}_1}{h},\quad \sigma_{T_2}=\frac{\operatorname{Re}\widetilde{T}_2}{h},\quad \sigma_{M_1}=-\frac{6}{3{,}3}\frac{\operatorname{Im}\widetilde{T}_2}{h},\quad \sigma_{M_2}=\nu\sigma_{M_1}.$$

Figure 32 shows the stress distribution. In this case the solid line designates the meridian stress on the internal surface (σ_1^-), while the broken line designates the ring tangential stress (σ_{T_2}).

To find the arbitrary complex constants which enter into the solution, we use the conditions of complex junction (when the shells are joined at an angle). In order to check the reliability of the results which are obtained and also to evaluate the error of the junction condition itself, we determine the arbitrary constants contained in the solution using the method of compliance coefficients. /189

With the adopted shell parameters the compliance ccefficients (8.16) assume the following values

for boiler	for spherical end plate
$a_{11}=-2570\frac{1}{E}$,	$a_{11}=2300\frac{1}{E}$,
$a_{12}=-330\frac{1}{Eh}$,	$a_{12}=-330\frac{1}{Eh}$,
$a_{22}=-85\frac{1}{Eh^2}$.	$a_{22}=92\frac{1}{Eh^2}$.

Equating the values of generalized displacements for the contiguous shells, we arrive at a system of two equations

$$\begin{aligned}85\frac{pr_0}{E}-\frac{2570}{E}Q_0-330\frac{M_0}{Eh}&=43.7\frac{pr_0}{E}+\frac{2300}{E}(Q_0-0.375pr_0)-330\frac{M_0}{Eh},\\ -330\frac{Q_0}{Eh}-82{,}5\frac{M_0}{Eh^2}&=-\frac{330}{Eh}(Q_0-0.375pr_0)+92.2\frac{M_0}{Eh^2}.\end{aligned}\qquad(17.14)$$

Its solution is

$$Q_0=0.185pr_0,\quad M_0=-0.696pr_0h.\qquad(17.15)$$

By means of these quantities, using equalities (8.18), we find

$$\begin{aligned}&(T_2-T_2^*)_{\xi=0}=-2{,}46pr_0,\\ &(M_1)_{\xi=0}=M_0=-0.696pr_0h\ \text{for boiler;}\\ &(T_2-T_2^*)_{\theta=\theta_0}=-2.08pr_0,\\ &(M_1)_{\theta=\theta_0}=M_0=-0.696pr_0h\ \text{for spherical end plate.}\end{aligned}$$

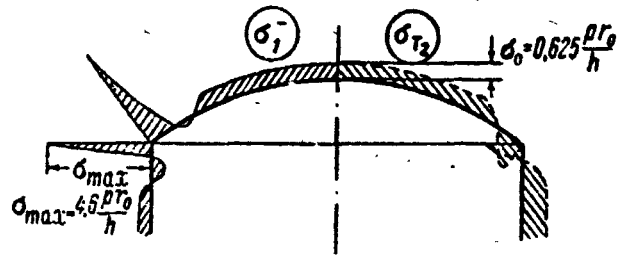

Figure 32

Analogous quantities, computed by means of expressions (17.12)-(17.13), are equal to

$$-2.49pr_0, \quad -0.688pr_0h;$$

$$-2.08pr_0, \quad -0.688pr_0h.$$

The coincidence is sufficiently good.

Now let us return to figure 32. We can see from this figure that the /190 total stresses which occur at the junction on the internal side are substantially greater than the membrane stresses. They are 4.6 times (4.62/1) greater for the shell and 7.6 times (4.75/0.625) greater for the spherical end plate.

The values of the additional flexural stresses are determined primarily by the value of the imaginary parts of constants $\tilde{A}$ and $\tilde{B}$. As we can easily see from system (17.11), large values for the imaginary parts are associated with the presence of thrust (the term corresponding to this thrust is underlined). In order to reduce the value of flexural stresses, efforts are made to design the vessels in such a way that the junction of the boiler with the end plate is smooth. As a result of this the thrust disappears, and the value of the flexural stresses is substantially reduced.

A basket curve end plate is an example of an end plate without thrust. This is the name usually given to end plates whose meridians are formed by a smooth junction of two circular arcs (ovals). In other words, a basket curve end plate (fig. 33) is obtained by a smooth junction of a spherical shell ($\theta < \theta_0$) with a so-called toroidal shell ($\theta_0 \leq \theta \leq \pi$ 2), whose geometry is characterized by two parameters (r_0-r_1) and r_1. As we can easily see from figure 32 for a toroidal end plate

$$R_1 = r_1 = \frac{\alpha}{1+\alpha} r_0, \quad R_2 = \frac{1+\alpha\sin\theta}{(1+\alpha)\sin\theta} r_0, \tag{17.16}$$

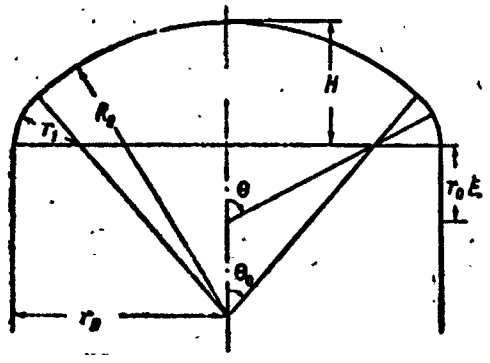

Figure 33

where

$$\alpha=\frac{r_1}{r_0-r_1}. \tag{17.17}$$

Thus the geometry of a basket curve end plate is characterized by parameters r_0, α, θ_0. Again by considering figure 33, it is easy to obtain the following relationships

$$R_0=\frac{1+\alpha\sin\theta_0}{(1+\alpha)\sin\theta_0}r_0,\quad H=\frac{1+\alpha\sin\theta_0-\cos\theta_0}{(1+\alpha)\sin\theta_0}r_0. \tag{17.18}$$

The membrane solution has the following form

for the spherical part

$$T_1^*=T_2^*=\frac{pR_0}{2}=\frac{pr_0}{2}\frac{1+\alpha\sin\theta_0}{(1+\alpha)\sin\theta_0}, \tag{17.19}$$

for the toroidal part

$$T_1^*=\frac{1+\alpha\sin\theta}{(1+\alpha)\cdot\sin\theta}\frac{pr_0}{2};\quad T_2^*=-\frac{1-\alpha^2\sin^2\theta}{\alpha(1+\alpha)\sin^2\theta}\frac{pr_0}{2}. \tag{17.20}$$

A typical distribution of stresses along the meridian is shown in figure 34. First of all, we become aware of a large discontinuity in the membrane /191

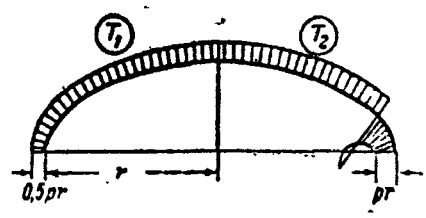

Figure 34

ring force at the point where the spherical and toroidal regions join. As a result of this discontinuity large bending stresses occur. Therefore at the present time a basket curve end plate is almost never used. Instead elliptical end plates are used, which are close to them in form but are free of this shortcoming.

Let us consider an elliptic bottom in more detail. The radii of curvature of an elliptic shell are determined by equalities (4.11) when $\gamma > -1$.

Assuming that in the first of these $\theta = \frac{\pi}{2}$, we obtain

$$r = r_0 = R_2\left(\frac{\pi}{2}\right) = \frac{R_0}{\sqrt{1+\gamma}}, \quad R_0 = \sqrt{1+\gamma}\, r_0.$$

Now equalities (4.11) may be written in the form

$$R_1 = \frac{r_0\sqrt{1+\gamma}}{(1+\gamma\sin^2\theta)^{3/2}}, \quad R_2 = \frac{r_0\sqrt{1+\gamma}}{(1+\gamma\sin^2\theta)^{1/2}}. \tag{17.21}$$

Furthermore

$$T_1^* = \frac{pR_2}{2} = \frac{pr_0}{2}\frac{\sqrt{1+\gamma}}{\sqrt{1+\gamma\sin^2\theta}},$$
$$T_2^* = \frac{pR_2}{2}\left(2 - \frac{R_2}{R_1}\right) = \frac{pr_0}{2}\sqrt{1+\gamma}\,\frac{1-\gamma\sin^2\theta}{\sqrt{1+\gamma\sin^2\theta}}. \tag{17.22}$$

The constant γ contained in these equations is associated with the height of the end plate by means of the following relationship

$$\gamma = \left(\frac{r_0}{H}\right)^2 - 1. \tag{17.23}$$

Assuming that $\theta = \frac{\pi}{2}$ in expressions (17.22), we obtain

$$T_1^*\left(\frac{\pi}{2}\right)=\frac{pr_0}{2},\quad T_2^*\left(\frac{\pi}{2}\right)=\frac{pr_0}{2}(1-\nu).$$

Comparing these expressions with the membrane forces for a boiler (17.4), /192 we see that the continuity of membrane ring forces cannot be provided for end plates of finite height. From this it follows that the horizontal displacement undergoes a discontinuity along the junction.

$$u_0=\frac{r_0}{Eh}(T_2^*-\nu T_1^*).$$

Thus although in the considered case the thrust is absent, moments exist in the shell because another condition for the existence of a membrane state of stress is violated: this condition is the smoothness of the middle surface. If we approach the cases considered by us from a geometric point of view, the elliptic end plate is substantially more advantageous than the spherical end plate. Indeed at the point where the elliptic end plate joins the boiler, there is a discontinuity in the curvature of the meridian. For a spherical end plate, not only the curvature, but also the slope angle of the tangent with respect to the meridian varies in a discontinuous manner at the junction. Thus the meridian of the vessel with an elliptic end plate is smoother. This makes it possible for us to maintain that the value of the additional flexural stresses which occur in it are substantially less than in a vessel with a spherical end plate. To confirm the general considerations which we have stated, we shall consider one more example.

EXAMPLE 2.

Let $\frac{H}{r_0} = \frac{1}{2}$, as in the preceding example. According to (17.23) this height is associated with $\gamma = 3$. The solution for the cylindrical part remains the same

$$\begin{gathered}\widetilde{T}_1=pr_0\{0.5\},\quad \widetilde{T}_2=pr_0\left\{1+\widetilde{A}e^{-(1-\eta)\beta_1}\right\},\\ \widetilde{T}=pr_0\left\{1.5+\widetilde{A}e^{-(1-\eta)\beta_1}\right\},\\ \frac{d\widetilde{T}}{d\xi}=pr_0\left\{-(1-i)\sqrt{\frac{r_0}{2c}}\widetilde{A}e^{-(1-\eta)\beta_1}\right\},\quad \beta_1=\sqrt{\frac{r_0}{2c}}\xi.\end{gathered}\qquad(17.24)$$

For the end plate

$$\widetilde{T}_1 = pr_0\left\{\frac{1}{\sqrt{1+3\sin^2\theta}}\right\},$$
$$\widetilde{T}_2 = pr_0\left\{\frac{1-3\sin^2\theta}{\sqrt{1+3\sin^2\theta}} + \widetilde{B}e^{(1-\theta)\beta_0}\right\},$$
$$\widetilde{T} = pr_0\left\{\frac{2-3\sin^2\theta}{\sqrt{1+3\sin^2\theta}} + \widetilde{B}e^{(1-\theta)\beta_0}\right\},$$
$$\frac{d\widetilde{T}}{d\theta} = pr_0\left\{\frac{d}{d\theta}\left(\frac{2-3\sin^2\theta}{\sqrt{1+3\sin^2\theta}}\right) + \widetilde{B}\sqrt{\frac{r_0}{c}}\frac{(1-\theta)}{(1+3\sin^2\theta)^{5/4}}e^{(1-\theta)\beta_0}\right\}, \quad (17.25)$$

$$\beta_2 = \sqrt{\frac{r_0}{c}}\int_{\theta_0}^{\theta}\frac{d\theta'}{(1+3\sin^2\theta')^{5/4}}.$$

Since the junction is smooth, we can use simpler conditions of complex junction (6.19). Substituting into these the expressions presented above, we obtain a system of equations /193

$$pr_0\left\{1.5+\widetilde{A}\right\} = pr_0\left\{-0.5+\widetilde{B}\right\},$$
$$pr_0\left\{-\frac{(1-\theta)}{r_0}\sqrt{\frac{r_0}{2c}}\widetilde{A}\right\} = pr_0\left\{\frac{(1-\theta)4}{r_0}\sqrt{\frac{r_0}{c}}\frac{1}{(4)^{5/4}}\widetilde{B}\right\}, \quad (17.26)$$

or

$$\widetilde{B}-\widetilde{A}=2, \quad \widetilde{A}+\widetilde{B}=0.$$

From this it follows that

$$\widetilde{A}=-1, \quad \widetilde{B}=1.$$

Thus for the boiler

$$\widetilde{T}_1 = pr_0\{0.5\}, \quad \widetilde{T}_2 = pr_0\{1-e^{-(1-\theta)\beta_1}\}, \quad \beta_1 = 12.9\xi, \quad (17.27)$$

and for the end plate

$$\widetilde{T}_1 = pr_0\left\{\frac{1}{\sqrt{1+3\sin^2\theta}}\right\}, \quad \widetilde{T}_2 = pr_0\left\{\frac{1-3\sin^2\theta}{\sqrt{1+3\sin^2\theta}} + e^{(1-\theta)\beta_2}\right\},$$
$$\beta_2 = 18.2\int_{\theta_0}^{\theta}\frac{d\theta'}{(1+3\sin^2\theta)^{5/4}} \approx -18.2(\theta_0-\theta). \quad (17.28)$$

Let us determine the maximum stress which occurs in the vessel. It follows from (17.27) that for the boiler

$$\sigma_{T_1} = \frac{pr_0}{h}\{1 - \cos\beta_1 e^{-\beta_1}\}.$$

It is easy to see that the ring stresses in the boiler even tend to decrease compared with the membrane values. Furthermore

$$\sigma_{M_1} = -\frac{6h\,\mathrm{Im}\,\widetilde{T}_2}{3{,}3h^2} = \frac{pr_0}{h}\{1.82\sin\beta_1 e^{-\beta_1}\}.$$

For figure 17 we find that

$$(\sigma_{M_1})_{\max} = \frac{pr_0}{h}\{1.82\cdot 0.32\} \approx 0.58\frac{pr_0}{h}.$$

In this case the maximum bending stress takes place when $\beta_1 \approx 0.8$, i.e., when $\xi = \frac{s}{r_0} \approx \frac{0.8}{12.9} \approx 0.06$. The total meridian stress on the external side of the boiler has the maximum value

$$(\sigma_1^+)_{\max} = \{0.5 + 0.58\}\frac{pr_0}{h} = 1{,}08\frac{pr_0}{h}. \qquad (17.29)$$

In a similar manner it is easy to show that the ring stresses in the end plate also decrease, and the maximum stress is the total meridian stress on the internal surface when $(\theta_0 - \theta) \approx 3^\circ$. Its magnitude is approximately equal to the value given by (17.29). /194

Thus although in the considered vessel there is a state of stress which involves moments, the total stresses exceed only slightly the corresponding membrane values. Calculation carried out by means of the membrane theory gives us no information on the true nature of stress distribution near the junction, but it may be used nevertheless to select the thickness of the vessel wall.

In order to draw a comparison with the case of a spherical end plate considered above, figure 35 shows the same diagram as figure 32. A comparison of both diagrams confirms the considerations expressed above concerning the advantages of an elliptical bottom.

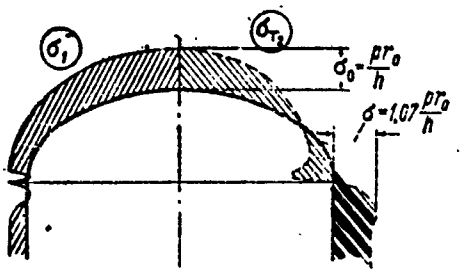

Figure 35

It is curious to note that the end plate considered by us is very close in its form to the Biezeno end plate. This is the name given to an end plate which is of uniform strength according to the theory of maximum tangential stresses (ref. 127).

In the last two examples, the edge effect was very pronounced. This is explained by the fact that the walls of the vessels are very thin ($r_0/h = 100$). Such large ratios r_0/h are characteristic for small pressures. In vessels which operate at high pressure, these ratios are much greater. In this case the stress concentration noted above is substantially reduced, and the damping zone of the edge effect is increased.

Let us consider further a vessel with a flat end plate having the same thickness as the boiler.

EXAMPLE 3.

As in the first example let us assume that $r_0/h = 100$. The solution for the boiler has the same form (see (17.9)-(17.10)). According to (13.14), for a flat end plate

$$\widetilde{T}_1 = pr_0\left\{i\frac{r_0}{c}\frac{\rho^2}{16} + \widetilde{B}\right\},\quad \widetilde{T}_2 = pr_0\left\{i\frac{r_0}{c}\frac{3\rho^2}{16} + \widetilde{B}\right\}. \tag{17.30}$$

Utilizing the complex conditions at the joints between shells (13.17) we arrive at the following system of equations

$$\begin{aligned} pr_0\left\{1 - \frac{\nu}{2} + \widetilde{A}\right\} &= pr_0\left\{i\frac{r_0}{c}\frac{3-\nu}{16} + (1-\nu)\widetilde{B}\right\}, \\ pr_0\left\{-(1+i)\sqrt{\frac{c}{2r_0}}\widetilde{A}\right\} &= pr_0\left\{i\frac{r_0}{c}\frac{1}{16} + \widetilde{B}\right\}. \end{aligned} \tag{17.31}$$

For the parameters which we have adopted, it takes the form

$$0{,}85+\widetilde{A}=i55.7+0.7\widetilde{B},$$
$$-(1+i)\,0.039\widetilde{A}=i20.6+\widetilde{B}.$$

The above system is satisfied by the following

$$\widetilde{A}=0.20+i\,40.2,\quad \widetilde{B}=1.56-i\,22.2.$$

Let us write down the complex forces which correspond to the established constants:

for the cylindrical part,

$$\widetilde{T}_1=pr_0\{0.5\},\quad \widetilde{T}_2=pr_0\{1+(0.20+i\,40.2)\,e^{-(1-i)\beta_1}\},\qquad \beta_1=12.9\xi, \tag{17.32}$$

for the flat end plate

$$\widetilde{T}_1=pr_0\{i20.6\rho^2+(1.56-i22.2)\},$$
$$\widetilde{T}_2=pr_0\{i61.8\rho^2+(1.56-i22.2)\}. \tag{17.33}$$

Let us use the expressions which we have obtained to determine the values of the edge bending moment:

for the boiler

$$M_0=M_1(0)=-\frac{40.2}{3.3}pr_0h=-12.2pr_0h, \tag{17.34}$$

for the end plate

$$M_0=M_1(1)=$$
$$=-\frac{(61.8-22.2)+0.3(20.6-22.2)}{3.3}pr_0h=-11.8pr_0h. \tag{17.35}$$

Different values for the same quantity are explained by the fact that in determining the complex quantities we use the approximate complex conditions at the joints between shells. According to the first of these

$$M_1 = -c\,\mathrm{Im}\left\{\widetilde{T}_2 - \nu\widetilde{T}_1\right\}.$$

The departure of this expression from the one used by us (13.15) explains the departure in the numerical values of the edge moment. We note, however, that even in this case which is rather unfavorable for the application of the complex conditions at the joints between shells, the relative error in the computation of M_0 is only approximately 3 percent $\left(\frac{12.2-11.8}{12.2}\right)$.

Comparing the diagram corresponding to this example (fig. 37) with the one shown in figure 32, we can see that in the case of a flat end plate the flexural stresses are several times greater than in the case of a spherical end plate. Thus a flat end plate is even more unfavorable. This is explained by the fact that the flat end plate which is devoid of spatial curvature has a state of stress which is close to the purely flexural state when the loading is transverse. Therefore when it is necessary for some reason or other to use a flat end plate, its thickness is taken so that it is several times greater than that of the boiler. /196

Let us consider in more detail the case of a vessel with a flat end plate of increased thickness (fig. 36). Applying the relationships of sections 11, 14 and 16 we have, first of all,

for a boiler (with a thickness h_1)

$$T_1^* = pr_0 0.5, \quad T_2^* = pr_0, \quad \sigma_1^* = \frac{pr_0}{h_1}0.5, \quad \sigma_2^* = \frac{pr_0}{h_1}, \quad u_0^* = \frac{pr_0^2}{Eh_1}0.85, \tag{17.36}$$

for an end plate (of thickness h_2)

$$\begin{gathered}\overline{w}^{**} = T_1^* = T_2^* = u_0^* = Q_0^* = 0,\\ w^* = \frac{10.9}{64}\frac{pr_0}{E}\left(\frac{r_0}{h_2}\right)^3\rho^4, \quad \vartheta^* = -\frac{10.9}{16}\frac{p}{E}\left(\frac{r_0}{h_2}\right)^3\rho^3,\\ M_1^* = -\frac{3.3}{16}pr_0^2\rho^2, \quad M_2^* = -\frac{1.9}{16}pr_0^2\rho^2,\\ \vartheta_0^* = -\frac{10.9}{16}\frac{p}{E}\left(\frac{r_0}{h_2}\right)^3, \quad M_0^* = -\frac{3.3}{16}pr_0^2.\end{gathered} \tag{17.37}$$

Utilizing equalities (16.16)-(16.17), (14.31)-(14.32), (11.19)-(11.20) and noting that for the boiler we should let $\mu = -1$, we arrive at a system of equations which determines Q_0 and M_0

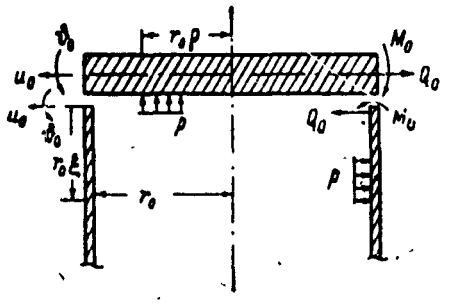

Figure 36.

$$-\frac{2.57}{E}\left(\frac{r_0}{h_1}\right)^{3/2}Q_0-\frac{3.30}{E}\left(\frac{r_0}{h_1}\right)\frac{M_0}{h_1}+\frac{0.85}{E}\left(\frac{r_0}{h_1}\right)pr_0=\frac{0.7}{E}\left(\frac{r_0}{h_2}\right),$$
$$-\frac{3.30}{E}\left(\frac{r_0}{h_1}\right)\frac{Q_0}{h_1}-\frac{8.48}{E}\sqrt{\frac{r_0}{h_1}}\frac{M_0}{h_1^2}=-\frac{0.68}{E}\left(\frac{r_0}{h_2}\right)^3p+ \qquad (17.38)$$
$$+\frac{8.40}{Eh_2}\left(\frac{r_0}{h_2}\right)\left[\frac{M_0}{h_2}+0.206\left(\frac{r_0}{h_2}\right)pr_0\right].$$

Let us introduce the following designations /197

$$\alpha=\frac{r_0}{h_1},\quad \beta=\frac{r_0}{h_2},\quad \gamma=\frac{\beta}{\alpha}=\frac{h_1}{h_2}. \qquad (17.39)$$

It is easy to verify that the system (17.38) satisfies the expressions

$$Q_0=pr_0\Phi_1,\quad M_0=-pr_0^2\Phi_2, \qquad (17.40)$$

where

$$\Phi_1=\frac{0.655\gamma^3+0.318\alpha\gamma^3+0.622\frac{1}{\sqrt{\alpha}}}{1+0.544\gamma\frac{1}{\sqrt{\alpha}}+1.98\gamma^3\sqrt{\alpha}+0.540\gamma^4},$$
$$\Phi_2=\frac{0.067\gamma^4+0.248\gamma^3\sqrt{\alpha}+0.258\frac{1}{\alpha}}{1+0.544\gamma\frac{1}{\sqrt{\alpha}}+1.98\gamma^3\sqrt{\alpha}+0.540\gamma^4}.$$

Let us consider a numerical example which is encountered in the calculation of the pipe panel of a heat exchanger. Let

$$E = 2{,}1 \cdot 10^6 \frac{\text{kg}}{\text{см}^2},\ p = 60 \frac{\text{kg}}{\text{см}^2},\ r_0 = 190\ \text{см},$$
$$h_1 = 7{,}6\ \text{см},\ h_2 = 53\ \text{см}. \tag{17.41}$$

Utilizing the relationships just obtained we find

$$\begin{gathered} \alpha = 25{,}0,\quad \beta = 3{,}58,\quad \gamma = 0{,}143, \\ \Phi_1 = 0{,}150,\quad \Phi_2 = 0{,}0133, \\ pr_0 = 1{,}140 \cdot 10^4 \frac{\text{kg}}{\text{см}},\quad pr_0^2 = 2{,}17 \cdot 10^6 \frac{\text{kg} \cdot \text{cm}}{\text{см}}, \\ Q_0 = 1{,}17 \cdot 10^3 \frac{\text{kg}}{\text{см}},\quad M_0 = -2{,}89 \cdot 10^4 \frac{\text{kg} \cdot \text{cm}}{\text{см}}. \end{gathered} \tag{17.42}$$

Let us consider the plate first. According to (16.18) and (17.37) we have the following for the plate

$$\sigma_{T_1} = \sigma_{T_2} = \frac{Q_0}{h_2} = 22 \frac{\text{kg}}{\text{см}^2}. \tag{17.43}$$

From (17.37) and (14.33)

$$\begin{gathered} M_1^* = -4{,}48 \cdot 10^4 \rho^2 \frac{\text{kg} \cdot \text{cm}}{\text{см}},\quad M_2^* = -2{,}58 \cdot 10^5 \rho^2 \frac{\text{kg} \cdot \text{cm}}{\text{см}}, \\ M_0^* = -4{,}48 \cdot 10^5 \frac{\text{kg} \cdot \text{cm}}{\text{см}}, \\ \sigma_{M_1}(\rho) = [-959\rho^2 + 896] \frac{\text{kg}}{\text{см}^2},\quad \sigma_{M_2}(\rho) = [-550\rho^2 + 896] \frac{\text{kg}}{\text{см}^2} \end{gathered} \tag{17.44}$$

The maximum flexural stresses (at the center of the bottom) are equal to

$$\sigma_1(0) = \sigma_2(0) = 896 \frac{\text{kg}}{\text{см}^2}. \tag{17.45}$$

The maximum total stress (radial) takes place at the external side of the end plate where /198

$$\sigma_{\max}^+ = 22 + 896 = 918 \frac{\text{kg}}{\text{см}^2}. \tag{17.46}$$

Furthermore

$$\vartheta_0^* = -0{,}893 \cdot 10^{-3} \text{ rad},$$
$$\vartheta_0 = 1{,}23\ 10^{-3} \text{ rad} \approx 4.2',$$
$$w^* = 4.24 \cdot 10^{-2} p^4 \text{ мм}.$$

The maximum flexure at the center of the plate according to $(14.33)_2$ is equal to

$$w(0) = 1{,}99 \text{ мм}. \tag{17.47}$$

As far as the radial displacement u_ρ is concerned, it achieves its maximum value at the edge and is equal to (see (17.38))

$$u_\rho(1) = u_0 = 2.04 \cdot 10^{-2} \text{ мм}. \tag{17.48}$$

Comparing the last quantity with the flexure (17.47), we see that the flexure is approximately 100 times greater than the tangential displacements of the points on the end plate. In the example considered, the general property which is common to thin-walled bodies is clearly illustrated: high rigidity during tangential deformation and small rigidity during flexural deformation. This is associated with the fact that the tangential stresses are substantially less than the flexural stresses (see (17.43), (17.44)). Indeed the value of the tangential stresses in a plate is determined by the thrust along its contour. A cylindrical shell on the other hand operates in flexure (along the edge) and cannot produce a substantial thrust. Furthermore, since the cylindrical shell is substantially thinner than the plate, it cannot produce a noticeable clamping of the plate edge.

Everything that we have said leads to the idea that the end plate must operate under conditions which are close to that of free support. We can verify this directly. Indeed, according to equations (14.37)-(14.43), we have the following values for a freely supported plate which has our parameters

$$\sigma_{\max}^{+} = 950 \text{kg/см}^2, \quad \vartheta^{\max} = \vartheta_0 = 1.37 \cdot 10^{-3} \text{ rad}, \tag{17.49}$$
$$w^{\max} = w(0) = 1{,}73 \text{ мм}.$$

In practice when making rough estimates for establishing the approximate thickness of the end plate, the latter is usually considered as being freely supported.

Let us now turn to the consideration of the boiler. The values obtained for Q_0 and M_0 as well as expressions (11.21) and (11.23) make it possible to determine the stresses and displacements in the boiler. The stress diagram is shown in figure 37.

Furthermore, for the edge of the boiler (see (17.48), (17.36))

$$u_0^* = 1.15 \text{ мм}, \quad u_0 \approx 0.02 \text{ мм}.$$

The first term pertains to the shell which is deformed in a membrane fashion. The second pertains to the horizontal displacement of the boiler edge which /199 actually takes place in the vessel.

A comparison of the stated quantities predicts a natural path for simplifying the relationships obtained above. We let

$$u_0 = 0. \tag{17.50}$$

In other words, we shall assume that the end plate cannot undergo an extension in its plane. Then it follows from (11.19) that

$$0 = u_0^* + \alpha_{11} Q_0 + \alpha_{12} M_0,$$
$$\vartheta_0 = \alpha_{12} Q_0 + \alpha_{22} M_0.$$

From the first equation we find

$$Q_0 = -\frac{u_0^*}{\alpha_{11}} - \frac{\alpha_{12}}{\alpha_{11}} M_0. \tag{17.51}$$

Substituting Q_0 into the second we obtain

$$\vartheta_0 = -\frac{\alpha_{12}}{\alpha_{11}} u_0^* + \frac{\alpha_{11}\alpha_{22} - \alpha_{12}^2}{\alpha_{11}} M_0. \tag{17.52}$$

If in the equalities which we have obtained we replace the compliance coefficients and u_0^* by their values, we arrive at the relationships

$$\vartheta_0 = \vartheta_0' + \gamma M_0, \tag{17.53}$$

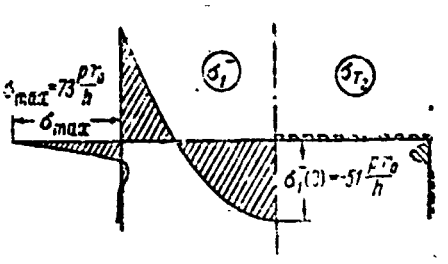

Figure 37

where

$$\gamma=\mu 2\left[3\left(1-\nu^2\right)\right]^{3/4}\sqrt{\frac{r_0}{h}}\frac{1}{Eh^2}, \tag{17.54}$$

$$\delta_0^{\nu}=\mu\frac{2-\nu}{2}\sqrt[4]{3\left(1-\nu^2\right)}\left(\frac{r_0}{h}\right)^{3/2}\frac{p}{E};$$

$$Q_0=\mu\left\{-\frac{2-\nu}{2}\frac{1}{2\sqrt[4]{3\left(1-\nu^2\right)}}\sqrt{\frac{h}{r_0}}pr_0+\right.$$

$$\left.+\sqrt[4]{3\left(1-\nu^2\right)}\sqrt{\frac{r_0}{h}}\frac{M_0}{r_0}\right\}. \tag{17.55}$$

By using the expressions which we have obtained we can determine Q_0 and M_0 in a simpler way by equating the angles of rotation at the edges of the boiler and of the end plate

$$-\frac{1.09}{E}\left(\frac{r_0}{h_1}\right)^{3/2}p-\frac{4.24}{E}\sqrt{\frac{r_0}{h_1}}\frac{M_0}{h_1^2}=-\frac{0.68}{E}\left(\frac{r_0}{h_2}\right)^3p+$$

$$+\frac{8.40}{Eh^2}\left(\frac{r_0}{h_2}\right)\left[\frac{M_0}{h_2^2}+0.206\left(\frac{r_0}{h_2}\right)pr_0\right]. \tag{17.56}$$

Determining M_0 from this expression and then determining Q_0 from (17.55), /200 we obtain

$$Q_0=pr_0\Phi_1,\quad M_0=-pr_0^2\Phi_2, \tag{17.57}$$

where we now have

$$\Phi_1 = \frac{0.655\gamma^3 + 0.318\alpha\gamma^3 + 0.662\frac{1}{\sqrt{\alpha}}}{1 + 1.98\gamma^3\sqrt{\alpha}},$$
$$\Phi_2 = \frac{0.248\gamma^3\sqrt{\alpha} + 0.2^{\text{ERR}}\frac{1}{\alpha}}{1 + 1.98\gamma^3\sqrt{\alpha}}. \qquad (17.58)$$

These simpler relations differ from (17.40) only in some of the small terms. For the considered example

$$\Phi_1 = 0.1525, \quad \Phi_2 = 0.0135, \qquad (17.59)$$

which differs little from the value (17.42).

This problem is of interest because, when used as an example, it illustrates how the problem can be simplified by utilizing physical considerations. Thus by assuming that the plate is freely supported, we have substantially simplified the problem of its calculation. Furthermore by assuming that the plate is infinitely rigid in its plane, we have replaced the system of two equations with one. The good coincidence of "exact" and approximate expressions confirms the validity of the adopted simplifications.

In our example the stresses are substantially less than in the preceding one $\left(\sigma_{M_0}^{\text{sh}} = -2\frac{pr_0}{h_1}, \sigma_k^{\text{sh}} = 1.3\frac{pr_0}{h_1}\right)$, which, however, is achieved by increasing the thickness of the plate by a factor of seven.[1] As a result, we must look upon the design of the vessel as extremely unsuccessful (from the standpoint of strength). Flat end plates are used only in cases when the flat shape is necessary because of special operating conditions associated with the structure. In the contrary case the form of the end plate is selected in a more rational manner (for example, a type considered in EXAMPLE 2).

18. The Calculation of Domes[2]

In the calculation of domes the design load is assumed to be equal to the natural weight (fig. 38)

$$q_1 = q\sin\theta, \quad q_n = -q\cos\theta. \qquad (18.1)$$

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[1] sh = shell.

[2] Written in collaboration with V. Ya. Pavilaynen.

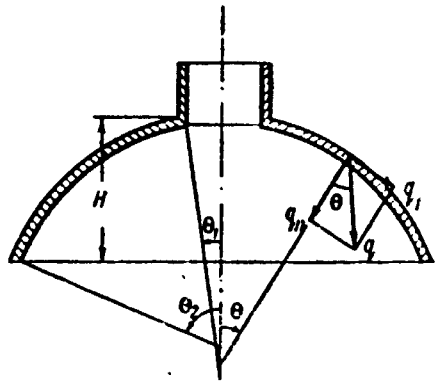

Figure 38

Here q is the weight of the dome per unit area of its surface. For the surface loading (18.1) the expressions for membrane forces (4.17) take the form

$$T_1^* = -\frac{q}{R_2 \sin^2\theta}\int_{\theta_1}^{\theta} R_1 R_2 \sin\theta d\theta + \frac{C_1}{R_2 \sin^2\theta},$$
$$T_2^* = -qR_2\cos\theta + \frac{q}{R_1 \sin^2\theta}\int_{\theta_1}^{\theta} R_1 R_2 \sin\theta d\theta - \frac{C_1}{R_1 \sin^2\theta}. \qquad (18.2)$$

Contrary to the situation in the preceding section, the integrals in /201 the right sides cannot be computed for an arbitrary form of the meridian. Therefore, it is the accepted practice to consider domes whose middle surfaces are formed by the rotation of second order curves around their axis of symmetry. As we have already shown in section 4, the radii of curvature of such surfaces are given by the relationships

$$R_1 = \frac{R_0}{(1+\gamma\sin^2\theta)^{3/2}},$$
$$R_2 = \frac{R_0}{(1+\gamma\sin^2\theta)^{1/2}}. \qquad (18.3)$$

In this case

$\gamma = 0$ corresponds to a sphere
$\gamma = -1$ corresponds to a paraboloid
$\gamma > -1$ corresponds to an ellipsoid
$\gamma < -1$ corresponds to a hyperboloid

A comparison of domes corresponding to different γ, but having the same base and height, shows (refs. 165 and 31) that the elliptical ones are most advantageous.

Let a and b be the horizontal (major) and the vertical semi-axes of the ellipse, whose arc is the meridian of the dome. The eccentricity of the ellipse ε is associated with the semi-axes and with the constant γ by the following relationships

$$\varepsilon^2 = 1 - \frac{b^2}{a^2} = \frac{\gamma}{1+\gamma}, \quad \gamma = \frac{\varepsilon^2}{1-\varepsilon^2}. \tag{18.4}$$

Assuming that $\theta = \frac{\pi}{2}$ in expression (18.3), we obtain

$$a = R_2\left(\frac{\pi}{2}\right) = \frac{kR_0}{(1+\gamma)^{1/2}}.$$

It follows from this that

$$R_0 = a(1+\gamma)^{1/2} = \frac{a}{\sqrt{1-\varepsilon^2}}.$$

By using the equalities which have been obtained, expressions (18.3) are reduced to the form /202

$$R_1 = \frac{(1-\varepsilon^2)a}{(1-\varepsilon^2\cos^2\theta)^{3/2}}, \quad R_2 = \frac{a}{(1-\varepsilon^2\cos^2\theta)^{1/2}}. \tag{18.5}$$

Let us substitute into (18.2) the expressions obtained for the radii of curvature. The integral contained in them is tabulated and easily determined. After simple transformations, and remembering the static meaning of the constant C_1 (see (6.3)), we obtain

$$\begin{aligned} T_1^* &= -\frac{qa}{2}\frac{\sqrt{1-\varepsilon^2\cos^2\theta}}{\sin^2\theta}\left\{\left[\frac{(1-\cos\theta)(1+\varepsilon^2\cos\theta)}{1-\varepsilon^2\cos^2\theta} + \right.\right. \\ &\left.\left. + \frac{1-\varepsilon^2}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\frac{1-\varepsilon\cos\theta}{1+\varepsilon\cos\theta}\right]_{\theta_1}^{\theta}\right\} + \frac{F_z^0}{2\pi a}\frac{\sqrt{1-\varepsilon^2\cos^2}}{\sin^2\theta}, \\ T_2^* &= -\frac{qa\cos\theta}{\sqrt{1-\varepsilon^2\cos^2\theta}} + \frac{qa(1-\varepsilon^2\cos^2\theta)^{3/2}}{(1-\varepsilon^2)\sin^2\theta}\left\{\left[\frac{(1-\cos\theta)(1+\varepsilon^2\cos\theta)}{1-\varepsilon^2\cos^2\theta} + \right.\right. \\ &\left.\left. + \frac{1-\varepsilon^2}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\frac{1-\varepsilon\cos\theta}{1+\varepsilon\cos\theta}\right]_{\theta_1}^{\theta}\right\} - \frac{F_z^0}{2\pi a}\frac{(1-\varepsilon^2\cos^2\theta)^{3/2}}{(1-\varepsilon^2)\sin^2\theta}. \end{aligned} \tag{18.6}$$

Here F_z^0 is the axial force applied to the internal contour of the dome, and its value is equal to the weight of the connecting piece. The connecting piece

contains the ventilation equipment, the apertures for transmitting light, etc. In the calculations, the increase in strength due to the effect of the connecting pi ce is neglected, because it is insignificant. Only the load transmitted by the connecting piece to the dome is taken into consideration.

For a dome which is closed at the apex (a dome without a connecting piece) we should let $\theta_1 = F_z^0 = 0$. In this case expressions (18.6) are somewhat simplified

$$T_1^* = -\frac{qa}{2}\frac{\sqrt{1-\varepsilon^2\cos^2\theta}}{\sin^2\theta}\left\{\frac{(1-\cos\theta)(1+\varepsilon^2\cos\theta)}{1-\varepsilon^2\cos^2\theta} + \right.$$
$$\left. + \frac{1-\varepsilon^2}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\frac{1-\varepsilon\cos\theta}{1+\varepsilon\cos\theta}\right\},$$
$$T_2^* = -\frac{qa\cos\theta}{\sqrt{1-\varepsilon^2\cos^2\theta}} - \frac{1-\varepsilon^2\cos^2\theta}{1-\varepsilon^2}T_1^*. \qquad (18.7)$$

It is easy to see that these equations are valid for a dome with a connecting piece, if the weight of the latter (following the recommendations by Dishinger (ref. 165)) is assumed to be equal to the part of the dome cut out by the connecting piece.

For a spherical dome ($\varepsilon = 0$) the stated relationships assume a much simpler form

$$T_1^* = -qR_0\frac{\cos\theta_1-\cos\theta}{\sin^2\theta} + \frac{F_z^0}{2\pi R_0}\frac{1}{\sin^2\theta},$$
$$T_2^* = -qR_0\left\{\cos\theta - \frac{\cos\theta_1-\cos\theta}{\sin^2\theta}\right\} - \frac{F_z^0}{2\pi R_0}\frac{1}{\sin^2\theta}, \qquad (18.8)$$

$$T_1^* = -\frac{qR}{1+\cos\theta},$$
$$T_2^* = -qR_0\left(\cos\theta - \frac{1}{1+\cos\theta}\right). \qquad (18.9)$$

The arbitrary constants are absent in all of the expressions obtained /203 above. Therefore the membrane forces are associated with definite values of the boundary quantities. The question as to whether there is or there is not a membrane state in the dome depends on whether we are able to select the edge element (supporting ring) so that it undergoes a specified relative extension when subjected to the assigned thrust.

In order to answer this question we shall determine the tensional and torsional rigidity of the ring. In doing so we shall assume that:

(a) the linear dimensions of the ring's cross section are small compared with its radius (the distance from the center of gravity of section 0 to the axis of rotation), and

(b) the ring is symmetric with respect to the middle horizontal plane.

Let us consider two cases of ring deformation.

1. Let us assume that the considered section of the ring is displaced in the radial direction by a distance u_k. Then the stress in part of the longitudinal fiber which adjoins the considered point of the section, is equal to

$$\sigma = E_k \varepsilon_k = E_k \frac{u_k}{r} \approx E_k \frac{u_k}{r_0}.$$

From this we find the longitudinal force acting at this cross section

$$F_k = \iint \sigma d\Omega_k = \frac{E_k \Omega_k}{r_0} u_k, \tag{18.10}$$

where Ω_k is the area of the ring's cross section.

2. Now let us assume that the cross section is rotated around the center of gravity by a small angle ϑ_k. Utilizing polar coordinates (fig. 39), we find an expression for the stresses in the longitudinal fibers

$$\sigma = E_k \varepsilon_k = E_k \frac{\rho \cos \theta \vartheta_k}{r} \approx \frac{E_k \vartheta_k}{r_0} z$$

and use it to compute the bending moment acting at this section

$$B_k = \iint \sigma (-z)\, d\Omega_k = -\frac{E_k I_k}{r_0} \vartheta_k. \tag{18.11}$$

Here $I_k = \iint z^2 d\Omega_k$ is the moment of inertia of the ring's cross section with respect to the horizontal axis which passes through its center of gravity.

In the symmetric case under consideration, the ring is subjected to /204 a thrust force Q_k and a twisting moment M_k (their positive directions are shown in figure 40a). Let us split the ring in half (fig. 40b). From the conditions

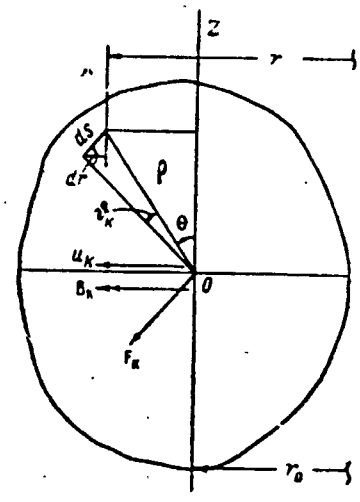

Figure 39

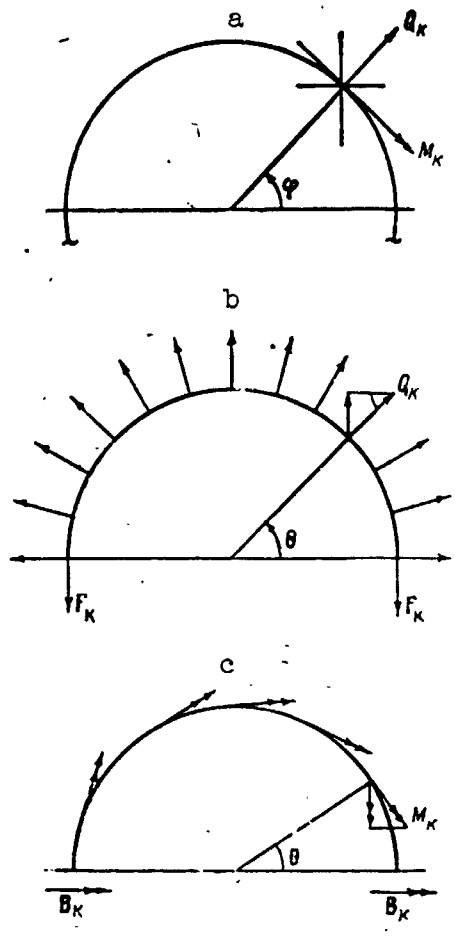

Figure 40

of equilibrium for one-half of the ring which is loaded with thrust forces, we obtain

$$F_k = \frac{1}{2}\int_0^{\pi} Q_k \sin\varphi r_0 d\varphi = r_0 Q_k. \tag{18.12}$$

In the same way, by applying a twisting moment to half of the ring (fig. 40c) we determine the bending moment.

$$B_k = -\frac{1}{2}\int_0^{\pi} M_k \sin\varphi r_0 d\varphi = -r_0 M_k. \tag{18.13}$$

Substituting the expressions established for F_k, B_k into equalities (18.10) and (18.11) and solving them with respect to u_k and ϑ_k, we obtain

$$u_k = \frac{r_0^2}{E_k \Omega_k} Q_k, \quad \vartheta_k = \frac{r_0^2}{E_k I_k} M_k. \tag{18.14}$$

By using the first relationship we shall try to select a connecting ring which provides for a membrane state of stress in the upper part of the dome. First of all, there is a thrust at the upper edge of the dome /205

$$Q_0^* = (T_1 \cos\theta)_{\theta=\theta_1} = \frac{F_z^0}{2\pi R_0} \frac{\cos\theta_1}{\sin^2\theta_1} \tag{18.15}$$

and a horizontal displacement

$$u_0^* = \frac{R_0 \sin\theta_1}{Eh} \left\{ -qR_0 \cos\theta_1 - \frac{F_z^0}{2\pi R_0} \frac{1+\nu}{\sin^2\theta_1} \right\}. \tag{18.16}$$

Since in the considered case (see fig. 43)

$$u_0^* = u_k, \quad Q_k^* = Q_k,$$

it follows from these relationships and from (18.14) that

$$\frac{r_1^2}{E_k \Omega_k} = \frac{u_k}{Q_k} = \frac{u_0^*}{Q_0^*} = \frac{r_1}{Eh} \delta, \tag{18.17}$$

where

$$\delta = -\frac{2\pi r_1^2 q}{F_z^0} \frac{1+\nu}{\cos\theta_1}. \tag{18.18}$$

The weight of the upper missing part of the dome is equal to

$$P(\theta_1) = 2\pi R_0^2 q (1 - \cos\theta_1). \tag{18.19}$$

We introduce the coefficient $k = \frac{F_z^0}{P(\theta_1)}$, which is equal to the ratio of the connecting piece weight to the weight of the removed part of the dome. Now it is easy to obtain an expression for the cross section area of the thrust ring

$$\Omega_k = r_1 h \frac{E}{E_k} \frac{1}{\delta}, \tag{18.20}$$

where

$$\delta = \frac{1+\cos\theta_1}{k} - \frac{1+\nu}{\cos\theta_1}. \tag{18.21}$$

To illustrate these relationships, we calculate a spherical dome having the following dimensions:

$$\begin{gathered} R_0 = 40\ м,\quad h = 0.1\ м\ \left(\frac{R_0}{h} = 400\right),\quad H = 26.0\ м, \\ \theta_1 = 7°30',\quad \sin\theta_1 = 0.130,\quad \cos\theta_1 = 0.991,\quad r_1 = 5.22\ м, \\ \theta_2 = 70°,\quad \sin\theta_2 = 0.940,\quad \cos\theta_2 = 0.342,\quad r_2 = 37.59\ м, \end{gathered} \tag{18.22}$$

fabricated from the Mark 300 concrete

$$\begin{aligned} &E = 2.7\cdot10^6\ t/m^2, \quad \nu = \frac{1}{6} = 0.167, \\ &\gamma = 2.5\ t/m^3, \quad q = \gamma\cdot h = 0.25\ t/m^2. \end{aligned} \tag{18.23}$$

Letter "t" = metric tons

We shall let k = 1, i.e., we shall assume that the weight of the connecting piece with the ring is equal to the weight of the missing part of the dome. Assuming further that the ring is made of the same material we obtain the following from (18.20)-(18.21)

$$\delta = 1 + 0.991 - \frac{1.167}{0.991} = 0.815,$$
$$\Omega_k = \frac{0.130\cdot40\cdot0.1}{0.815} = 0.641\ м^2.$$

It is easy to compute the weight of such a ring which is equal to 52.6 metric tons, while according to the hypothesis the weight of the entire connecting piece with the dome is only $P(\theta_1)$ = 21.5 metric tons. The situation /206

is not saved by decreasing the value of k. Thus, assuming that k = 0.5, we find that a ring which provides for a membrane state of stress must weigh 13.5 metric tons. However, according to the proposition, the ring together with the connecting piece weighs only $\frac{1}{2} P(\theta_1) = 10.75$ metric tons.

Thus, as a rule it is not possible to provide for the membrane state in the region of the connecting piece. Therefore, the cut for the connecting piece is reinforced with a ring whose dimensions are structurally acceptable, and the upper part of the dome is computed by a theory which takes into account the effect of moments.

In line with the adopted positive direction, for the boundary quantities (fig. 41) the condition for the elastic junction of the dome's edge with the connecting piece ring has the form

$$M_k = M_0, \quad Q_k = Q_0, \quad u_k = u_0, \quad \vartheta_k = \vartheta_0. \tag{18.24}$$

Using this information, equalities (18.14) and expressions for the compliance coefficients (18.15), we arrive at the following system of equations

$$\begin{aligned} \frac{r_1^2}{E_k \Omega_k} Q_0 &= \alpha_{11} Q_0 + \alpha_{12} M_0 - \alpha_{11} Q_0^* + u_0^*, \\ \frac{r_1^2}{E_k I_k} M_0 &= \alpha_{12} Q_0 + \alpha_{22} M_0 - \alpha_{12} Q_0^*, \end{aligned} \tag{18.25}$$

which determine Q_0 and M_0.

For the dome which we considered we used a connecting ring of cross section 0.3 r × 0.2 m, for which $E_k = E = 2.7 \times 10^6$ t/m, $\Omega_k = 0.3 \cdot 0.2 = 0.06$ m^2, $I_k = \frac{0.3 \cdot 0.2^3}{12} = 0.0002$ m^4.

Furthermore, for the upper edge of the dome we had

$$r_1 = 5.22 \text{ m}, \quad Q_0^* = -4.98 \text{ t/m}, \quad u_0^* = -0.0784 \cdot 10^{-3} \text{ m},$$
$$\alpha_{11} = -0.132 \cdot 10^{-3} \text{ m}^2/\text{t}, \quad \alpha_{12} = -0.660 \cdot 10^{-3} \text{ m/t},$$
$$\alpha_{22} = -6.613 \cdot 10^{-3} \frac{1}{\text{t}}.$$

Substituting the computed values into the conditions for the elastic junction we obtain a system

$$\begin{aligned} 0{,}168 \cdot 10^{-3} Q_0 &= -0{,}132 \cdot 10^{-3} Q_0 - 0{,}660 \cdot 10^{-3} M_0 - 0{,}735 \cdot 10^{-3}, \\ 50{,}48 \cdot 10^{-3} M_0 &= -0{,}660 \cdot 10^{-3} Q_0 - 6{,}613 \cdot 10^{-3} M_0 - 0{,}330 \cdot 10^{-3}, \end{aligned}$$

which is satisfied by the following

$$M_0 = -0.0302 \text{ t m/m}, \quad Q_0 = -2.38 \text{ t/m}. \tag{18.26}$$

The stresses and strains corresponding to the additional moment state are determined by means of equations presented in section 8. /207

Let us attempt to select a thrust ring for the lower part of the dome which would provide for a membrane state of stress in the lower part of the dome. Equating the horizontal displacements of the shell and the ring edges, we obtain

$$\frac{r_2^2 Q_k}{E_k \Omega_k} = u_k = u_0 = \frac{r_2}{Eh}\left(T_2^* - \nu T_1^*\right)\Big|_{\theta=\theta_1}.$$

From this, noting that $E_k = E$ and $Q_k = -Q_0^*$ (fig. 42), we obtain an expression for the cross section area of the supporting ring

$$\Omega_k = \frac{r_2 h\left(-Q_0^*\right)}{\left(T_2^* - \nu T_1^*\right)_{\theta=\theta_1}}. \tag{18.27}$$

For the considered edge ($r_2 = 37.6$ m)

$$Q_0^* = -2.54 \text{ t/m}, \quad T_2^* = 4.03 \text{ t/m},$$
$$T_1^* = -7.45 \text{ t/m}.$$

Therefore

$$\Omega_k = \frac{37.6 \cdot 0.1 \cdot 2.54}{4.03 + 0.167 \cdot 7.45} = 1.82 \text{ m}^2.$$

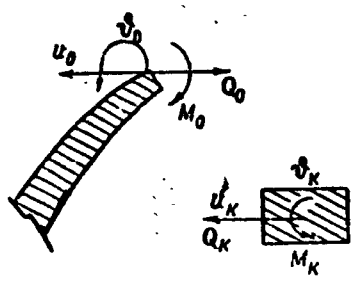

Figure 41

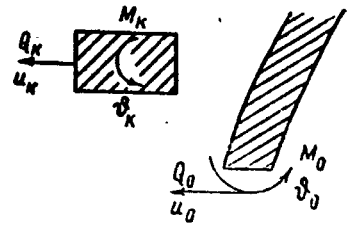

Figure 42

Figure 43 shows the stress diagrams: the solid line shows the tangential stresses, the broken line shows the stresses on the exterior surface, while the dot and dash line shows the stresses at the interior surface. We also note that the displacements and the angle of rotation of the dome edge are rather insignificant quantities.

As we can see from these diagrams, a membrane state of stress is realized far away from the upper edge. This is achieved by installing a rather rigid supporting ring (Ω_k = 1.82 m^2). Although such a ring can be achieved structurally, it is not desirable. The situation is also aggravated by the fact that in real domes the thickness is uniformly increased as the lower supporting ring is approached. (This, in particular, is necessary for a more uniform distribution of temperature strains.) In this case there is a slight increase in the forces T_1^* and T_2^*, but the thickness h increases at a substantially greater rate. It then follows from equality (18.27) that the cross section area of the ring in a real dome is even larger. To lighten the ring, it is desirable to select a more rational shape for the dome (elliptic). If, however, a spherical dome is constructed, then a ring of acceptable dimensions is installed at the lower edge and the calculations are carried out with the theory which takes into account moments. /208

This is done in almost the same manner as the calculation carried out above for the connecting piece ring. The difference is only that, in accordance with figure 43, the conditions for elastic junction (18.24) are replaced by

$$M_k = -M_0, \quad Q_k = -Q_0, \quad u_k = u_0, \quad \vartheta_k = \vartheta_0 \tag{18.28}$$

and in computing the quantities which enter into this condition, it must be assumed that the thickness of the dome is equal to the thickness at its lower edge. Since the lower part of the dome is made thicker, the additional bending stresses in this region are usually quite small.

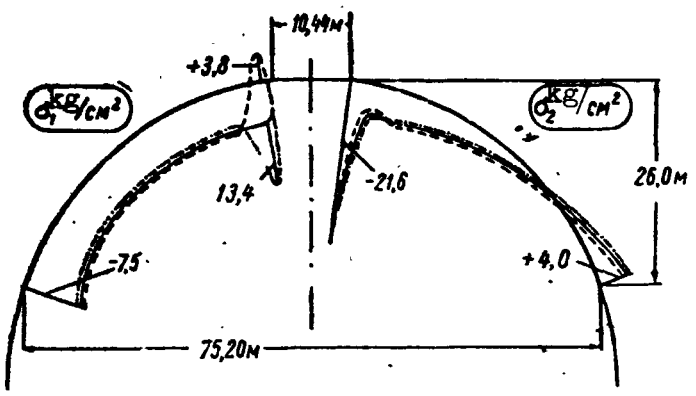

Figure 43

In addition to the symmetric loading (natural weight), an important rule in the calculation of domes is played by wind loading, which may be expressed approximately by the components

$$q_1 = q_2 = 0, \quad q_n = -p \sin\theta \cos\varphi. \tag{18.29}$$

Let us consider the computation of a spherical dome with a connecting piece subjected to a load of type (18.29) (see fig. 44). As in the above case, we shall assume that the connecting piece in the form of a short cylinder does not resist the deformations of the shell and of the upper ring, and that its influence is reduced to the transmission of forces to the upper ring which act in the direction of the wind, and to the transmission of the tumbling moment.

Calculations show that for a closed spherical dome subjected to wind /209
pressure in accordance with law (18.29), the supporting forces (forces in the region of the lower supporting ring) are small, compared with the basic forces produced by symmetric loading. Therefore, as in the symmetric case, the calculation of the dome region which is in direct contact with the upper ring is of prime interest, because it is precisely here that the membrane state of stress is disrupted.

The idea behind the solution of this problem is that the membrane solution is used to satisfy all of the static boundary conditions at the upper edge of the dome; then by considering the general solution of the homogeneous system of equations associated with the moment theory, a provision is made for the common deformation of the shell edge with the reinforcing ring.

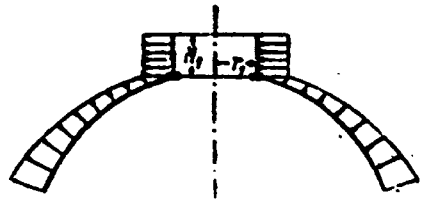

Figure 44

Determination of Forces and Displacements in the Dome According to the Membrane Theory

The membrane forces T_1^*, T_2^* and S^* (see section 4) in the considered problem are determined from the expressions

$$\begin{aligned}
T_1^* &= \frac{1}{R_0^2 \sin^3\theta}\left\{C_2 + C_1 R_0(\cos\theta_1 - \cos\theta) - \right. \\
&\left. - pR_0^3 \cos\theta\left[(\cos\theta_1 - \cos\theta) + \frac{1}{3}(\cos^3\theta - \cos^3\theta_1)\right]\right\}\cos\varphi, \\
T_2^* &= -pR_0 \sin\theta\cos\varphi - \frac{1}{R_0^2\sin^3\theta}\left\{C_2 + C_1 R_0(\cos\theta_1 - \cos\theta) - \right. \\
&\left. - pR_0^3\cos\theta\left[(\cos\theta_1 - \cos\theta) + \frac{1}{3}(\cos^3\theta - \cos^3\theta_1)\right]\right\}\cos\varphi, \\
S^* &= \left\{\frac{\cos\theta}{R_0\sin^3\theta}[C_2 + C_1 R_0(\cos\theta_1 - \cos\theta)] - \frac{C_1}{R_0\sin\theta} - \right. \\
&\left. - \frac{pR_0}{\sin^3\theta}\left[(\cos\theta_1 - \cos\theta) + \frac{1}{3}(\cos^3\theta - \cos^3\theta_1)\right]\right\}\sin\varphi.
\end{aligned} \tag{18.30}$$

The constants of integration C_1 and C_2 contained in solution (18.30) have a clear static meaning, which was evaluated in (9.24). It is easy to verify that in the given problem their values are equal to

$$C_1 = -pr_1H_1, \quad C_2 = -\frac{pr_1H_1^2}{2} \tag{18.31}$$

(where p is the intensity of the normal wind pressure on the connecting piece at the section $\varphi = 0$, while r_1 and H_1 are the radius and height of the connecting piece, respectively), i.e., within the accuracy of the factor $\frac{1}{\pi}$ they

/210

represent the values of the horizontal force and of the tumbling moment which are transmitted by the connected piece to the upper edge of the dome.

The initial system for determining the displacements (4.7) is now reduced to the form

$$\begin{aligned} &\frac{\partial \xi^*}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \zeta^*}{\partial \varphi} = f_1(\theta)\cos\varphi, \\ &\sin\theta\frac{\partial \gamma^*}{\partial \theta} + \frac{\partial \xi^*}{\partial \varphi} = f_2(\theta)\sin\varphi, \end{aligned} \tag{18.32}$$

where

$$\begin{aligned} f_1(\theta) &= \frac{2pR_0^2(1+\nu)}{Eh}\left\{\frac{1}{\sin^4\theta}\left[C_2' + C_1'(\cos\theta_1 - \cos\theta) - \right.\right. \\ &\left.\left. - \cos\theta\left(\cos\theta_1 - \cos\theta + \frac{1}{3}(\cos^3\theta - \cos^3\theta_1)\right)\right] + \frac{1}{2}\right\}, \\ f_2(\theta) &= \frac{2pR_0^2(1+\nu)}{Eh}\left\{\frac{\cos\theta}{\sin^3\theta}\left[C_2' + C_1'(\cos\theta_1 - \cos\theta)\right] - \right. \\ &\left. - \frac{C_1^*}{\sin\theta} - \frac{1}{\sin^3\theta}\left[\cos\theta_1 - \cos\theta + \frac{1}{3}(\cos^3\theta - \cos^3\theta_1)\right]\right\}. \end{aligned} \tag{18.33}$$

Here, for convenience of the calculation steps, we introduce the dimensionless constants C_1' and C_2', which are related with (18.31) by the following equalities

$$C_1 = pR_0^2C_1', \quad C_2 = pR_0^3C_2'$$

and are designated in the future by C_1 and C_2.

To solve the homogeneous systems (18.32), we use the method of the separation of variables, assuming that

$$\xi^* = \xi^*_{,1}(\theta)\cos\varphi, \quad \zeta^* = \zeta^*_{,1}(\theta)\sin\varphi. \tag{18.34}$$

As a result we obtain the following equations for determining $\xi^*_{,1}(\theta)$, $\zeta^*_{,1}(\theta)$

$$\begin{aligned} &\frac{d\xi^*_{,1}}{d\theta} - \frac{1}{\sin\theta}\zeta^*_{,1} = f_1(\theta), \\ &\sin\theta\frac{d\zeta^*_{,1}}{d\theta} - \xi^*_{,1} = f_2(\theta). \end{aligned} \tag{18.35}$$

The general solution of the homogeneous system has the form

$$\xi_1^* = \frac{D_1}{\sin\theta} - D_2 \operatorname{ctg}\theta,$$
$$\zeta_1^* = -D_1 \operatorname{ctg}\theta + D_2 \frac{1}{\sin\theta}, \tag{18.36}$$

therefore the particular solution (18.35) may be sought by the method of the variation of integration constants /211

$$\xi_1^* = \frac{1}{\sin\theta} D_1(\theta) - \operatorname{ctg}\theta D_2(\theta),$$
$$\zeta_1^* = -\operatorname{ctg}\theta D_1(\theta) + \frac{1}{\sin\theta} D_2(\theta). \tag{18.37}$$

Differentiating (18.37) and substituting the result into (18.35), we arrive at an algebraic system of equations containing $D_1'(\theta)$, $D_2'(\theta)$

$$D_1'(\theta) - \cos\theta D_2'(\theta) = \sin\theta f_1(\theta),$$
$$-\cos\theta D_1'(\theta) + D_2'(\theta) = f_2(\theta).$$

From this it follows that

$$D_1(\theta) = \int \frac{1}{\sin^2\theta} [\sin\theta f_1(\theta) + \cos\theta f_2(\theta)]\, d\theta,$$
$$D_2(\theta) = \int \frac{1}{\sin^2\theta} [\sin\theta \cos\theta f_1(\theta) + f_2(\theta)]\, d\theta, \tag{18.38}$$

and after computing the integrals contained in (18.38), determining $\xi_{,1}^*$ and $\zeta_{,1}^*$ according to equations (18.37) and transforming from the auxiliary functions ξ^*, ζ^* to the unknown displacements u^*, v^*, w^*, we shall have the final expressions

$$u^* = \frac{pR_0^2(1+\nu)}{2Eh}\left\{K_1 \ln\operatorname{tg}\frac{\theta}{2} - \left(K_1 + \frac{2}{3}\right)\frac{\cos\theta}{\sin^2\theta} + K_2\frac{1}{\sin^2\theta} + K_2 + \right.$$
$$\left. + K_2\cos\theta \ln\operatorname{tg}\frac{\theta}{2} - \frac{2}{3}\cos\theta \ln\sin\theta\right\}\cos\varphi, \tag{18.39}$$
$$v^* = -\frac{pR_0^2(1+\nu)}{2Eh}\left\{K_2 \ln\operatorname{tg}\frac{\theta}{2} - K_2\frac{\cos\theta}{\sin^2\theta} + \left(K_1 + \frac{2}{3}\right)\left(1 + \frac{1}{\sin^2\theta}\right) + \right.$$
$$\left. + K_1\cos\theta \ln\operatorname{tg}\frac{\theta}{2} - \frac{2}{3}\ln\sin\theta\right\}\sin\varphi,$$

$$w^* = \frac{pR_0^2(1+\nu)}{Eh}\left\{\frac{1+\sin^2\theta}{3\sin\theta} - \frac{1}{2}K_2\frac{\cos\theta}{\sin\theta} + \frac{1}{2}K_2\sin\theta\ln\operatorname{tg}\frac{\theta}{2} - \right.$$
$$\left. - \frac{1}{3}\sin\theta\ln\sin\theta - \frac{1}{1+\nu}\sin\theta\right\}\cos\varphi.$$

Here K_1 and K_2 are constants which are determined from the following equalities

$$K_1 = C_1\cos\theta_1 + C_2,$$
$$K_2 = C_1 + \cos\theta_1 - \frac{1}{3}\cos^2\theta_1.$$

The angles of rotation ϑ^* and ψ^* (equations (1.11)) in this case will be equal to

$$\vartheta^* = \frac{pR_0(1+\nu)}{2Eh}\left[K_1\left(\ln\operatorname{tg}\frac{\theta}{2} - \frac{\cos\theta}{\sin^2\theta}\right) + \frac{2}{1+\nu}\cos\theta\right]\cos\varphi,$$
$$\psi^* = -\frac{pR_0(1+\nu)}{2Eh}\left[K_1\left(1 + \frac{1}{\sin^2\theta} + \cos\theta\ln\operatorname{tg}\frac{\theta}{2}\right) + \frac{2}{1+\nu}\right]\sin\varphi. \qquad (18.40)$$

In conclusion we note that the solution of the homogeneous problem (18.36) determines the rigid displacements of the shell, which will not be of interest to us in the future. The kinematic meaning of the constants D_1 and D_2 contained in solution (18.36) was also established above in (9.38) and (9.39). /212

Calculation of the Upper Ring

To find the solution of the problem of the combined deformation of the dome edge with the upper ring, we calculate the circular ring for the action of edge forces and moments acting from the side of the shell. The arrows in figure 45 show the positive directions of the forces and moments along the edge of the dome: $Q_z = Q^0_{z,1} \times \cos\varphi$ and $Q_\rho = Q^0_{\rho,1}\cos\varphi$ represent the vertical and radial forces, while $S = S^0_{,1}\sin\varphi$ represents the tangential force and $M_1 = M^0_{1,1}\cos\varphi$ represents the meridian bending moment.

By letting $Q^0_{\rho,1} \quad q_0$, $M^0_{1,1} = M_0$ and formulating conditions for the self-balancing of the given load along the contour of the shell, we find that (9.22)

$$S^0_{,1} = Q_{0,1}, \quad Q^0_{z,1} = -\frac{M_0}{r_1}. \tag{18.41}$$

To calculate the ring we introduce the unit vectors of the local coordinate system $\vec{e}_\tau$ (tangential), $\vec{e}_\nu$ (principal normal), $\vec{e}_b$ (binormal), as is customary in the theory of thin rods. We note that this group of three vectors is associated with the orthogonal unit vectors (figs. 15 and 16) which were introduced earlier in the theory of shells of revolution, by the following relationships

$$\vec{e}_\nu = -\vec{e}_\rho, \quad \vec{e}_b = \vec{e}_z, \quad \vec{e}_\tau = \vec{e}_\varphi.$$

It is obvious that the positive directions of the forces and moments acting from the side of the shell under a ring are opposite to those shown in figure 45. Furthermore, taking into account the position of the local system of coordinates along the axis of the ring (fig. 45) and taking into account equations (18.41), we obtain the following components for the external loads on the ring

$$\begin{aligned} q_\nu &= -Q_0 \cos\frac{s}{r_1}, \quad m_\nu = m_b = 0, \\ q_b &= \frac{M_0}{r_1}\cos\frac{s}{r_1}, \quad m_\tau = M_0 \cos\frac{s}{r_1}, \\ q_\tau &= Q_0 \sin\frac{s}{r_1}, \end{aligned} \tag{18.42}$$

Here s is the length of the arc of the ring's axial line measured from the plane $\varphi = 0$, while r_1 is its radius. /213

The periodic solution obtained by integrating the well-known Kirchhoff-Klebysh system of differential equations (ref. 89) for a thin circular rod subjected to a load of type (18.42) is shown in table 2.

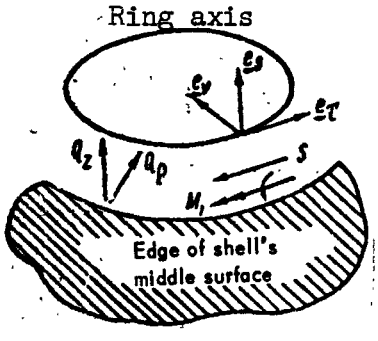

Figure 45

TABLE 2

	Tangential	Principal normal	Binormal
Force	$\frac{Q_0 F r_1^3}{F r_1^2 + I_b} \cos \frac{s}{r_1}$	$\frac{Q_0 r_1 I_b}{F r_1^2 + I_b} \sin \frac{s}{r_1}$	$-M_0 \sin \frac{s}{r_1}$
Moment	$-\frac{M_0 r_1 G I_d}{E I_v + G I_d} \sin \frac{s}{r_1}$	$\frac{M_0 r_1 E I_v}{E I_v + G I_d} \cos \frac{s}{r_1}$	$\frac{Q_0 r_1^2 I_b}{F r_1^2 + I_b} \cos \frac{s}{r_1}$
Displacement	$\left[\frac{Q_0 r_1^4}{E\left(F r_1^2 + I_b\right)} + A\right] \sin \frac{s}{r_1}$	$A \cos \frac{s}{r_1}$	$B r_1 \cos \frac{s}{r_1}$
Rotation	$\left[\frac{M_0 r_1^2}{E I_v + G I_d} - B\right] \cos \frac{s}{r_1}$	$B \sin \frac{s}{r_1}$	$\frac{Q_0 r_1^3}{E\left(F r_1^2 + I_b\right)} \sin \frac{s}{r_1}$

The designations in table 2 are as follows: F is the area of the ring's cross section; I_ν, I_b, I_d are the moments of inertia of this section with respect to the horizontal, vertical axis and during torsion, respectively; E is the modulus of normal elasticity and G is the shear modulus of the ring material. For concrete cases these are related by the expression G = 0.425 E; A and B are arbitrary constants of integration corresponding to the rigid displacement and rotation of the ring.

The values of these constants can be established after determining the rigid displacements of the upper edge of the shell. Since the displacements and rotations of the ring which depend on A and B do not produce a deformation, the corresponding terms in table 2 may be dropped in the future.

Conditions for Joining the Shell With the Upper Ring Results of Calculations

In section 10 we obtained the strain boundary conditions (10.24), taking into account the elastic rotation and the relative elongation of the shell's edge. We recall that along the junction the positive directions for the rotation of the upper edge of the dome and the rotation of the ring section around the $\vec{e}_\tau$ axis coincide (fig. 41).

Therefore, by using equations (10.12), (10.26), (18.39) and (18.40) and also table 2 and by equating the elastic rotations and the relative elongation of the ring's edge on one side and of the ring on the other, we obtain the following condition for elastic joining

$$\begin{aligned}
\frac{r_1^4}{E(Fr_1^2+I_b)}Q_0 &= \frac{pr_1}{Eh}\left\{(1+\nu)\frac{H_1^2}{2r_1\sin\theta_1}-r_1\right\}+\alpha_{11}(Q_0-Q_0^*)+\alpha_{12}M_0,\\
\frac{r_1^2}{EI_\nu+GI_d}M_0 &= \frac{pR_0(1+\nu)}{2Eh}\left[\frac{2\cos\theta_1}{3\sin^2\theta_1}-\frac{2K_2}{\sin^2\theta_1}+\frac{2\cos\theta_1(1+\sin^2\theta_1)}{3\sin^2\theta_1}\right]+\\
&\quad+\alpha_{21}(Q_0-Q_0^*)+\alpha_{22}M_0.
\end{aligned} \tag{18.43}$$

Having determined Q_0 and M_0 from this system, we can use equations (8.11) and (10.21) to find all of the forces and moments in the upper part of the dome.

Let us carry out a numerical calculation of the same dome as in the symmetric case. The basic parameters of this dome are presented in equations (18.22) and (18.23).

Assuming that the height of the connecting piece H_1 = 2.0 m, while /215 the intensity of the normal wind pressure is p = 0.07 t/m^2, we find that at the upper edge of the dome

$$Q_0^* = -0.407 \text{ t/m}; \; u_{0,1}^* = -2.433 \cdot 10^{-6} \text{ m}; \; \chi_0^* = 4.18 \cdot 10^{-6} \text{ rad}.$$

Substituting the computed values into the conditions for elastic joints (18.43), we obtain a system

$$0.168 \cdot 10^{-3} Q_0 = -0.132 \cdot 10^{-3} Q_0 - 0.660 \cdot 10^{-3} M_0 - 0.056 \cdot 10^{-3},$$
$$25.24 \cdot 10^{-3} M_0 = -0.660 \cdot 10^{-3} Q_0 - 6.613 \cdot 10^{-3} M_0 - 0.265 \cdot 10^{-3},$$

whose solution will be

$$M_0 = -0.00465 \frac{\text{t} \cdot \text{m}}{\text{m}}, \; Q_0 = -0.177 \text{ t/m}. \tag{18.44}$$

Comparing these quantities with the analogous ones obtained in the symmetric case (18.26), we see that the maximum stresses occurring in the considered dome under the action of wind loading will be approximately one order of magnitude less than those which took place in the symmetric case. This is quite natural. However, a change in the dimensions of the connecting piece (for example, an increase in its height) may make the wind stresses compatible with the basic (symmetric) loads; in computing the domes it is, therefore, expedient to verify the stresses due to the wind pressure on the basis of the presented solution.

19. Examples

In concluding the chapter, we shall consider several examples for computing shells of revolution. Below we present examples of both theoretical and certain practical interest.

Example 1

Let us consider a spherical vessel designed to store a liquid, (with a specific weight γ_1). The vessel is installed on a circular cylindrical support, whose length is selected in such a way that on the one hand we can neglect the mutual effects of the edges and on the other hand we need not fear a loss of stability in the cylindrical support. The vessel is shown schematically in figure 47.

The design load in this case is the hydrostatic pressure of the liquid

$$q_n = R_0 \gamma_1 (1 - \cos \theta) \quad (q_1 = 0). \tag{19.1}$$

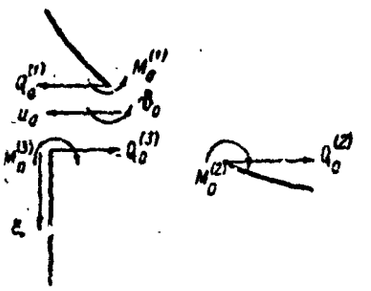

Figure 46

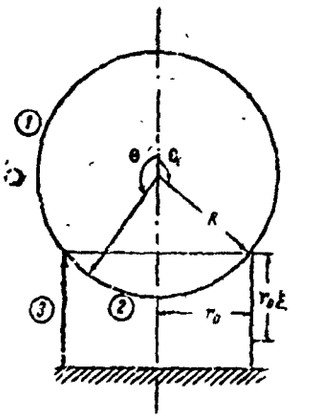

Figure 47

To make the calculations convenient, we break up the shell into three parts (fig. 47).

For the first part, according to (6.2), (4.5) and (8.14)

$$U^{(1)}=\int_0^\theta q_n\cos\theta R_1R_2\sin\theta d\theta=R_0^3\gamma t\int_0^\theta(\cos\theta-\cos^2\theta)\sin\theta d\theta$$
$$=\frac{R_0^3\gamma t}{6}(1-3\cos^2\theta+2\cos^3\theta),$$
$$T_1^{*(1)}=\frac{R_0^2\gamma t}{6}\frac{1-3\cos^2\theta+2\cos^3\theta}{\sin^2\theta},\qquad(19.2)$$
$$T_2^{*(1)}=R_0^2\gamma t\left\{1-\cos\theta-\frac{1}{6}\frac{1-3\cos^2\theta+2\cos^3\theta}{\sin^2\theta}\right\},$$
$$Q_0^{*(1)}=(T_1^{*(1)}\cos\theta)_{\theta=\theta_0},\quad u_0^{*(1)}=\frac{R_0\sin\theta_0}{Eh_1}(T_2^{*(1)}-\nu T_1^{*(1)})_{\theta=\theta_0}.$$

For the second /216

$$U^{(2)}=R_0^2\gamma t\int_0^\theta(\cos\theta-\cos^2\theta)\sin\theta d\theta=\frac{R_0^3\gamma t}{6}(5-3\cos^2\theta+2\cos^3\theta)$$
$$T_1^{*(2)}=\frac{R_0\gamma t}{6}\frac{5-3\cos^2\theta+2\cos^3\theta}{\sin^2\theta},\qquad(19.3)$$
$$T_2^{*(2)}=R_0\gamma t\left\{1-\cos\theta-\frac{1}{6}\frac{5-3\cos^2\theta+2\cos^3\theta}{\sin^2\theta}\right\},$$
$$Q_0^{*(2)}=(T_1^{*(2)}\cos\theta)_{\theta=\theta_0},\quad u_0^{*(2)}=\frac{R_0\sin\theta_0}{Eh_2}(T_2^{*(2)}-\nu T_1^{*(2)})_{\theta=\theta_0}.$$

For the third (cylindrical part)

$$T_1^{*(3)}=-\frac{P}{2\pi r_0},\quad T_2^{*(3)}=0,$$
$$Q_0^{*(3)}=0,\quad u_0^{*(3)}=\frac{w_0}{Eh_3}\frac{P}{2\pi r_0}.\qquad(19.4)$$

Here $P = P_l + P_v$ is the weight of the vessel with the liquid and in this case

$$P_l = \frac{4\pi R_0^3}{3}\gamma_l, \quad P_v = 4\pi R_0^2 h \gamma_v \quad (h = h_1 = h_2). \tag{19.5}$$

If we take into account the positive direction of the boundary quantities (fig. 46), the conditions at the joints have the form /217

$$u_0^{(1)} = u_0^{(2)} = u_0^{(3)} = u_0, \quad \vartheta_0^{(1)} = \vartheta_0^{(2)} = \vartheta_0^{(3)} = \vartheta_0, \tag{19.6}$$

$$M_0^{(1)} - M_0^{(2)} - M_0^{(3)} = 0, \quad Q_0^{(1)} - Q_0^{(2)} - Q_0^{(3)} = 0. \tag{19.7}$$

Let us carry out the numerical calculation of a specific vessel for which

$$\begin{gathered} E = 2\cdot 10^6 \text{ kg/cm}^2, \ \gamma_l = 1.03 \text{ g/cm}^3, \ \gamma_v = 7.8 \text{ g/cm}^3, \\ R = 525 \text{ cm}, \ r_0 = 432.5 \text{ cm}, \ h_1 = h_2 = 2.5 \text{ cm}, \ h_3 = 2 \text{ cm}. \end{gathered} \tag{19.8}$$

First of all

$$P_l = \frac{4\cdot\pi\cdot(525)^3\cdot 1{,}03}{3\cdot 1000} = 623000 \text{ kg},$$

$$P_v = \frac{4\cdot\pi\cdot(525)^2\cdot 2{,}5\cdot 7{,}8}{1000} = 67\,600 \text{ kg}, \quad \vartheta_0 = \arcsin\frac{r_0}{R_0} = 125^\circ.$$

Then by using equalities (8.14) and (8.16) we determine the compliance coefficients and the membrane edge quantities

For shell number 1

$$a_{11}^1 = 2{,}63\cdot 10^{-3}\frac{cm^2}{kg}, \quad a_{12}^1 = -1{,}14\cdot 10^{-4}\frac{cm}{kg}, \quad a_{22}^1 = -0{,}985\cdot 10^{-5}\frac{1}{kg},$$

$$Q_0^{*(1)} = +14{,}63\frac{kg}{cm}, \quad u_0^{*(1)} = +4{,}13\cdot 10^{-2} \text{ cm}.$$

For shell number 2

$$a_{11}^2 = -2{,}63\cdot 10^{-3}\frac{cm^2}{kg}, \quad a_{12}^2 = -1{,}14\cdot 10^{-4}\frac{cm}{kg}, \quad a_{22}^2 = -0{,}985\cdot 10^{-5}\frac{1}{kg},$$

$$Q_0^{*(2)} = -147\frac{kg}{cm}, \quad u_0^{*(2)} = 0{,}955\cdot 10^{-2} \text{ cm}.$$

For shell number 3

$$a_{11}^3 = -4{,}08\cdot 10^{-3}\frac{cм^2}{kg},\quad a_{12}^3 = -1.78\cdot 10^{-4}\frac{cм}{kg},\quad a_{22}^3 = -1{,}56\cdot 10^{-5}\frac{1}{kg},$$
$$Q_0^{*(3)} = 0,\quad u_0^{*(3)} = 0.743\cdot 10^{-2}\ cм.$$

By using these values, we can write the continuity conditions (19.6) by utilizing expression (8.15). In this case, as we have already stated, we let $\vartheta^{*(i)} = 0$. Adding the equilibrium conditions for the junction of the three shells (19.7) to the continuity conditions, we arrive at a system of six algebraic equations

$$\begin{aligned}
&26.3\left(Q_0^{(1)} + Q_0^{(2)}\right) - 1{,}14\left(M_0^{(1)} - M_0^{(2)}\right) + 3800 = 0,\\
&-26.3Q_0^{(2)} + 40.8Q_0^{(3)} - 1.14M_0^{(2)} + 1.78M_0^{(3)} - 3840 = 0,\\
&-11.4\left(Q_0^{(1)} - Q_0^{(2)}\right) + 0{,}985\left(M_0^{(1)} + M_0^{(2)}\right) + 1840 = 0,\\
&-11.4Q_0^{(2)} + 17.8Q_0^{(3)} - 0{,}985M_0^{(2)} + 1.56M_0^{(3)} - 1676 = 0,\\
&Q_0^{(1)} - Q_0^{(2)} - Q_0^{(3)} = 0,\\
&M_0^{(1)} - M_0^{(2)} - M_0^{(3)} = 0.
\end{aligned} \tag{19.9}$$

Simplifying it by means of the last equations and solving it, we arrive at the following values for the boundary static quantities /218

$$\begin{aligned}
&Q_0^{(1)} = -64{,}1\frac{kg}{cм},\quad M_0^{(1)} = -926\ kg,\\
&Q_0^{(2)} = -96{,}0\frac{kg}{cм},\quad M_0^{(2)} = -573\ kg,\\
&Q_0^{(3)} = 31{,}9\frac{kg}{cм},\quad M_0^{(3)} = -353\ kg.
\end{aligned} \tag{19.10}$$

Figure 48 shows the diagrams of the meridian and ring stresses on the external surface of the vessel.

Example 2[1]

Let us consider a truncated conic cantilever (fig. 23), which is built into a wide base and covered at the other end by a very rigid diaphragm. A force P and a bending moment M are applied to the diaphragm. In problems of this type the rigidity of the structure is usually of more interest than the value of the stresses. Therefore, contrary to the examples considered earlier, we devote prime attention to the determination of the angle of rotation and of the radial displacement of the diaphragm. /219

[1]The calculations were carried out by V. A. Shamina.

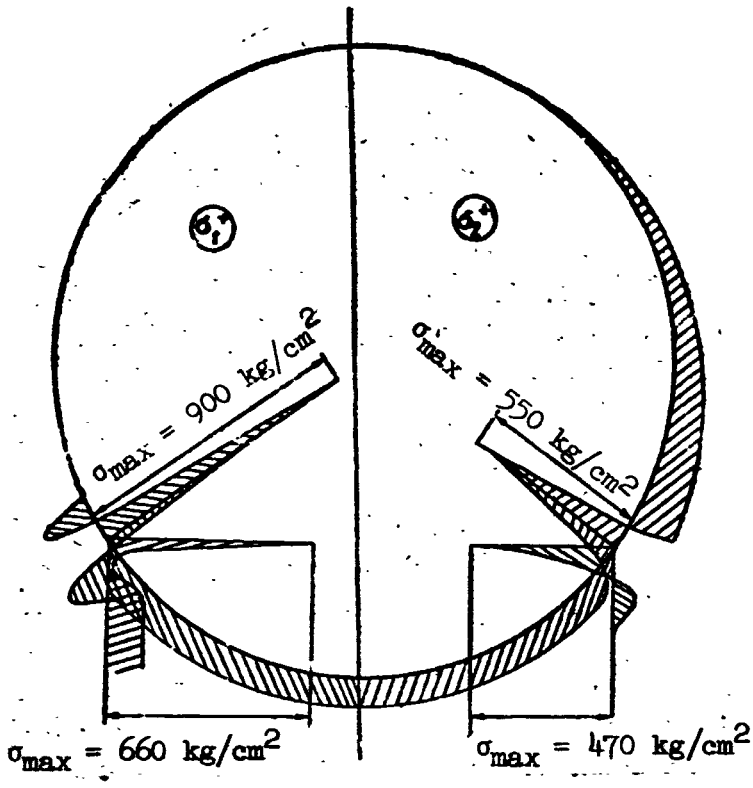

Figure 48

This problem falls within the scope of the first case of axial symmetry associated with the deformation of a conic shell. Surface loading is absent. Therefore, by introducing a variable which is more convenient in this problem

$$r = s\cos\theta_0$$

and noting that according to (12.18) and figure 23

$$C_3 = \frac{1}{\pi}F_x(s_1) = -\frac{P}{\pi}, \quad C_4 = \frac{1}{\pi}B_y(s_1) = -\frac{M}{\pi}.$$

we obtain from (12.17), (12.18) and (12.23)

$$W = -\frac{M + P(s - s_1)\sin\theta_0}{\pi r},$$

$$T_{1,1}^* = -\frac{M + P(s - s_1)\sin\theta_0}{\pi r^2 \sin\theta_0}, \quad T_{2,1}^* = 0,$$

$$S_{,1}^* = -\frac{M + P(s - s_1)\sin\theta_0}{\pi r^2 \sin\theta_0}\cos\theta_0 + \frac{P}{\pi r}.$$

It follows from this that

$$T^{\circ}_{1,1}(s_1)=-\frac{M}{\pi r_1^2\sin\theta_0},\qquad T^{\circ}_{1,1}(s_2)=-\frac{M+HP}{\pi r_2^2\sin\theta_0},$$
$$u^{\circ}_0(s_1)=-\frac{\nu r_1}{Eh}T^{\circ}_{1,1}(s_1),\qquad u^{\circ}_0(s_2)=-\frac{\nu r_2}{Eh}T^{\circ}_{1,1}(s_2)$$

At the edge $s = s_2$ we have the following constraint

$$u_{\rho,1}=u_{z,1}=v_{,1}=\vartheta_{,1}=0.$$

These conditions may be written in a more convenient form

$$\gamma(s_2)=0,\quad \chi(s_2)=0,\quad u_{\rho,1}(s_2)=0,\quad u_{z,1}(s_2)=0. \tag{19.11}$$

The first two conditions, as we have already stated, are formulated in terms of the strain components. This fixes the rigid displacement of the shell (constants a_3, a_4). At the other edge we have conditions for a rigid edge

$$\gamma(s_1)=0,\quad \chi(s_1)=0,$$

since the rigid diaphragm prevents ring deformation and elastic rotation of the shell's edge.

We are interested in the values of the radial displacement and of the angle of rotation of the diaphragm, i.e., in the quantities

$$\Delta=u_{\rho,1}(s_1),$$
$$\vartheta=\vartheta_{,1}(s_1)=-\frac{u_{z,1}(s_1)}{r_0}$$
$$(\chi(s_1)=0).$$

First of all, by using equality (12.10) the conditions at the edges /220 are written in the form

$$a_{11}(Q_0-Q_0^{\circ})+a_{12}M_0+u_0^{\circ}=0,$$
$$a_{12}(Q_0-Q_0^{\circ})+a_{22}M_0=0.$$

Solving this equality, we find for $s = s_1$

$$Q_0 - Q_0^* = -2\nu T_{1,1}^*(s_1)\sqrt{\frac{c}{2r_1\sin\theta_0}},$$
$$M_0 = c\nu T_{1,1}^*(s_2).$$

These values, according to (12.23), correspond to

$$\sigma_{M_0} = \frac{3\nu}{\sqrt{3(1-\nu^2)}}\frac{T_{1,1}^*(s_1)}{h}, \quad \sigma_k = -2\nu\frac{T_{1,1}^*(s_1)}{h}.$$

In the same way, for the edge $s = s_2$, we have

$$Q_0 - Q_0^* = 2\nu T_{1,1}^*(s_2)\sqrt{\frac{c}{2r_2\sin\theta_0}},$$
$$M_0 = c\nu T_{1,1}^*(s_2),$$
$$\sigma_{M_0} = \frac{3\nu}{\sqrt{3(1-\nu^2)}}\frac{T_{1,1}^*(s_2)}{h}, \quad \sigma_k = -2\nu\frac{T_{1,1}^*(s_2)}{h}.$$

It is now easy to use expressions (12.12) and compute the stresses in the shell.

Furthermore, by introducing these expressions into equalities (12.13) and subjecting them to boundary conditions (19.11), we obtain, after a few simple steps, the compliance coefficients of the cantilever

$$\Delta = \gamma_{11}P + \gamma_{12}M,$$
$$\theta = \gamma_{12}P + \gamma_{22}M,$$

where

$$\gamma_{11} = \frac{1}{\pi Eh\cos\theta_0}\left\{(1+\nu)\frac{r_2^2-r_1^2}{r_2^2} + \frac{1}{\cos^2\theta_0}\left[-\frac{r_2-r_1}{r_2} - \frac{(r_1-r_2)^2}{2r_2^2} - \ln\frac{r_1}{r_2}\right]\right\},$$
$$\gamma_{12} = \frac{1}{2\pi Eh\sin\theta_0\cos^2\theta_0}\frac{(r_1-r_2)^2}{r_1\cdot r_2^2}\left[1 + 2(1+\nu)\cos^2\theta_0\frac{r_1+r_2}{r_1-r_2}\right],$$
$$\gamma_{22} = \frac{1+2(1+\nu)\cos^2\theta_0}{2\pi Eh\sin^2\theta_0\cos\theta_0}\frac{r_2^2-r_1^2}{r_1^2r_2^2}.$$

From the relationships established above, it is easy to obtain the corresponding relationships for a cylindrical cantilever. For this purpose we must

approach the limit in all of these equalities, letting θ_0 tend to $\pi/2$. As a result we transform to the following equalities

$$T^*_{1,1}=-\frac{M+Pr_0(\xi-\xi_1)}{\pi r_0^2},\quad T^*_{2,1}=0,\quad S^*_{,1}=\frac{P}{\pi r_0}.$$

$$\gamma_{11}=\frac{1}{\pi Eh}\left\{2(1+\nu)(\xi_2-\xi_1)+\frac{1}{3}(\xi_2-\xi_1)^3\right\}.$$

$$\gamma_{12}=\frac{1}{2\pi Ehr_0}(\xi_2-\xi_1)^2,$$

$$\gamma_{22}=\frac{1}{\pi Eh r_0^2}(\xi_2-\xi_1)^3.$$

We strongly urge the reader to obtain the last relationships directly from equalities of section 11. /221

Let us apply the equations which we have obtained to the calculation of the following cantilever

$$\theta_0=19°30',\quad h=1.1\ см,\quad r_1=7.17\ см,\quad r_2=41.7\ см,$$
$$E=2.1\cdot10^6\ \text{kg}/см^2,\quad \nu=0.3,$$
$$M=6620\ \text{kg}\cdot см,\quad P=0.$$

From this it follows that

$$\sin\theta_0=0.334,\quad \cos\theta_0=0.942,$$

$$\sigma_{T^*_{1,1}}(s_1)=-\frac{M}{\pi\sin\theta_0 r_1^2 h}=-\frac{6620}{\pi\cdot0.334\cdot7.17^2\cdot1.1}=-111\ \text{kg}/\text{см}^2.$$

The maximum stress is the total meridian stress at the edge $s = s_1$ (in the cross section $\varphi = 0$). It is relatively small

$$\sigma_{\max}=\sigma_{T^*_{1,1}}(s_1)+\sigma_{M_0}=\sigma_{T^*_{1,1}}(s_1)\left(1+\frac{3\nu}{\sqrt{3(1-\nu^2)}}\right)=-172\ \text{kg}/\text{cm}^3.$$

Computing the angle of rotation we obtain

$$\vartheta=M\frac{1+2(1+\nu)\cos^2\theta_0}{2\pi Eh\sin^2\theta_0\cos\theta_0}\frac{r_2^2-r_1^2}{r_1^2r_2^2}=$$
$$=\frac{(1+2.6\cdot0.942^2)\cdot6620}{2\pi\cdot2.1\cdot10^6\cdot1.1\cdot0.334^2\cdot0.942}\cdot\frac{41.7^2-7.7^2}{7.17^2\cdot41.7^2}=0.273\cdot10^{-3}.$$

Example 3[1]

Let us assume that a sufficiently large plate which has been weakened by a circular hole is extended along the outer contour by forces of intensity ph and qh in the direction of X and Y axes (fig. 49). If a connecting pipe is attached to the hole, it reduces the stress concentration in the plate and thereby makes it stronger. By considering the connecting pipe to be a sufficiently long cylindrical shell we shall consider the question of its strengthening effect. We shall select the positive directions of the coordinates as shown in figure 44.

We shall construct solutions for the plate and for the cylindrical shells and then join them using the conditions of complex junction.

We start with the plate. The boundary conditions at the external edge of the plate may be replaced approximately by conditions at infinity. /222

$$T_x^{(-)}=ph,\quad T_y^{(-)}=qh,\quad T_{xy}^{(-)}=0.$$

In terms of the polar coordinates which we utilize these conditions are written in the form

$$\begin{aligned} T_1^{(-)} &= \frac{h}{2}[(p+q)+(p-q)\cos 2\varphi], \\ T_2^{(-)} &= \frac{h}{2}[(p+q)-(p-q)\cos 2\varphi], \\ S^{(-)} &= -\frac{h}{2}(p-q)\sin 2\varphi. \end{aligned} \tag{19.12}$$

Since the surface load in the problem considered by us is absent, the written boundary conditions determine the form of relationships for the unknown quantities as functions of the angle φ. Thus

$$\begin{aligned} \widetilde{T}_1(\rho,\varphi) &= \widetilde{T}_1+\widetilde{T}_{1,2}\cos 2\varphi, \\ \widetilde{T}_2(\rho,\varphi) &= \widetilde{T}_2+\widetilde{T}_{2,2}\cos 2\varphi, \\ \widetilde{S}(\rho,\varphi) &= \widetilde{S}+\widetilde{S}_{,2}\sin 2\varphi. \end{aligned}$$

According to relationships (13.14) and (13.15), in the symmetric case

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[1] The calculations were carried out by V. M. Mal'kov.

$$\tilde{T}_1 = -\frac{i}{c}\frac{F_z^0}{2\pi}\left(\frac{1}{2}\ln\rho - \frac{1}{4}\right) + \frac{1}{2}\tilde{C}_1 + \frac{\tilde{C}_2}{\rho^2},$$
$$\tilde{T}_2 = -\frac{i}{c}\frac{F_z^0}{2\pi}\left(\frac{1}{2}\ln\rho + \frac{1}{4}\right) + \frac{1}{2}\tilde{C}_1 - \frac{\tilde{C}_2}{\rho^2}.$$

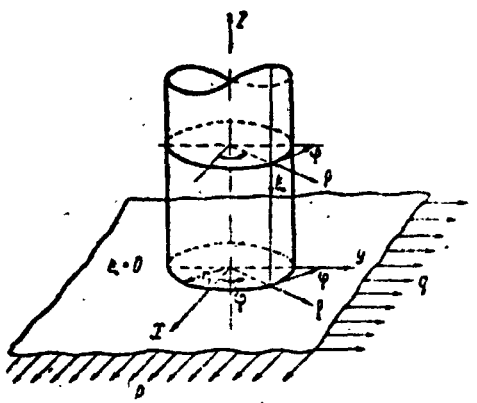

Figure 49

We note, first of all, that due to the absence of the vertical loading both in the plate and in the cylindrical shell $F_z^0 = 0$. Therefore, ($\tilde{C}_2 = \tilde{A}$;

$$\tilde{T}_1 = \frac{\tilde{C}_1}{2} + \frac{\tilde{A}}{\rho^2}, \quad \tilde{T}_2 = \frac{\tilde{C}_1}{2} - \frac{\tilde{A}}{\rho^2}, \quad \tilde{S} = 0. \tag{19.13}$$

In the case k = 2, by seeking a particular solution of the homogeneous system (13.6) (in the same way as carried out in section 14) in the form $X = \rho^s$ and retaining terms which are finite when $\rho \to \infty$, we obtain the following expressions for the complex forces

$$\tilde{T}_1 = \left[-\frac{\tilde{C}_2}{2} + \tilde{B}\rho^{-2} + \tilde{D}\rho^{-4}\right]\cos 2\varphi,$$
$$\tilde{T}_2 = \left[\frac{\tilde{C}_2}{2} - \tilde{D}\rho^{-4}\right]\cos 2\varphi,$$
$$\tilde{S} = \left[\frac{\tilde{C}_2}{2} + \frac{\tilde{B}}{2}\rho^{-2} + \tilde{D}\rho^{-4}\right]\sin 2\varphi.$$

Adding these expressions to (19.13) and subjecting them to conditions /223 at infinity (19.12), we arrive at the following relationships

$$\widetilde{T}_1=\frac{h}{2}(p+q)+\widetilde{A}\rho^{-2}+\left[\frac{h}{2}(p-q)+\widetilde{B}\rho^{-2}+\widetilde{D}\rho^{-4}\right]\cos 2\varphi,$$
$$\widetilde{T}_2=\frac{h}{2}(p+q)-\widetilde{A}\rho^{-2}+\left[-\frac{h}{2}(p-q)-\widetilde{D}\rho^{-4}\right]\cos 2\varphi, \qquad (19.14)$$
$$\widetilde{S}=\left[-\frac{h}{2}(p-q)+\frac{\widetilde{B}}{2}\rho^{-2}+\widetilde{D}\rho^{-4}\right]\sin 2\varphi,$$
$$\widetilde{T}=h(p+q)+B\rho^{-2}\cos 2\varphi.$$

The expressions which have been established, in particular, make it possible for us to obtain the well-known solution of the problem of the concentration of stresses in a plane weakened by a circular hole. Indeed, by subjecting the solution to conditions at the free end

$$T_1\ (1) = S\ (1) = 0,$$

we find from (19.14)

$$\mathrm{Re}\,\widetilde{A}=-\frac{h}{2}(p+q),\quad \mathrm{Re}\,\widetilde{B}=-2h(p-q),\quad \mathrm{Re}\,\widetilde{D}=\frac{3}{2}h(p-q).$$

Let us substitute the values obtained for the constants into the expression for $T_2 = \mathrm{Re}\ \widetilde{T}_2$. Assuming that in this expression $\rho = 1$, we obtain an equality for the boundary value of the ring force

$$T_2(\varphi)=h(p+q)-2h(p-q)\cos 2\varphi.$$

Let us consider three specific cases

(a) p = q.

In this case

$$T_2=2hp,\quad \sigma_{T_2}=2p.$$

The concentration coefficient (K) is equal to 2.

(b) q = 0.

$$T_2=hp-2hp\cos^2\varphi,$$

From this it follows that

$$(T_2)_{max}=T_2(\pm\pi/2)=3hp,\quad (s_2)_{max}=3p,\quad K=3.$$

(c) q = -p.

$$T_2=-4hp\cos 2\varphi,$$
$$(T_2)_{max}=T_2(\pm\pi/2)=4hp,\quad (s_2)_{max}=4p,\quad K=4.$$

Now let us turn to the cylindrical shell. For a symmetric case (axial load is absent) /224

$$\widetilde{T}_2=\widetilde{T}=\widetilde{C}e^{b(1-i)\xi},$$
$$\widetilde{T}_1=\widetilde{S}=0\quad \left(b=\sqrt{\frac{r_0}{2t}}\right). \tag{19.15}$$

(Assuming that the connecting pipe is sufficiently long, we retain only the solution associated with the edge $\xi = 0$.)

The case k = 2 requires special consideration. According to (11.4) the system which determines the complex forces assumes the following form when k = 2

$$\Delta_2\widetilde{U}_{,2}+4r_0\left(1+\frac{i}{2b^2}\right)\widetilde{T}_{,2}=0,$$
$$-\frac{i}{2b^2}\Delta_2\widetilde{T}_{,2}+\widetilde{T}_{,2}-\frac{1}{r_0}\widetilde{U}_{,2}=0$$
$$\left(\Delta_2=\frac{d^2}{d\xi^2}-4\right).$$

Eliminating $\widetilde{U}_{,2}$ from the above and expanding the equation obtained in this manner, we have

$$\frac{d^4\widetilde{T}_{,2}}{d\xi^4}+(12b^2-4)\frac{d^2\widetilde{T}_{,2}}{d\xi^2}+12\widetilde{T}_{,2}=0. \tag{19.16}$$

We seek its particular solution in the form

$$\widetilde{T}_{,2}=e^{\gamma\xi}.$$

Substituting the adopted expression into (19.16), we arrive at the following characteristic equation

$$\gamma^4+(i2b^2-4)\gamma^2+12=0.$$

Solving this equation we find

$$\gamma^2=(4-ib^2)\left[1\pm\sqrt{1-\frac{12}{(4-i2b^2)^2}}\right].$$

The underlined term is small. Therefore

$$\sqrt{1-\frac{12}{(4-i2b^2)^2}}\approx 1-\frac{6}{(4-i2b^2)^2}$$

and

$$\gamma^2=(4-ib^2)\left[1\pm\left(1-\frac{6}{(4-i2b^2)^2}\right)\right].$$

From this we have

$$\gamma_{1,3}^2=(4-ib^2)\left[2-\frac{6}{(4-i2b^2)^2}\right]\approx 2(4-ib^2),$$
$$\gamma_{2,4}^2=\frac{6}{4-ib^2},$$

and

$$\gamma_1=\sqrt{8-i2b^2},\quad \gamma_3=-\gamma_1,$$
$$\gamma_2=\sqrt{\frac{6}{4-ib^2}},\quad \gamma_4=-\gamma_2.$$

Since we are considering only the solution associated with the edge $\xi = 0$, it is necessary to retain the terms corresponding to γ_1 and γ_2. Then /225

$$\widetilde{T}_{,2}=\widetilde{C}_1e^{\gamma_1\xi}+\widetilde{C}_2e^{\gamma_2\xi}.$$

Having obtained the expression for the basic complex function, it is not difficult to use relationships (11.5) and compute the complex forces

$$\begin{aligned}
\widetilde{T}_{1,2} &= -i\frac{4}{2b^2}\widetilde{C}_1 e^{\gamma_1\xi} + \left(1+i\frac{4}{2b^2}\right)\widetilde{C}_2 e^{\gamma_2\xi}, \\
\widetilde{T}_{2,2} &= \left(1+i\frac{4}{2b^2}\right)\widetilde{C}_1 e^{\gamma_1\xi} - i\frac{4}{2b^2}\widetilde{C}_2 e^{\gamma_2\xi}, \\
\widetilde{S}_{,2} &= i\frac{1}{2}\frac{4}{2b^2}\gamma_1\widetilde{C}_1 e^{\gamma_1\xi} - \frac{1}{2}\left(1+i\frac{4}{2b^2}\right)\gamma_2\widetilde{C}_2 e^{\gamma_2\xi}.
\end{aligned} \tag{19.17}$$

To join the constructed solutions, we use the conditions of complex junction which assume the following form for the selected positive directions

$$\begin{aligned}
\{\widetilde{M}_1\}^{pl} &= \{\widetilde{M}_1\}^{cyl}, \quad \{\widetilde{Q}_{12}\}^{pl} = \{\widetilde{Q}_{12}\}^{cyl}, \\
\{\widetilde{T}_1\}^{pl} &= \{\widetilde{Q}_{1n}\}^{cyl}, \quad -\{\widetilde{Q}_{1n}\}^{pl} = \{\widetilde{T}_1\}^{cyl}.
\end{aligned} \tag{19.18}$$

Replacing the complex forces and moments by their simplified expression (11.24) we rewrite equalities (19.18) in the following manner

$$\begin{aligned}
\{\widetilde{T}_2 - \nu\widetilde{T}_1\}^{pl} &= \{\widetilde{T}_2 - \nu\widetilde{T}_1\}^{cyl}, \quad \{\widetilde{S}\}^{pl} = \{\widetilde{S}\}^{cyl}, \\
\{\widetilde{T}_1\}^{pl} &= \left\{\frac{i}{2b^2}\frac{d\widetilde{T}}{d\xi}\right\}^{cyl}, \quad \left\{-\frac{i}{2b^2}\frac{d\widetilde{T}}{d\rho}\right\}^{pl} = \{\widetilde{T}_1\}^{cyl}.
\end{aligned} \tag{19.19}$$

For the symmetric terms this system is reduced to two equalities ($\xi = 0$, $\rho = 1$)

$$\begin{aligned}
\frac{k}{2}(p+q) - \widetilde{A} - \nu\left[\frac{k}{2}(p+q) + \widetilde{A}\right] &= \widetilde{C}, \\
\frac{k}{2}(p+q) + \widetilde{A} &= \frac{i}{2b^2}(1-i)b\widetilde{C}.
\end{aligned} \tag{19.20}$$

For terms corresponding to k = 2, we obtain the equations

$$\begin{aligned}
&-\frac{k}{2}(p-q) - \widetilde{D} - \nu\left[\frac{k}{2}(p-q) + \widetilde{B} + \widetilde{D}\right] = \\
&= \left(1+\frac{4}{2b^2}\right)\widetilde{C}_1 - i\frac{4}{2b^2}\widetilde{C}_2 - \nu\left[-i\frac{4}{2b^2}\widetilde{C}_1 + \left(1+i\frac{4}{2b^2}\right)\widetilde{C}_2\right],
\end{aligned} \tag{19.21}$$

$$-\frac{h}{2}(p-q)+\frac{\widetilde{B}}{2}+\widetilde{D}=i\frac{2}{2b^2}\gamma_1\widetilde{C}_1-\frac{1}{2}\left(1+i\frac{4}{2b^2}\right)\gamma_2\widetilde{C}_2,$$
$$\frac{h}{2}(p-q)+\widetilde{B}+\widetilde{D}=\frac{i}{2b^2}(\gamma_1\widetilde{C}_1+\gamma_2\widetilde{C}_2),$$
$$\frac{i}{2b^2}2\widetilde{B}=-i\frac{4}{2b^2}+\left(1+i\frac{4}{2b^2}\right)\widetilde{C}_2.$$

It is easy to see that the above system contains small terms which may be dropped. It is proposed that in order to achieve the necessary skill the reader simplify system (19.21) and thereby expressions (19.17). The guideline for this may be the closeness of the values for the established constants to values of quantities obtained by us for the following specific example /226

Let

$$\frac{h}{r_0}=\frac{1}{30};\quad \nu=0.3.$$

It is easy to compute that

$$2b^2=99.15,\quad b=7.04,$$
$$\gamma_1=7.33-i6.72,\quad \gamma_2=0.255+i0.235.$$

Substituting these quantities in the systems (19.20) and (19.21) and solving these, we obtain

$$\widetilde{A}=(-0.440+i0.061)\,h\,(p+q),\quad \widetilde{C}=(0.922-i0.078)\,h\,(p+q),$$
$$\widetilde{C}_1=(-1.448+i0.510)\,h\,(p-q),\quad \widetilde{C}_2=(-0.015-i0.025)\,h\,(p-q),$$
$$\widetilde{B}=(-1.627+i0.326)\,h\,(p-q),\quad \widetilde{D}=(1.041-i0.302)\,h\,(p-q).$$

The established constants were substituted into the expressions for the complex forces (19.14), (19.15) and (19.17). By using the latter the stresses were computed in the usual manner, together with the respective diagrams. The most interesting of these are presented below.

Let us assume that p = q. For the case of a plate not weakened by a hole, this corresponds to uniform extension in all directions. If the reinforcing connecting pipe is absent, the coefficient of stress concentration near the opening is equal to 2. As we can see from figure 50, the reinforcing role of the connecting pipe is very small. However, substantial ring stresses occur in the connecting pipe itself. This is explained by the fact that the edge of the connecting pipe is subject to flexure, while the plate is extended in its plane. As we have already stated, the rigidity of a thin plate during tangential

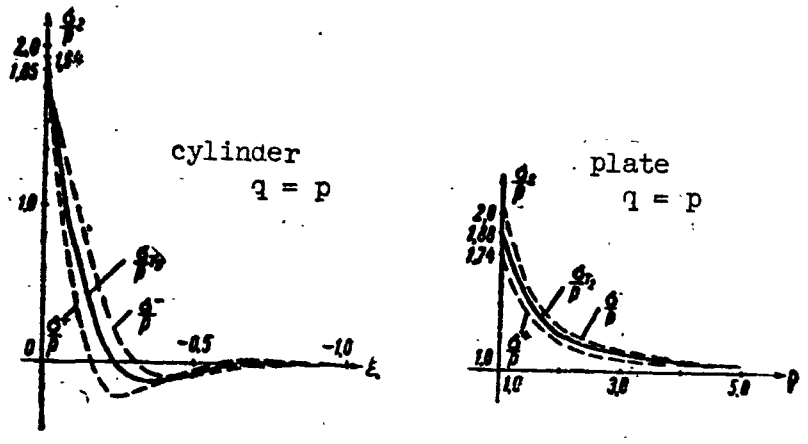

Figure 50

deformation is substantially greater than its flexural rigidity. Therefore the edge of the connecting pipe is almost unable to retain the contour of the plate and is itself subject to ring deformation, which is approximately equal to the deformation of the contour of an unsupported plate. Since vertical forces are absent we have $T_1|_{\xi=0} = 0$ at the edge of the connecting pipe. In addition, since the resistance of the connecting pipe edge is small as indicated above, we may assume approximately that $T_1|_{\rho=1} = 0$ at the edge of the plate. On the other hand, when the value of ring deformation coincides with force T_1, the ring tangential stresses at the edges of the joined shells become equal. This follows from the equality

$$\sigma_{T_2} = \nu \frac{T_1}{h} + E\varepsilon_2,$$

which can be easily established from the relationships of Hooke's law (IV 4.9). The good agreement between the ring stresses, which can be observed from figure 50, serves as a confirmation that the obtained solution is correct. We should like to indicate one other property of the solutions which can be observed from the diagrams: while for a cylindrical shell the edge perturbation attenuates very quickly as we move away from the edge, for a plate this process takes place at a much slower rate. This confirms once again the remark made above concerning the absence of the edge effect in a plate. /227

The assumption q = 0 corresponds to a uniaxial extension of the unweakened plate by forces of intensity ph. Assuming, on the other hand, q = -p, we deal with pure shear. Figures 51-52 show diagrams for the ring stress which is of prime interest. To establish the strengthening effect of the connecting pipe we should recall that for uniaxial extension and pure shear the coefficients

of stress concentration at the unriveted circular hole are equal, respectively, to 3 and 4. Our remarks in connection with the case of the uniform extension of a plate in all directions also pertain to the last two cases. Thus the reinforcing role of a connecting pipe is relatively small, and the pipe itself operates under rather difficult conditions. Therefore, connecting pipes of heavy thickness are usually used when reinforcing holes.

Example 4[1]

Let us consider the problem concerning the flexure of a thin-walled cylindrical cantilever by a concentrated normal force applied at the edge /229 (fig. 53). The concentrated force may be represented in the form of the following (diverging) series

$$P(\varphi)=\frac{P}{2\pi r_0}\left(1+2\sum_{k=1}^{\infty}\cos k\varphi\right)\quad(-\pi\leqslant\varphi\leqslant\pi). \tag{19.22}$$

Figure 54 shows that if we retain a finite number of terms in series (19.22), we obtain a "tongue" which becomes more constricted as the number of terms is increased. To facilitate visualization all of the quantities in /230 this figure are reduced to one ordinate $\frac{P}{2\pi r_0}(1+2n)$, which represents the value of the sum

$$P_n=\frac{P}{2\pi r_0}\left(1+2\sum_{k=1}^{n}\cos k\varphi\right) \tag{19.23}$$

at the point where the force is applied ($\varphi = 0$).

Below we shall consider the problem of determining the displacements and stresses in a shell loaded along its edge by a normal load distributed according to the law (19.23). In this case we shall limit ourselves by retaining only 11 terms of the expansion.

Such a small number of terms, of course, does not simulate the concentrated force too well. However, a comparison of results obtained with subsequent increase in the number n demonstrates the tendency produced by the constriction of the region where the force is applied. As a length of such a region we may take, for example, the distance between two depressions in figure 54. In practice it is just as interesting to obtain solutions of this

[1] A short presentation of the work carried out by V. I. Kruglyakova (1957) in connection with a university degree.

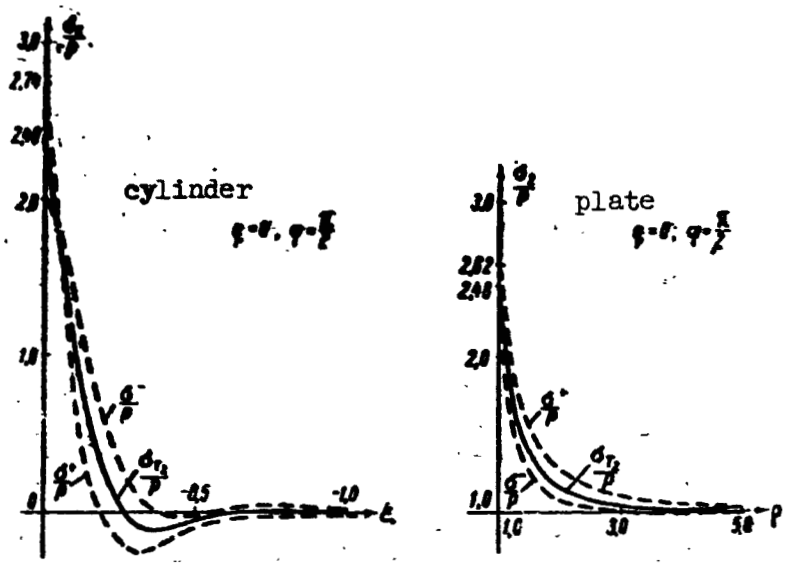

Figure 51

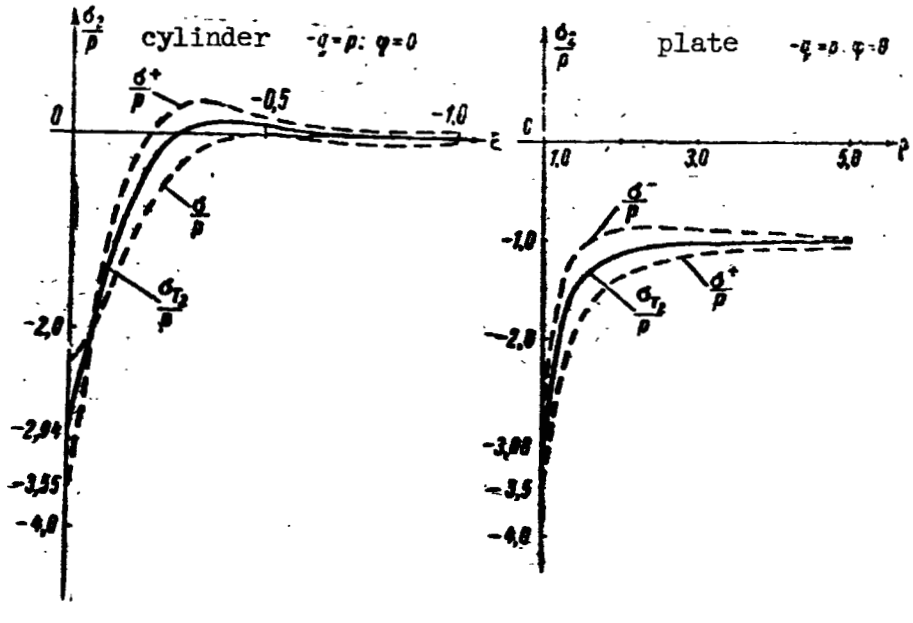

Figure 52

type as it is to have an exact solution for a concentrated force, because the force is always applied over some finite region.

To solve the formulated problem, it is necessary to use relationships of section 11 and to construct particular solutions corresponding to k = 0, 1, 2, ..., n. In this connection in the future we shall assume that the shell is sufficiently long so that the interaction of the edges can be neglected.

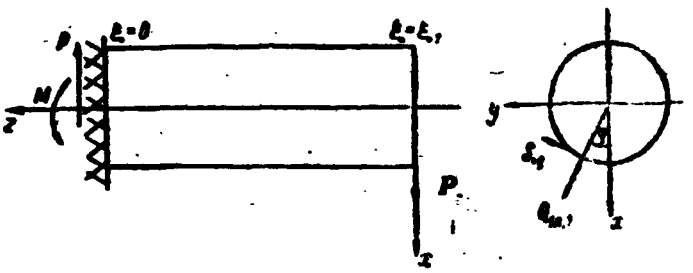

Figure 53

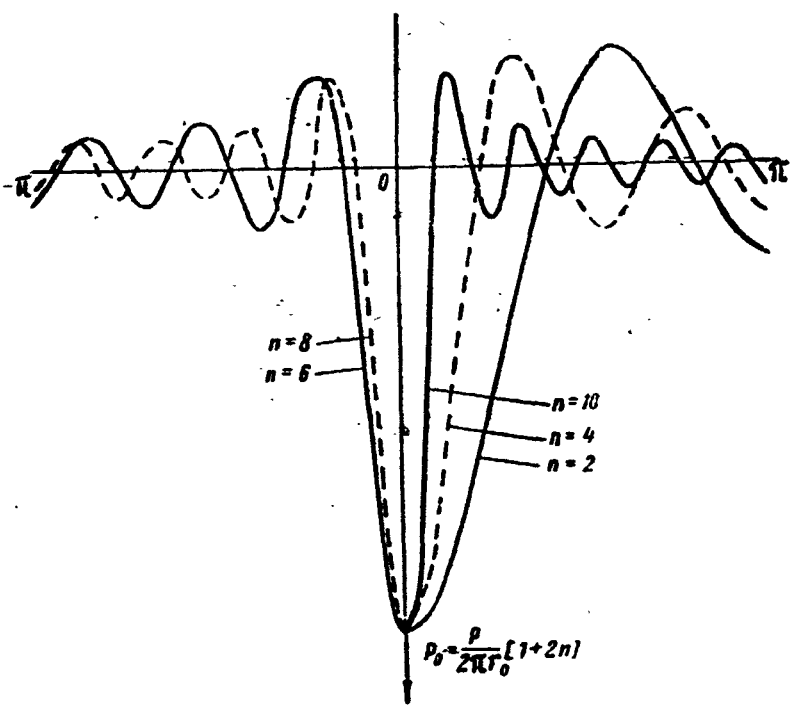

Figure 54

Symmetric Case

Since the surface load is absent and the edge load is self-balanced, the membrane solution need not be introduced. Then the homogeneous boundary conditions at the fixed edge ($\xi = 0$)

$$u_0 = 0, \quad \psi_0 = 0 \tag{19.24}$$

lead to the absence of stresses and deformation at this edge. Therefore, in the symmetric case we need consider only part of the solution associated with

the edge $\xi = \xi_1$ where according to (19.23) the following boundary conditions exist

$$Q_0 = \frac{P}{2\pi r_0}; \quad M_0 = 0. \tag{19.25}$$

According to relationships (11.21)-(11.23) these boundary conditions /231 are associated with the stresses

$$\sigma_{M_1} = -\frac{6}{\sqrt[4]{3(1-\nu^2)}} \sqrt{\frac{r_0}{h}} \frac{P}{2\pi r_0 h} \sin\beta e^{-\beta}, \quad \sigma_{M_2} = \nu\sigma_{M_1}$$
$$\sigma_{T_2} = 2\sqrt[4]{3(1-\nu^2)} \sqrt{\frac{r_0}{h}} \frac{P}{2\pi r_0 h} \cos\beta e^{-\beta}, \quad \sigma_{T_1} = 0 \tag{19.26}$$
$$\left(\beta = \sqrt[4]{3(1-\nu^2)} \sqrt{\frac{r_0}{h}} (\xi_1 - \xi)\right),$$

with the flexure

$$w = 2\sqrt[4]{3(1-\nu^2)} \sqrt{\frac{r_0}{k}} \frac{P}{2\pi Eh} \cos\beta e^{-\beta} \tag{19.27}$$

and with an angle of rotation

$$\vartheta = -2\sqrt{3(1-\nu^2)} \left(\frac{r_0}{h}\right) \frac{P}{2\pi Ehr_0} (\cos\beta + \sin\beta) e^{-\beta}. \tag{19.28}$$

The Antisymmetric Case

From the equilibrium of the cantilever as a whole (fig. 53), following the rule of signs (section 9) we obtain

$$F_z^0 = P, \quad B_y^0 = -Pr_0\xi. \tag{19.29}$$

From this and from (11.27), (11.26) and (11.33) it follows that ($\xi_0 = 0$)

$$C_3 = \frac{1}{\pi} P, \quad C_4 = -\frac{1}{\pi} r_0 \xi_1 P,$$
$$W = -\frac{P}{\pi}(\xi_1 - \xi), \tag{19.30}$$
$$T_{1,1}^* = -\frac{P}{\pi r_0}(\xi_1 - \xi), \quad T_{2,1}^* = 0, \quad S_{,1}^* = -\frac{P}{\pi r_0}.$$

From system (11.13) we find the membrane displacements

$$u^*_{,1} = \frac{P}{\pi Eh}\left(\frac{\xi^2}{2} - \xi_1\xi\right),$$
$$v^*_{,1} = \frac{P}{\pi Eh}\left(\frac{\xi^3}{6} - \frac{\xi_1\xi^2}{2} - 2(1+\nu)\xi\right), \qquad (19.31)$$
$$w^*_{,1} = \frac{P}{\pi Eh}\left(-\frac{\xi^3}{6} + \frac{\xi_1\xi^2}{2} + (2+\nu)\xi + \nu\xi_1\right).$$

In this case the constants of integration are selected in such a way that

$$u^*_{,1}(0) = v^*_{,1}(0) = 0. \qquad (19.32)_{1,\,2}$$

From (19.31) it also follows that

$$w^*_{,1}(0) = \frac{P}{\pi Eh}(\nu\xi_1), \quad \vartheta^*_{,1}(0) = \frac{P}{\pi Ehr_0}[-(2+\nu)]. \qquad (19.32)_{3,\,4}$$

$$u^*_{,1}(\xi_1) = \frac{P}{\pi Eh}\left\{-\frac{\xi_1^2}{2}\right\},$$
$$v^*_{,1}(\xi_1) = \frac{P}{\pi Eh}\left\{-\frac{\xi_1^2}{3} - 2(1+\nu)\xi_1\right\},$$
$$w^*_{,1}(\xi_1) = \frac{P}{\pi Eh}\left\{\frac{\xi_1^3}{3} + 2(1+\nu)\xi_1\right\}, \qquad (19.33)$$
$$\vartheta^*_{,1}(\xi_1) = \frac{P}{\pi Ehr_0}\left\{-\frac{\xi_1^2}{2} - (2+\nu)\right\}.$$

At the fixed edge the following conditions must be satisfied. /232

$$u_0 = 0, \quad \vartheta_0 = 0. \qquad (19.34)$$

However, according to (19.31)

$$u^*_0 = r_0 u^*_{\nu,1}(0) = \frac{P}{\pi Eh}(\nu\xi_1),$$
$$\vartheta^*_0 = \left(\vartheta^*_{,1} + \frac{w^*_{,1}}{r_0}\right)_{\xi=0} = \frac{P}{\pi Ehr_0}[-2 + \nu(\xi_1 - 1)]. \qquad (19.35)$$

Therefore at the edge of the cantilever ($\xi = 0$) a small edge effect occurs. As it follows from (11.29), the quantities Q_0 and M_0 are determined from the system ($\mu = -1$)

$$\begin{gathered} -2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)^{3/2} Q_0 - \\ -2\sqrt{3(1-\nu^2)}\,\frac{M_0}{h}\left(\frac{r_0}{h}\right) = -Eu_0^*, \\ -2\sqrt{3(1-\nu^2)}\left(\frac{r_0}{h}\right) Q_0 - \\ -4\left[3(1-\nu^2)\right]^{3/4}\sqrt{\frac{r_0}{h}}\,\frac{M_0}{h} = -Eh\vartheta_0^*. \end{gathered} \tag{19.36}$$

Solving this system we obtain

$$Q_0 = \frac{P}{\pi h}\left\{\frac{\nu\xi_1}{\sqrt[4]{3(1-\nu^2)}}\left(\frac{h}{r_0}\right)^{3/2} + \underline{\frac{1-\frac{\nu}{2}(\xi_1-1)}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{r_0}\right)^2}\right\},$$

$$\frac{M_0}{h} = \frac{P}{\pi h}\left\{\underline{\frac{-1+\frac{\nu}{2}(\xi_1-1)}{\left[3(1-\nu^2)\right]^{3/4}}\left(\frac{h}{r_0}\right)^{3/2}} - \frac{\nu\xi_1}{2\sqrt{3(1-\nu^2)}}\left(\frac{h}{r_0}\right)\right\}.$$

The terms which are underlined in the above equations (corresponding to ϑ_0^*) are small and may be dropped. This again confirms the remark concerning the role of the quantity ϑ_0^*, made in section 8.

Dropping the underlined terms we find

$$\sigma_{M_0} = \frac{P}{\pi r_0 h}\left\{-\frac{3\nu\xi_1}{\sqrt{3(1-\nu^2)}}\right\}, \quad \sigma_h = \frac{P}{\pi r_0 h}\{2\nu\xi_1\}. \tag{19.37}$$

Now it is easy to use equations (11.21)-(11.23) and to compute the stresses and displacements for the part of the shell which adjoins the edge $\xi = 0$.

At the other end of the cantilever we have the boundary conditions

$$M_0 = 0, \quad Q_0 = \frac{P}{\pi r_0}. \tag{19.38}$$

Assuming in the same relationships that $\mu = +1$ we obtain expressions for the stresses and displacements from them. In particular at the edge itself ($\xi = \xi_1$)

$$\sigma_{r_1} = \sigma_{M_1} = \sigma_{M_2} = 0,$$
$$\sigma_{T_1} = 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\frac{P}{\pi r_0 h}. \tag{19.39}$$

$$u_{,1} = \frac{P}{\pi E h}\left\{-\frac{\xi_1^2}{2}\right\},$$
$$v_{,1} = \frac{P}{\pi E h}\left\{-\frac{\xi_1^3}{3} - 2(1+\nu)\xi_1\right\},$$
$$w_{,1} = \frac{P}{\pi E h}\left\{\frac{\xi_1^2}{3} + 2(1+\nu)\xi_1 + 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\right\}, \tag{19.40}$$
$$\psi_{,1} = \frac{P}{\pi E h r_0}\left\{-\frac{\xi_1^2}{2} - (2+\nu) - 2\sqrt{3(1-\nu^2)}\frac{r_0}{h}\right\}.$$

Self-Balanced Load ($k \geq 2$)

For terms corresponding to the values $k \geq 2$, all of the quantities are expressed in terms of the solution of system (11.4). Eliminating $\widetilde{U}_{,k}$ we arrive at the basic complex equation

$$\frac{d^4\widetilde{T}_{,k}}{d\xi^4} + (12b^2 - 2k^2)\frac{d^2\widetilde{T}_{,k}}{d\xi^2} + k^2(k^2-1)\widetilde{T}_{,k} = 0\left(2b^2 = \frac{r_0}{c}\right). \tag{19.41}$$

We seek its particular solution in the form

$$T_{,k} = e^{\alpha\xi}, \tag{19.41'}$$

and arrive at the characteristic equation

$$\alpha^4 + (12b^2 - 2k^2)\alpha^2 + k^2(k^2-1) = 0, \tag{19.}$$

whose solution is written in the form

$$\alpha_1=\frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{m_1^2+n_1^2}+m_1}-i\sqrt{\sqrt{m_1^2+n_1^2}-m_1}\right],$$
$$\alpha_2=-\alpha_1,$$
$$\alpha_3=\frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{m_2^2+n_2^2}+m_2}-i\sqrt{\sqrt{m_2^2+n_2^2}-m_2}\right], \qquad (19.43)$$
$$\alpha_4=-\alpha_3,$$

where

$$m_1=k^2-b^2\frac{\sqrt{\sqrt{1+\frac{4k^4}{b^4}}-1}}{\sqrt{2}},$$
$$m_2=k^2+b^2\frac{\sqrt{\sqrt{1+\frac{4k^4}{b^4}}-1}}{\sqrt{2}},$$
$$n_1=b^2\left(1-\frac{\sqrt{\sqrt{1+\frac{4k^4}{b^4}}+1}}{\sqrt{2}}\right),$$
$$n_2=b^2\left(1+\frac{\sqrt{\sqrt{1+\frac{4k^4}{b^4}}+1}}{\sqrt{2}}\right).$$

Now the general solution of equation (19.41) may be represented in the form /234

$$\widetilde{T}^0_{,k}=\sum_{j=1,3}\widetilde{C}_j e^{-\alpha_j(\xi_1-\xi)}+\sum_{j=2,4}\widetilde{C}_j e^{\alpha_j\xi}. \qquad (19.44)$$

From this, by using the second equation (11.4) and relationships (11.5), we obtained the following expression for the complex forces

$$\widetilde{T}_{1,k}=\sum_{j=1,3}\left(1-i\frac{\alpha_j^2-k^2}{2b^2}\right)\widetilde{C}_j e^{-\alpha_j(\xi_1-\xi)}+$$
$$+\sum_{j=2,4}\left(1-i\frac{\alpha_j^2-k^2}{2b^2}\right)\widetilde{C}_j e^{\alpha_j\xi},$$
$$\widetilde{T}_{2,k}=\frac{i}{2b^2}\left[\sum_{j=1,3}(\alpha_j^2-k^2)\widetilde{C}_j e^{-\alpha_j(\xi_1-\xi)}+\right. \qquad (19.45)$$
$$\left.+\sum_{j=2,4}(\alpha_j^2-k^2)\widetilde{C}_j e^{\alpha_j\xi}\right].$$

$$\tilde{S}_{,k}=\frac{1}{2}\left\{\sum_{j=1,3}\left(i\frac{a_j^3-k^2}{2k^2}-a_j\right)\tilde{C}_j e^{-a_j(\xi_1-\xi)}+\right.$$
$$\left.+\sum_{j=2,4}\left(i\frac{a_j^3-k^2}{2k^2}-a_j\right)\tilde{C}_j e^{a_j\xi}\right\}.$$

Then we find the complex displacements from system (11.6). We write down only the expression for $\tilde{w}_{,k}$

$$\tilde{w}_{,k}=\frac{r_0}{Eh}\left\{\sum_{j=1,3}\left[(1+\nu)k-\nu-\frac{k^2}{a_j^2}+i(1+\nu)\frac{(1-k)a_j^3}{2k^2}+\right.\right.$$
$$\left.+i(1+\nu)\frac{k^2(a_j-k)}{2k^2a_j^2}\right]\tilde{C}_j e^{-a_j(\xi_1-\xi)}+$$
$$+\sum_{j=2,4}\left[(1+\nu)k-\nu-\frac{k^2}{a_j^2}+i(1+\nu)\frac{(1-k)a_j^3}{2k^2}+\right.$$
$$\left.\left.+i(1+\nu)\frac{k^2(a_j-k)}{2k^2a_j^2}\right]\tilde{C}_j e^{a_j\xi}\right\}. \qquad (19.46)$$

We agreed previously to consider only a sufficiently long shell so as /235 not to take into account the interaction of the edges. Since surface loading is absent, while the edge loading is self-balanced when $k \geq 2$, we need not consider the membrane solution. Then the conditions for the built-in edge $\xi = 0$

$$u_{,k}=0,\quad v_{,k}=0,\quad w_{,k}=0,\quad \vartheta_{,k}=0 \qquad (19.47)$$

lead to the equality

$$\tilde{C}_2=\tilde{C}_4=0.$$

We introduced the designations

$$\tilde{C}_1=A_1+iB_1,\quad \tilde{a}_1=a_1-ib_1,$$
$$\tilde{C}_3=A_2+iB_2,\quad \tilde{a}_2=a_2-ib_2; \qquad (19.48)$$

$$\begin{aligned}
\Phi_1 &= \operatorname{ch} a_1(\xi_1-\xi)\cos b_1(\xi_1-\xi),\\
\Phi_2 &= \operatorname{sh} a_1(\xi_1-\xi)\sin b_1(\xi_1-\xi),\\
\Phi_3 &= \operatorname{ch} a_1(\xi_1-\xi)\sin b_1(\xi_1-\xi),\\
\Phi_4 &= \operatorname{sh} a_1(\xi_1-\xi)\cos b_1(\xi_1-\xi);
\end{aligned} \tag{19.49}$$

$$\begin{aligned}
\Psi_1 &= \operatorname{ch} a_2(\xi_1-\xi)\cos b_2(\xi_1-\xi),\\
\Psi_2 &= \operatorname{sh} a_2(\xi_1-\xi)\sin b_2(\xi_1-\xi),\\
\Psi_3 &= \operatorname{ch} a_2(\xi_1-\xi)\sin b_2(\xi_1-\xi),\\
\Psi_4 &= \operatorname{sh} a_2(\xi_1-\xi)\cos b_2(\xi_1-\xi).
\end{aligned} \tag{19.50}$$

By separating the real and imaginary parts in the above expressions we find that the quantities

$$T_{1,\,k};\quad S_{,k};\quad w_{,k};\quad v_{,k}$$

have the form

$$\begin{aligned}
&\delta\{(\gamma A_1-\gamma' B_1)(\Phi_1-\Phi_4)-(\gamma B_1+\gamma' A_1)(\Phi_3-\Phi_2)+\\
&+(\beta A_2-\beta' B_2)(\Psi_1-\Psi_4)-(\beta B_2+\beta' A_2)(\Psi_3-\Psi_2)\},
\end{aligned} \tag{19.51}$$

while

$$\begin{aligned}
&T_{2,k};\quad M_{1,k};\quad Q_{1n,k};\quad M_{2,k}-\\
&\delta\{(\gamma' A_1+\gamma B_1)(\Phi_1-\Phi_4)+(\gamma A_1-\gamma' B_1)(\Phi_3-\Phi_2)+\\
&+(\beta' A_2+\beta B_2)(\Psi_1-\Psi_4)+(\beta A_2-\beta' B_2)(\Psi_3-\Psi_2)\}.
\end{aligned} \tag{19.52}$$

The values of the coefficients for each of the specified quantities are shown in table 3.

The arbitrary constants of the particular solutions are determined for each value of k from the boundary conditions at the edge $\xi = \xi_1$

$$S_{,k}=0;\quad T_{1,k}=0;\quad M_{1,k}=0;\quad Q_{1n,k}=\frac{P}{2\pi r_0}. \tag{19.53}$$

TABLE 3. EQUATIONS FOR COMPUTING THE COEFFICIENTS γ, γ', β, β'

	a	γ	γ'	β	β'
T_1	1	$1-\frac{a_1b_1}{b^2}$	$-\frac{a_1^2-b_1^2-k^2}{2b^2}$	$1-\frac{a_3b_3}{b^2}$	$-\frac{a_3^2-b_3^2-k^2}{2b^2}$
T_2	$-\frac{1}{2b^2}$	$a_1^2-b_1^2-k^2$	$-2a_1b_1$	$a_3^2-b_3^2-k^2$	$-2a_3b_3$
S	$\frac{1}{2}$	$-\left(a_1+b_1\frac{b_1^2-3a_1^2}{2b^2}\right)$	$b_1+\frac{a_1\left(a_1^2-3b_1^2\right)-k^2}{2b^2}$	$-\left(a_3+b_3\frac{b_3^2-3a_3^2}{2b^2}\right)$	$b_3+\frac{a_3\left(a_3^2-3b_3^2\right)-k^2}{2b^2}$
Q_{1n}	$-\frac{c}{r_0}$	a_1	$-b_1$	a_3	$-b_3$
M_1	$-c$	$(1-\nu)\frac{a_1b_1}{b^2}+\nu$	$(1-\nu)\frac{a_1^2-b_1^2-k^2}{2b^2}$	$(1-\nu)\frac{a_3b_3}{b^2}+\nu$	$(1-\nu)\frac{a_3^2-b_3^2-k^2}{2b^2}$
M_2	$-c$	$1-(1-\nu)\frac{a_1b_1}{b^2}$	$-(1-\nu)\frac{a_1^2-b_1^2-k^2}{2b^2}$	$1-(1-\nu)\frac{a_3b_3}{b^2}$	$-(1-\nu)\frac{a_3^2-b_3^2-k^2}{2b^2}$
v	$\frac{r_0}{Eh}$	$\frac{kb^2\left(a_1^2-b_1^2\right)-(1+\nu)k^3a_1b_1}{b^2\left[\left(a_1^2-b_1^2\right)^2+4a_1^2b_1^2\right]}+$ $+(1+\nu)\left(\frac{a_1b_1}{b^2}-1\right)+$ $+\frac{(1+\nu)k^2b_1}{2b^2\left(a_1^2+b_1^2\right)}$	$\frac{4b^2a_1b_1k+(1+\nu)k^3\left(a_1^2-b_1^2\right)}{2b^2\left[\left(a_1^2-b_1^2\right)^2+4a_1^2b_1^2\right]}+$ $+(1+\nu)\frac{a_1^2-b_1^2-k}{2b^2}-$ $-(1+\nu)\frac{a_1k^2}{2b^2\left(a_1^2+b_1^2\right)}$	$\frac{kb^2\left(a_3^2-b_3^2\right)-(1+\nu)k^3a_1b_1}{b^2\left[\left(a_3^2-b_3^2\right)^2+4a_3^2b_3^2\right]}+$ $+(1+\nu)\left(\frac{a_3b_3}{b^2}-1\right)+$ $+\frac{(1+\nu)k^2b_3}{2b^2\left(a_3^2+b_3^2\right)}$	$\frac{4b^2a_3b_3k+(1+\nu)k^3\left(a_3^2-b_3^2\right)}{2b^2\left[\left(a_3^2-b_3^2\right)^2+4a_3^2b_3^2\right]}+$ $+(1+\nu)\frac{a_3^2-b_3^2-k}{2b^2}-$ $-(1+\nu)\frac{a_3k^2}{2b^2\left(a_3^2+b_3^2\right)}$
w	$\frac{r_0}{Eh}$	$\frac{(1+\nu)k^3}{2b^2}\times$ $\times\frac{\left[b_1\left(a_1^2-b_1^2\right)-2a_1b_1(a_1-k)\right]}{\left(a_1^2-b_1^2\right)^2+4a_1^2b_1^2}-$ $-\frac{k^2\left(a_1^2-b_1^2\right)}{\left(a_1^2-b_1^2\right)^2+4a_1^2b_1^2}+$ $+(1+\nu)k-\nu+$ $+(1+\nu)\frac{(1-k)a_1b_1}{b^2}$	$\frac{(1+\nu)k^3}{2b^2}\times$ $\times\frac{\left[(a_1-k)\left(a_1^2-b_1^2\right)+2a_1b_1^2\right]}{\left(a_1^2-b_1^2\right)^2+4a_1^2b_1^2}-$ $-\frac{2a_1b_1k^2}{\left(a_1^2-b_1^2\right)^2+4a_1^2b_1^2}+$ $+(1+\nu)\frac{(1-k)\left(a_1^2-b_1^2\right)}{2b^2}$	$\frac{(1+\nu)k^3}{2b^2}\times$ $\times\frac{\left[b^2\left(a_3^2-b_3^2\right)-2a_3b_3(a_3-k)\right]}{\left(a_3^2-b_3^2\right)^2+4a_3^2b_3^2}-$ $-\frac{k^2\left(a_3^2-b_3^2\right)}{\left(a_3^2-b_3^2\right)^2+4a_3^2b_3^2}+$ $+(1+\nu)k-\nu+(1+\nu)\frac{(1-k)a_3b_3}{b^2}$	$\frac{(1+\nu)k^3}{2b^2}\times$ $\times\frac{\left[(a_3-k)\left(a_3^2-b_3^2\right)+2a_3b_3^2\right]}{\left(a_3^2-b_3^2\right)^2+4a_3^2b_3^2}-$ $-\frac{2a_3b_3k^2}{\left(a_3^2-b_3^2\right)^2+4a_3^2b_3^2}+$ $+(1+\nu)\frac{(1-k)\left(a_3^2-b_3^2\right)}{2b^2}$

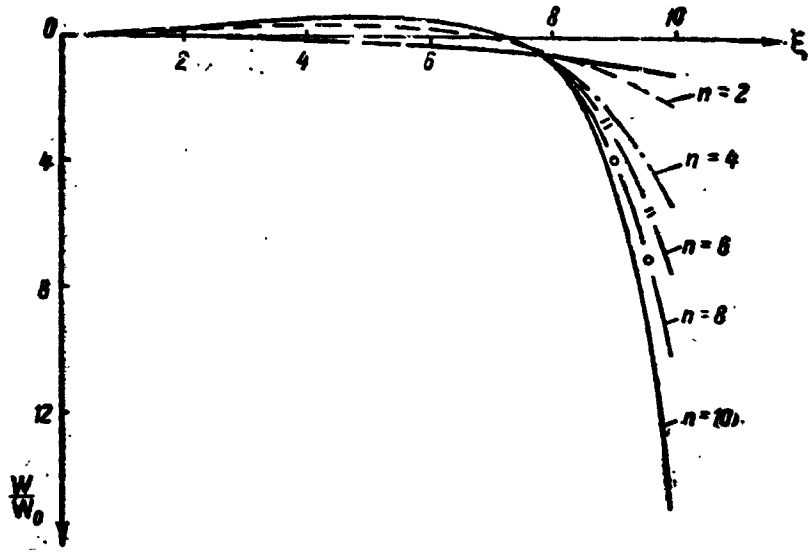

Figure 55

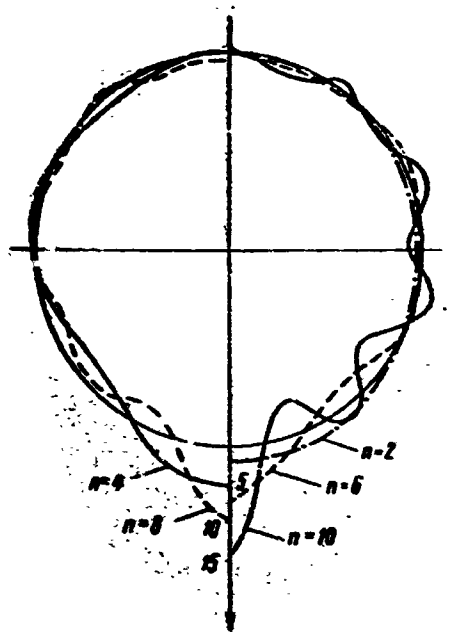

Figure 56

The equations presented have been used to carry out the calculation /239 for n = 1, 2, ..., 10 in the case of a cantilever with parameters $\xi_1 = 10$, $\frac{h}{r_0} = \frac{1}{50}$.

In figure 55 the flexure of the lower generatrix for the cylinder $w|_{\varphi=0}$ is compared with the flexure of a beam axis when the beam has the same section w_0. The number n determines the form of the edge loading (19.23). Figure 56 shows the deformation of the end section. The flexure of an end section of the cantilever covered with a rigid diaphragm is taken as the unit measure of the scale. Finally figure 57 shows the values of the tangential and flexural ring stresses.

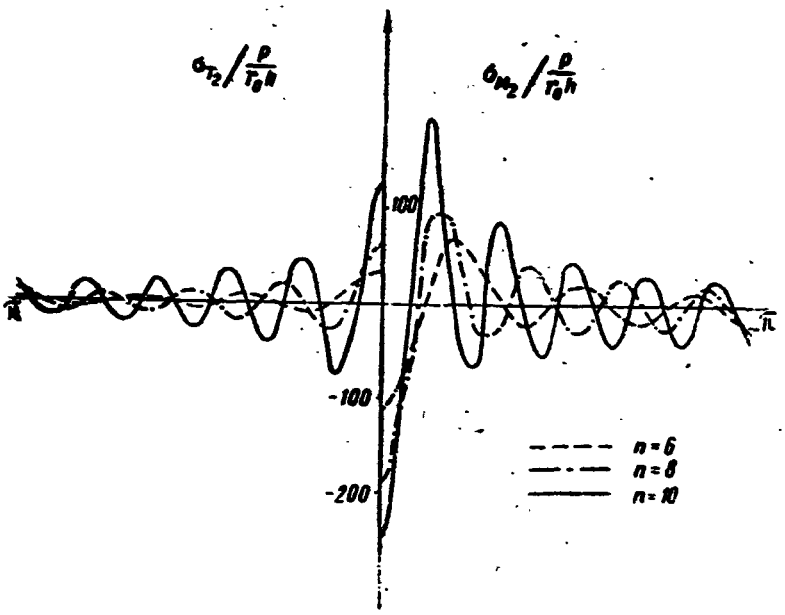

Figure 57

CHAPTER VI. THE THEORY OF CYLINDRICAL SHELLS

The large interest shown in the calculation of cylindrical shells is explained by the wide application of these shells in the most diverse branches of technology. This in turn is due to the fact that in cylindrical cells there is a successful combination of advantages associated with the curvature of the middle surface, and because cylindrical shells are easy to fabricate. In addition to this, the relative simplicity of the relationships obtained in the theory of cylindrical shells makes their calculation much easier. /240

In the preceding chapter, we considered closed circular cylindrical cells. Below we derive relationships for cylindrical shells of the general type.

1. The Determination of Forces, Moments and Displacements

Let us refer the middle surface of a cylindrical shell to the lines of curvature. The latter consist of the generatrixes and directrixes (fig. 58). We introduce the dimensionless Cartesian coordinates

$$\alpha_1=\xi=\frac{x}{r_0},\quad \alpha_2=\eta=\frac{y}{r_0},\quad A_1=A_2=r_0 \tag{1.1}$$

and the dimensionless radius of curvature of the directrix

$$\rho_1(\eta)=\frac{R_2(\eta)}{r_0}\quad\left(\frac{1}{R_1}=0\right). \tag{1.2}$$

Here r_0 is some characteristic linear dimension of the middle surface of the shell.

The system of equations expressed in terms of the complex forces (IV, 9.1) has the following form in the adopted coordinates

$$\begin{cases}\dfrac{\partial\widetilde{T}_1}{\partial\xi}+\dfrac{\partial\widetilde{S}}{\partial\eta}=-r_0q_1,\\[2ex] \dfrac{\partial\widetilde{T}_2}{\partial\eta}+\dfrac{\partial\widetilde{S}}{\partial\xi}+\dfrac{t}{2k^2\rho}\dfrac{\partial\widetilde{M}}{\partial\eta}=-r_0q_2,\\[2ex] \widetilde{T}_2-t\dfrac{k^2}{2k^2}\Delta\widetilde{T}=r_0\rho q_n. \end{cases} \tag{1.3}$$

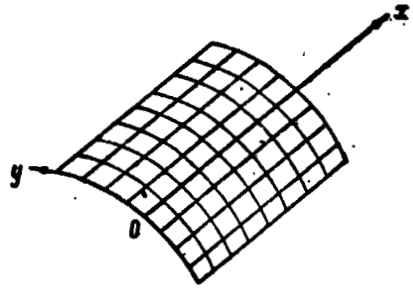

Figure 58

Here and below

$$\Delta=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2},\quad 2b^2=\frac{r_0}{c}=2\sqrt{3(1-\nu^2)}\frac{r_0}{h}. \tag{1.4}$$

It is easy to reduce system (1.3) to one equation. Indeed, if we replace $\widetilde{T}_1$ in the first equation with the expression /241

$$\widetilde{T}_1=\widetilde{T}-\widetilde{T}_2 \tag{1.5}$$

and eliminate $\widetilde{S}$ from the first two equations, we arrive at the following equation

$$\frac{\partial^2\widetilde{T}}{\partial\xi^2}-\Delta\widetilde{T}_2-\frac{i}{2b^2}\frac{\partial}{\partial\eta}\frac{1}{\rho}\frac{\partial\widetilde{T}}{\partial\eta}=$$
$$=r_0\left(\frac{\partial q_2}{\partial\eta}-\frac{\partial q_1}{\partial\xi}\right).$$

By replacing $\widetilde{T}_2$ with its expression given by $(1.3)_3$, we arrive at the basic complex equation of the theory of cylindrical shells

$$\Delta\rho\Delta\widetilde{T}+\frac{\partial}{\partial\eta}\frac{1}{\rho}\frac{\partial\widetilde{T}}{\partial\eta}+i2b^2\frac{\partial^2\widetilde{T}}{\partial\xi^2}=i2b^2r_0\left[\Delta\rho q_n+\frac{\partial q_2}{\partial\eta}-\frac{\partial q_1}{\partial\xi}\right]. \tag{1.6}$$

According to equalities $(1.3)_3$ and (1.5) the complex forces are expressed in terms of the basic complex function $\widetilde{T}$ by means of relationships

$$\widetilde{T}_2 = -\frac{i}{2b^2}\rho\Delta\widetilde{T} + r_0\rho q_n, \quad \widetilde{T}_1 = \widetilde{T} - \frac{i}{2b^2}\rho\Delta\widetilde{T} - r_0\rho q_n. \tag{1.7}$$

By using these we obtain expressions for the arbitrary functions $\widetilde{S}$ from the first two equations of (1.3)

$$\begin{aligned}\frac{\partial\widetilde{S}}{\partial\xi} &= -\frac{i}{2b^2}\left\{\frac{\partial}{\partial\eta}[\rho\Delta\widetilde{T}] + \frac{1}{\rho}\frac{\partial\widetilde{T}}{\partial\eta}\right\} - r_0\left(\frac{\partial\rho q_n}{\partial\eta} + q_2\right),\\ \frac{\partial\widetilde{S}}{\partial\eta} &= -\frac{\partial\widetilde{T}}{\partial\xi} + \frac{i}{2b^2}\rho\frac{\partial\Delta\widetilde{T}}{\partial\xi} + r_0\left(\rho\frac{\partial q_n}{\partial\xi} - q_1\right).\end{aligned} \tag{1.8}$$

It is easy to verify that they are compatible in view of the basic equation (1.6). The strain components (IV, 9.5) take on the following form for a cylindrical shell ($u_1 = u$, $u_2 = v$, $\vartheta_1 = \vartheta$, $\vartheta_2 = \psi$)

$$\begin{aligned} r_0\varepsilon_1 &= \frac{\partial u}{\partial\xi}, & r_0^2\varkappa_1 &= r_0\frac{\partial\vartheta}{\partial\xi} = -\frac{\partial^2 w}{\partial\xi^2},\\ r_0\varepsilon_2 &= \frac{\partial v}{\partial\eta} + \frac{w}{\rho}, & r_0^2\varkappa_2 &= r_0\frac{\partial\psi}{\partial\eta} = -\frac{\partial^2 w}{\partial\eta^2} + \frac{\partial}{\partial\eta}\left(\frac{v}{\rho}\right),\\ r_0\omega &= \frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}, & r_0^2\tau &= r_0\frac{\partial\psi}{\partial\xi} = -\frac{\partial^2 w}{\partial\xi\partial\eta} + \frac{1}{\rho}\frac{\partial v}{\partial\xi}. \end{aligned} \tag{1.9}$$

In the present chapter we adopt the membrane solution for the functions of the static system such that in accordance with (IV, 9.34) this solution satisfies equations /242

$$\frac{\partial T_1^*}{\partial\xi} + \frac{\partial S^*}{\partial\eta} = -r_0 q_1, \quad \frac{\partial S^*}{\partial\xi} + \frac{\partial T_2^*}{\partial\eta} = -r_0 q_2, \quad T_2^* = r_0\rho q_n. \tag{1.10}$$

Then the complex Hooke's law (IV, 7.1) is written in the form

$$\begin{aligned} \frac{\partial\widetilde{u}}{\partial\xi} &= \frac{r_0}{Eh}(\widetilde{T}_1 - \nu\widetilde{T}_2), & -\frac{\partial^2\widetilde{w}}{\partial\xi^2} &= i\frac{2b^2 r_0}{Eh}(\widetilde{T}_2 - T_2^*),\\ \frac{\partial\widetilde{v}}{\partial\eta} + \frac{\widetilde{w}}{\rho} &= \frac{r_0}{Eh}(\widetilde{T}_2 - \nu\widetilde{T}_1), & -\frac{\partial^2\widetilde{w}}{\partial\eta^2} + \frac{\partial}{\partial\eta}\left(\frac{\widetilde{v}}{\rho}\right) &= i\frac{2b^2 r_0}{Eh}(\widetilde{T}_1 - T_1^*),\\ \frac{\partial\widetilde{u}}{\partial\eta} + \frac{\partial\widetilde{v}}{\partial\xi} &= \frac{r_0}{Eh}2(1+\nu)\widetilde{S}, & -\frac{\partial^2\widetilde{w}}{\partial\xi\partial\eta} + \frac{1}{\rho}\frac{\partial\widetilde{v}}{\partial\xi} &= -i\frac{2b^2 r_0}{Eh}(\widetilde{S} - S^*). \end{aligned} \tag{1.11}$$

By using equations (1.6) and (1.7) it is easy to obtain the following expression from the first three relationships of this system

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial \xi} = \frac{r_0}{Eh}\left\{\tilde{T} - i\frac{1+\nu}{2b^2}\rho\Delta\tilde{T} - (1+\nu)r_0\rho q_n\right\},\\
&\frac{\partial^2 \tilde{v}}{\partial \xi^2} = \frac{r_0}{Eh}\left\{-\left(1+i\frac{1+\nu}{b^2\rho}\right)\frac{\partial \tilde{T}}{\partial \eta} - i\frac{1+\nu}{2b^2}\frac{\partial}{\partial \eta}(\rho\Delta\tilde{T}) + \right.\\
&\qquad \left. + r_0\left[-(1+\nu)\frac{\partial \rho q_n}{\partial \eta} - 2(1+\nu)q_2\right]\right\},\\
&\frac{\partial^2 \tilde{w}}{\partial \xi^2} = \frac{r_0\rho}{Eh}\left\{\Delta\tilde{T} + r_0(1+\nu)\left(\frac{\partial q_1}{\partial \xi} + \frac{\partial q_2}{\partial \eta}\right)\right\}
\end{aligned} \tag{1.12}$$

(the underlined term is small and may be dropped).

The equalities which have been obtained make it possible to determine the complex displacements by direct integration over ξ. It is interesting to note that these expressions, which were discovered by V. V. Novozhilov, are not random quantities. In the next section we shall show that within the accuracy of constant factors these quantities are the strain components of a normal element associated with the generatrix (η = const). Separating the real parts in the established complex displacements we obtain the values for the displacements. In regard to the stresses, they are computed by means of the usual equations (IV, 9.2 and 9.3).

2. Boundary Quantities. Conditions at the Joints of the Shells

Let us consider on the middle surface some line whose direction (fig. 59) is characterized by angle γ (Chapter I, section 2). According to (I, 2.10) and (1.2) /243

$$\frac{1}{R_n} = \frac{\sin^2\gamma}{r_0\rho}, \quad \frac{1}{R_{nt}} = -\frac{\sin\gamma\cos\gamma}{r_0\rho}, \quad \frac{1}{R_t} = \frac{\cos^2\gamma}{r_0\rho}. \tag{2.1}$$

According to section 3 of Chapter II the curve Γ is associated with the displacements

$$\begin{aligned}
&\vec{U} = u_n\vec{\nu} + u_t\vec{t} + w\vec{n},\\
&u_n = u\cos\gamma + v\sin\gamma,\\
&u_t = -u\sin\gamma + v\cos\gamma
\end{aligned} \tag{2.2}$$

and with the angle of rotation

$$\begin{aligned}\vartheta_* &= \cos\gamma\,\vartheta + \sin\gamma\,\psi,\\ \vartheta &= -\frac{1}{r_0}\frac{\partial w}{\partial \xi},\\ \psi &= -\frac{1}{r_0}\left(\frac{\partial w}{\partial \eta} - \frac{v}{\rho}\right).\end{aligned} \tag{2.3}$$

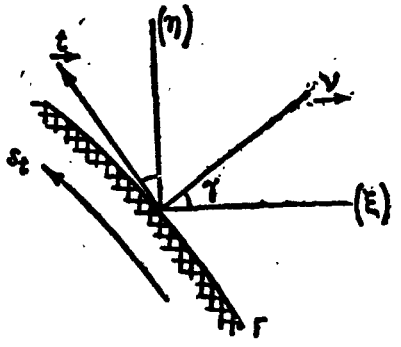

Figure 59

The deformation boundary quantities on the other hand (II, 3.18) take the form

$$\begin{aligned}\varepsilon_t &= \cos^2\gamma\,\varepsilon_2 - \sin\gamma\cos\gamma\,\omega + \sin^2\gamma\,\varepsilon_1,\\ \vec{\varkappa}_t &= -\varkappa_{tt}\vec{\nu} + \varkappa_{t\tau}\vec{t} - \varkappa_{tn}\vec{n},\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\varkappa_{tt} &= \cos^2\gamma\,\varkappa_2 - 2\sin\gamma\cos\gamma\,\tau + \sin^2\gamma\,\varkappa_1 +\\ &\quad + \frac{\sin\gamma\cos\gamma}{r_0\rho}\left[\sin\gamma\cos\gamma(\varepsilon_2-\varepsilon_1) + \cos^2\gamma\,\omega\right],\\ \varkappa_{t\tau} &= \sin\gamma\cos\gamma(\varkappa_2-\varkappa_1) + (\cos^2\gamma - \sin^2\gamma)\,\tau -\\ &\quad - \frac{\cos^2\gamma}{r_0\rho}\left[\sin\gamma\cos\gamma(\varepsilon_2-\varepsilon_1) + \cos^2\gamma\,\omega\right],\\ \varkappa_{tn} &= -\frac{\sin\gamma}{r_0}\left\{\frac{\partial\varepsilon_1}{\partial\eta} + \frac{\partial}{\partial\xi}\left[\sin\gamma\cos\gamma(\varepsilon_2-\varepsilon_1) - \sin^2\gamma\,\omega\right]\right\} +\\ &\quad + \frac{\cos\gamma}{r_0}\left\{-\frac{\partial\varepsilon_2}{\partial\xi} + \frac{\partial}{\partial\eta}\left[\sin\gamma\cos\gamma(\varepsilon_2-\varepsilon_1) + \cos^2\gamma\,\omega\right]\right\}.\end{aligned} \tag{2.5}$$

Similar expressions exist for the boundary static quantities

$$M_{\nu}=\cos^2\gamma M_1+2\sin\gamma\cos\gamma H+\sin^2\gamma M_2,$$
$$\vec{Q}_{\nu}=Q_{\nu\nu}\vec{\nu}+Q_{\nu t}\vec{t}+Q_{\nu n}\vec{n}, \tag{2.6}$$

where

$$\begin{aligned}
Q_{\nu\nu}&=\cos^2\gamma T_1+2\sin\gamma\cos\gamma S+\sin^2\gamma T_2+\\
&+\frac{\sin\gamma\cos\gamma}{r_{0\rho}}[\sin\gamma\cos\gamma(M_2-M_1)+\cos^2\gamma 2H];\\
Q_{\nu t}&=\sin\gamma\cos\gamma(T_2-T_1)+(\cos^2\gamma-\sin^2\gamma)S+\\
&+\frac{\cos^2\gamma}{r_{0\rho}}[\sin\gamma\cos\gamma(M_2-M_1)+\cos^2\gamma 2H];\\
Q_{\nu n}&=\frac{\sin\gamma}{r_{0\rho}}\left\{\frac{\partial M_2}{\partial\eta}-\frac{\partial}{\partial\xi}[\sin\gamma\cos\gamma(M_2-M_1)-\sin^2\gamma 2H]\right\}+\\
&+\frac{\cos\gamma}{r_{0\rho}}\left\{\frac{\partial M_1}{\partial\xi}+\frac{\partial}{\partial\eta}[\sin\gamma\cos\gamma(M_2-M_1)+\cos^2\gamma 2H]\right\}.
\end{aligned} \tag{2.7}$$

Finally, the following approximate equalities are valid for the complex boundary quantities /244

$$\begin{aligned}
&\widetilde{Q}_{\nu\nu}=\cos^2\gamma\widetilde{T}_1+2\sin\gamma\cos\gamma\widetilde{S}+\sin^2\gamma\widetilde{T}_2,\\
&\widetilde{Q}_{\nu t}=\sin\gamma\cos\gamma(\widetilde{T}_2-\widetilde{T}_1)+(\cos^2\gamma-\sin^2\gamma)\widetilde{S},\\
&\widetilde{Q}_{\nu n}=ic\frac{d\widetilde{T}}{ds_{\nu}}\quad\left(\frac{d}{ds_{\nu}}=\frac{\cos\gamma}{r_0}\frac{\partial}{\partial\xi}+\frac{\sin\gamma}{r_0}\frac{\partial}{\partial\eta}\right),\\
&\widetilde{M}_{\nu}+i(1+\nu)c\widetilde{Q}_{\nu\nu}=ic\widetilde{T}.
\end{aligned} \tag{2.8}$$

If the considered boundary element is associated with the directrix of the cylinder (ξ = const), then, assuming that $\gamma = 0$ in these relationships, we

$$u_{\nu}=u,\quad u_t=v,\quad w,\quad \vartheta_{\nu}=\vartheta=-\frac{1}{r_0}\frac{\partial w}{\partial\xi}; \tag{2.9}$$

$$\begin{aligned}
&\varepsilon_{tt}=\varepsilon_2=\frac{1}{r_0}\left(\frac{\partial v}{\partial\eta}+\frac{w}{\rho}\right),\\
&\varkappa_{tt}=\varkappa_2=\frac{1}{r_0^2}\left(-\frac{\partial^2 w}{\partial\eta^2}+\frac{\partial}{\partial\eta}\left(\frac{v}{\rho}\right)\right),\\
&\varkappa_{t\nu}=\varepsilon-\frac{\omega}{r_{0\rho}}=\frac{1}{r_0^2}\left\{-\frac{\partial^2 w}{\partial\xi\partial\eta}-\frac{1}{\rho}\frac{\partial u}{\partial\xi}\right\},\\
&\varkappa_{tn}=\frac{1}{r_0}\left(\frac{\partial\omega}{\partial\eta}-\frac{\partial\varepsilon_2}{\partial\xi}\right)=\frac{1}{r_0^2}\left\{\frac{\partial^2 u}{\partial\eta^2}-\frac{1}{\rho}\frac{\partial w}{\partial\xi}\right\},\\
&\left(\vec{\varkappa}_t=-\varkappa_{tt}\vec{e}_1+\varkappa_{t\nu}\vec{e}_2-\varkappa_{t\gamma}\vec{n}\right);
\end{aligned} \tag{2.10}$$

$$M_{\nu}=M_1,\quad Q_{\nu}=T_1,\quad Q_{\tau}=S+\frac{2H}{r_0\rho},\quad Q_{\nu n}=\frac{1}{r_0}\left(\frac{\partial M_1}{\partial\xi}+\frac{\partial 2H}{\partial\eta}\right) \tag{2.11}$$

$$\left(\vec{Q}_{\nu}=Q_{\nu}\vec{e}_1+Q_{\tau}\vec{e}_2+Q_{\nu n}\vec{n}\right)\ldots$$

The following simplified expressions exist for the complex boundary quantities

$$\tilde{Q}_{\nu}=\tilde{T}_1,\quad \tilde{Q}_{\tau}=\tilde{S},\quad \tilde{Q}_{\nu n}=\frac{i}{2b^2}\frac{\partial\tilde{T}}{\partial\xi},\quad \tilde{M}_{\nu}=ic(\tilde{T}_2-\nu\tilde{T}_1). \tag{2.12}$$

On the other hand, the boundary element is associated with the generatrix /245 ($\eta = \eta_0 = \text{const}$); then, assuming that $\gamma = \pi/2$, we find

$$u_{\nu}=v,\quad u_t=-u,\quad w,\quad \vartheta_{\nu}=\psi=-\frac{1}{r_0}\left(\frac{\partial w}{\partial\eta}-\frac{v}{\rho}\right); \tag{2.13}$$

$$\varepsilon_{tt}=\varepsilon_1=\frac{1}{r_0}\frac{\partial u}{\partial\xi}; \tag{2.14}$$

$$\varkappa_{tt}=\varkappa_1=-\frac{1}{r_0^2}\frac{\partial^2 w}{\partial\xi^2},$$

$$\varkappa_{t\nu}=\tau=\frac{1}{r_0}\frac{\partial\psi}{\partial\xi}=-\frac{1}{r_0^2}\left(\frac{\partial^2 w}{\partial\xi\partial\eta}-\frac{1}{\rho}\frac{\partial v}{\partial\xi}\right),$$

$$\varkappa_{tn}=\frac{1}{r_0}\left(\frac{\partial\vartheta}{\partial\xi}-\frac{\partial\varkappa_1}{\partial\eta}\right)=\frac{1}{r_0^2}\frac{\partial^2 v}{\partial\xi^2}. \tag{2.15}$$

Comparing the expressions which have been obtained with (2.12), we become convinced that the latter, within the accuracy of the [illegible], coincide with the strain components of the normal element associated with the generatrix of the cylinder.

Furthermore,

$$Q_{\nu}=T_2,\quad Q_{\tau}=-S,\quad Q_{\nu n}=\frac{1}{r_0}\left(\frac{\partial M_2}{\partial\eta}+\frac{\partial 2H}{\partial\xi}\right),\quad M_{\nu}=M_2. \tag{2.16}$$

For complex boundary conditions

$$\tilde{Q}_{\nu}=T_2-iEhc\varkappa_1=T_2^{\circ}+i\frac{Eh}{2b^2r_0}\frac{\partial^2\tilde{w}}{\partial\xi^2},$$

$$\tilde{Q}_{\tau}=-S+iEhc\tau=-S^{\circ}-i\frac{Eh}{2b^2r_0}\left(\frac{\partial^2\tilde{w}}{\partial\xi\partial\eta}-\frac{1}{\rho}\frac{\partial\tilde{v}}{\partial\xi}\right), \tag{2.17}$$

$$\tilde{Q}_{vn}=\frac{1}{r_0}\left(\frac{\partial M_1}{\partial\eta}+\frac{\partial 2H}{\partial\xi}\right)-i\frac{Ehc}{r_0}\left\{\frac{\partial w}{\partial\xi}-\frac{\partial\varepsilon_1}{\partial\eta}\right\}=-i\frac{Eh}{2b^2r_0}\frac{\partial^2\tilde{v}}{\partial\xi^2},$$

$$\tilde{M}_{vv}=M_2+iEhc\varepsilon_1=l\frac{Eh}{2b^2}\frac{\partial\tilde{u}}{\partial\xi}$$

$$\left(\vec{\tilde{Q}}_v=\tilde{Q}_{vv}\vec{e}_2-\tilde{Q}_{vt}\vec{e}_1+\tilde{Q}_{vn}\vec{n}\right)$$

the following simplified expressions are valid

$$\tilde{Q}_{vv}=\tilde{T}_2,\ \tilde{Q}_{vt}=-\tilde{S},\ \tilde{Q}_{vn}=\frac{i}{2b^2}\frac{\partial\tilde{T}}{\partial\eta},\ \tilde{M}_{vv}=ic\left\{\tilde{T}_1-\nu\tilde{T}_2\right\}. \tag{2.18}$$

By using this relationship it is particularly simple to formulate the conditions at the joints of the shells for the rectilinear edges. Let us consider in more detail the most important practical problem of joining two cylindrical plates of the same thickness. If the junction is smooth (without a break), then according to (2.18) the conditions at the joints of the shells in the complex form are formulated in the following manner

$$\left\{\tilde{T}_2\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\tilde{S}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\frac{1}{r_0}\frac{\partial\tilde{T}}{\partial\eta}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\tilde{T}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}}. \tag{2.19}$$

As we can see from (2.17), these conditions are equivalent to the require- /246
ment that the four static quantities

$$\left\{T_2\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{S\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\frac{1}{r_0}\left(\frac{\partial M_2}{\partial\eta}+\frac{\partial 2H}{\partial\xi}\right)\right\}^{\mathrm{I}}=\left\{\ldots\right\},\ \left\{M_2\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}} \tag{2.20}$$

and the four deformation quantities

$$\left\{\frac{\partial u}{r_0\partial\xi}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\frac{\partial^2 w}{r_0^2\partial\xi^2}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\frac{\partial^2 v}{r_0^2\partial\xi^2}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}},\ \left\{\frac{\partial\psi}{r_0\partial\xi}\right\}^{\mathrm{I}}=\left\{\ldots\right\}^{\mathrm{II}}. \tag{2.21}$$

be continuous along the junction.

If the thickness of the joined plates is different, the 8 real conditions (2.20)-(2.21) may be used as conditions at the joints of the shells. The latter (deformation) conditions may be replaced by the geometric ones

$$\{u\}^{I}=\{\ldots\}^{II},\ \{v\}^{I}=\{\ldots\}^{II},\ \{w\}^{I}=\{\ldots\}^{II},\ \{\psi\}^{I}=\{\ldots\}^{II}. \tag{2.22}$$

It is easy to see that the deformation conditions may be obtained from the latter by differentiating along the generatrix. The deformation conditions, as we have stated, are usually more desirable, because (2.15) they can be expressed in terms of the strain components. Therefore, when they are investigated it is not necessary to determine the displacements first.

If cylindrical shells are joined at an angle (with a break), the complex conditions at the joints of the shells should be formulated in vector form

$$\{\vec{\tilde{Q}}_{\nu}\}^{I}=\{\vec{\tilde{Q}}_{\nu}\}^{II},\ \{\tilde{T}_{1}-\nu\tilde{T}_{2}\}^{I}=\{\ldots\}^{II}, \tag{2.23}$$

where $\vec{\tilde{Q}}_{\nu}$ is determined by relationship (2.17).

If the thicknesses are different, the simplified conditions at the joints of the shells are obtained by separating the real and imaginary parts in the complex relationships (2.23).

3. Method of Separating Variables

The fact that the coefficients obtained above are independent of ξ makes it possible for us to apply the method of the separation of variables by expanding all quantities in series of $\sin\lambda_m\xi$ and $\cos\lambda_m\xi$. The most widely used is the expansion of $\tilde{T}$ in functions of $\sin\lambda_m\xi$, where

$$\lambda_m=\frac{\pi m r_0}{L}, \tag{3.1}$$

m is a whole number, L is the length of the shell. As we shall see in the future, this dependence on ξ provides for a condition where each term of the expansion is satisfied by the condition of free support along the curvilinear edges. In this case the plate is broken down into m parts (fig. 60), each of which may be looked upon as being freely supported. Since for cylindrical plates free support is the most interesting practical version of the boundary conditions, we shall limit ourselves below to the consideration of this case. /247

We shall seek the solution of the basic equation (1.6) in the form

$$\tilde{T}=\sum_{m=1}^{\infty}\tilde{T}_{m}(\eta)\sin\lambda_m\xi. \tag{3.2}$$

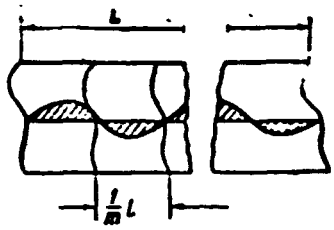

Figure 60

Then we obtain the characteristic equations for the function $\widetilde{T}_{,m}$

$$\Delta_m \rho \Delta_m \widetilde{T}_{,m} + \frac{d}{d\eta}\frac{1}{\rho}\frac{d\widetilde{T}_{,m}}{d\eta} - i2b^2\lambda_m^2\widetilde{T}_{,m} = \\ = i2b^2 r_0\left[\Delta_m(\rho q_{n,m}) + \frac{dq_{2,m}}{d\eta} + \lambda_m q_{1,m}\right]. \tag{3.3}$$

Here

$$\Delta_m = \frac{d^2}{d\eta^2} - \lambda_m^2,$$

while $q_{1,m}$; $q_{2,m}$; $q_{n,m}$ are the coefficients of the expansions

$$q_1 = \sum_{m=1}^{\infty} q_{1,m}\cos\lambda_m\xi, \quad q_2 = \sum_{m=1}^{\infty} q_{2,m}\sin\lambda_m\xi,$$

$$q_n = \sum_{m=1}^{\infty} q_{n,m}\sin\lambda_m\xi.$$

Furthermore, by representing the quantities

$$T_1,\ T_2,\ M_1,\ M_2,\ v,\ w,\ \psi$$

(and the analogous ones with different signs) in the form of expansions

$$\Phi(\xi,\eta) = \sum_{m=1}^{\infty} \Phi_{,m}(\eta)\sin\lambda_m\xi,$$

and the quantities

in the form

$$\Phi(\xi, \eta) = \sum_{m=1}^{\infty} \Phi_{\cdot m}(\eta) \cos \lambda_m \xi,$$

we obtain, from relationships of sections 1 and 2,

$$\begin{aligned}
&\widetilde{T}_{2,m} = i\frac{\rho}{2b^2}\Delta_m \widetilde{T}_{\cdot m} + r_0 \rho q_{n,m},\\
&\widetilde{T}_{1,m} = \widetilde{T}_{\cdot m} - i\frac{\rho}{2b^2}\Delta_m \widetilde{T}_{\cdot m} - r_0 \rho q_{n,m},\\
&\widetilde{S}_{\cdot m} = \frac{i}{2b^2\lambda_m}\left(\frac{d}{d\eta}\rho\Delta_m \widetilde{T}_{\cdot m} + \frac{1}{\rho}\frac{d\widetilde{T}_{\cdot m}}{d\eta}\right) + \frac{r_0}{\lambda_m}\left[\frac{d\rho q_{n,m}}{d\eta} + q_{2,m}\right];\\
&\widetilde{u}_{\cdot m} = \frac{r_0}{Eh\lambda_m}\left\{-\widetilde{T}_{\cdot m} + i\frac{1+\nu}{2b^2}\rho\Delta_m \widetilde{T}_{\cdot m} + (1+\nu) r_0 \rho q_{n,m}\right\},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\widetilde{v}_{\cdot m} = \frac{r_0}{Eh\lambda_m^2}\Bigg\{&\frac{d\widetilde{T}_{\cdot m}}{d\eta} + i\frac{1+\nu}{2b^2}\frac{d}{d\eta}\rho\Delta_m \widetilde{T}_{\cdot m} +\\
&+ (1+\nu) r_0\left(\frac{d\rho q_{n,m}}{d\eta} + 2q_{2,m}\right)\Bigg\},
\end{aligned} \tag{3.5}$$

$$\widetilde{w}_{\cdot m} = \frac{r_0\rho}{Eh\lambda_m^2}\left\{-\Delta\widetilde{T}_{\cdot m} - (1+\nu) r_0\left(-\lambda_m q_{1,m} + \frac{dq_{2,m}}{d\eta}\right)\right\}.$$

The equations of the membrane theory are written in the following form

$$\begin{aligned}
&\lambda_m T^0_{1,m} + \frac{dS^0_{\cdot m}}{d\eta} = -r_0 q_{1,m},\\
&-\lambda_m S^0_{\cdot m} + \frac{dT^0_{2,m}}{d\eta} = -r_0 q_{2,m},\\
&T^0_{2,m} = r_0 \rho q_{n,m}.
\end{aligned} \tag{3.6}$$

Let us consider the boundary conditions in more detail. Along the curvilinear edges the following conditions

$$w = v = T_1 = M_1 = 0.$$

are satisfied because of the adopted relationship between the quantities and ξ. Thus the shell is freely supported along the curvilinear edges. Along the rectilinear edges the following complex conditions at the joints of the shells take place

$$\{\tilde{T}_{2,m}\}^{I}=\{\ldots\}^{II},\ \{\tilde{S}_{,m}\}^{I}=\{\ldots\}^{II},\ \left\{\frac{1}{r_0}\frac{d\tilde{T}_{,m}}{d\eta}\right\}^{I}=\{\ldots\}^{II},$$
$$\{\tilde{T}_{,m}\}=\{\ldots\}^{II},\ (\{\tilde{T}_{1,m}\}^{I}=\{\ldots\}^{II}). \tag{3.7}$$

The above conditions are equivalent to 8 real conditions at the joints of the shells

$$\{T_{2,m}\}^{I}=\{\ldots\}^{II},\ \{S_{,m}\}^{I}=\{\ldots\}^{II},\ \left\{\frac{1}{r_0}\left(\frac{dM_{2,m}}{d\eta}-\lambda_m 2H_{,m}\right)\right\}^{I}=\{\ldots\}^{II} \tag{3.8}$$
$$\{M_{2},_{m}\}^{I}=\{\ldots\}^{II},$$
$$\{u_{,m}\}^{I}=\{\ldots\}^{II},\ \{v_{,m}\}^{I}=\{\ldots\}^{II},\ \{w_{,m}\}^{I}=\{\ldots\}^{II},$$
$$\{\psi_{,m}\}^{I}=\{\ldots\}^{II}. \tag{3.9}$$

We should note that in the considered specific case the deformation conditions at the joints of the shells coincide with the geometric ones. /249

In case of plates which are joined at an angle it is necessary to utilize relationships (2.23) and to replace the complex quantities contained in them with their expansion coefficients.

4. Circular Cylindrical Plates

For a circular plate whose directrix is the arc of a circle the quantity r_0 may be identified with the radius of the circle and the quantity η with the angle φ (fig. 61). In this case $\rho = \frac{R_2}{r_0} = 1$. Therefore for a circular plate the basic relationships are substantially simplified. Thus the basic complex equation takes the form

$$\Delta_m\Delta_m\tilde{T}_{,m}+\frac{d^2\tilde{T}_{,m}}{d\varphi^2}-i2b^2\lambda_m^2\tilde{T}_{,m}=$$
$$=i2b^2r_0\left\{\Delta_m q_{n,m}+\right.$$
$$\left.+\frac{dq_{2,m}}{d\varphi}+\lambda_m q_{1,m}\right\}, \tag{4.1}$$

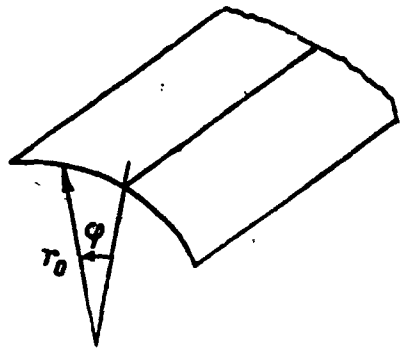

Figure 61

where

$$\Delta_m = \frac{d^2}{d\varphi^2} - \lambda_m^2.$$

Since the coefficients of the equation are constant, the particular solution of the homogeneous equation

$$\Delta_m \Delta_m \widetilde{T}_{.m} + \frac{d^2 \widetilde{T}_{.m}}{d\varphi^2} - i2b^2 \lambda_m^2 \widetilde{T}_{.m} = 0 \tag{4.2}$$

may be sought in the form

$$\widetilde{T}_{.m} = e^{\gamma\varphi}. \tag{4.3}$$

Substituting this expression into the homogeneous equation, we obtain the following characteristic equation for determining the constant γ

$$(\gamma^2 - \lambda_m^2)^2 + \gamma^2 - i2b^2\lambda_m^2 = 0. \tag{4.4}$$

It is easy to obtain its solution

$$\gamma^2 = \lambda_m^2 - \frac{1}{2} \pm \sqrt{i2b^2\lambda_m^2 - \left(\lambda_m^2 - \frac{1}{4}\right)}. \tag{4.5}$$

As we can see from equalities (3.1) the small values of the quantity λ_m correspond to very long shells (thin-walled rods).

The theory of thin-walled shells is presented in the monograph by V. Z. Vlasov (ref. 72), who has made many contributions to this area. Excluding the long cylindrical shells from our consideration, we may assume that the quantities λ_m are not small. Then we can neglect the term $(\lambda_m^2-1/4)$ under the radical sign compared with the preceding term. By doing this we obtain

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$$\gamma^2=\lambda_m^2-\frac{1}{2}\pm(1+i)\,b\lambda_m.$$

By finding the roots we arrive at four values for the constant γ

$$\begin{aligned}\gamma_{1,2}&=\pm(c_1+id_1),\\ \gamma_{3,4}&=\pm(c_2-id_2),\end{aligned}\tag{4.6}$$

where

$$c_1=\sqrt{\frac{b\lambda_m}{2}}\sqrt{\sqrt{(1+f_m)^2+1}+1+f_m},$$
$$d_1=\sqrt{\frac{b\lambda_m}{2}}\sqrt{\sqrt{(1+f_m)^2+1}-1-f_m},$$
$$c_2=\sqrt{\frac{b\lambda_m}{2}}\sqrt{\sqrt{(1-f_m)^2+1}-1+f_m},$$
$$d_2=\sqrt{\frac{b\lambda_m}{2}}\sqrt{\sqrt{(1-f_m)^2+1}+1-f_m}\quad\left(f_m=\frac{\lambda_m^2-\frac{1}{2}}{b\lambda_m}\right).$$

Thus we obtain the following general solution for the homogeneous equation

$$\widetilde{T}_{,m}=\sum_{k=1}^{4}\widetilde{C}_k e^{\gamma_k\varphi}.\tag{4.7}$$

The four complex constants contained in this solution are determined from the boundary conditions at the rectilinear edges.

We can see from expressions (4.6) that the real parts of the constants γ are quantities which in their order of magnitude are not greater than $\sqrt{b}\sim\sqrt[4]{\frac{r_0}{2c}}$, i.e., they are quantities which are substantially less than the

exponent in the theory of shells of revolution $\left(\sqrt{\frac{R_0}{2c}}\right)$. Therefore in cylindrical shells the edge effect associated with the rectilinear edges is exhibited less vividly than in shells of revolution. In practice the zone of the edge effect envelops the entire plate.

In regard to the particular solution of the homogeneous equation, when the loading is uniform and when the length of the shell is moderate we may assume that

$$\widetilde{T}_{1,m}^{P} = -\frac{r_0}{\lambda_m^2}\left\{\Delta_m q_{n,m} + \frac{dq_{2,m}}{d\varphi} + \lambda_m q_{1,m}\right\} = T_{1,m}^{*}. \tag{4.8}$$

(P = particular)

Substituting this expression into relationships (3.4) we obtain /251

$$\widetilde{T}_{2,m}^{P} = -\frac{ir_0}{2b^2\lambda_m^2}\left\{\frac{d^2}{d\varphi^2} - \lambda_m^2\right\}\left(\Delta_m q_{n,m} + \frac{dq_{2,m}}{d\varphi} + \lambda_m q_{1,m}\right) + r_0 q_{n,m},$$

$$\widetilde{T}_{1,m}^{P} = -\frac{r_0}{\lambda_m^2}\left\{\Delta_m q_{n,m} + \frac{dq_{2,m}}{d\varphi} + \lambda_m q_{1,m}\right\} + i\frac{r_0}{2b^2\lambda_m^2}\left\{\frac{d^2}{d\varphi^2} - \lambda_m^2\right\}\left(\Delta_m q_{n,m} + \frac{dq_{2,m}}{d\varphi} + \lambda_m q_{1,m}\right) - r_0 q_{n,m},$$

$$\widetilde{S}_{1,m}^{P} = -i\frac{r_0}{2b^2\lambda_m^3}\left\{\frac{d^3}{d\varphi^3} + (1-\lambda_m^2)\frac{d}{d\varphi}\right\}\left(\Delta_m q_{n,m} + \frac{dq_{2,m}}{d\varphi} + \lambda_m q_{1,m}\right) + \frac{r_0}{\lambda_m}\left[\frac{dq_{n,m}}{d\varphi} + q_{2,m}\right].$$

If we drop the small terms, we arrive at the following simplified expressions

$$\widetilde{T}_{2,m}^{P} = r_0 q_{n,m} = T_{2,m}^{*},$$

$$\widetilde{T}_{1,m}^{P} = -\frac{r_0}{\lambda_m^2}\left\{\frac{c^2 q_{n,m}}{d\varphi^2} + \frac{dq_{2,m}}{d\varphi} + \lambda_m q_{1,m}\right\} = T_{1,m}^{*}, \tag{4.9}$$

$$\widetilde{S}_{1,m}^{P} = \frac{r_0}{\lambda_m}\left\{\frac{dq_{n,m}}{d\varphi} + q_{2,m}\right\} = S_{1,m}^{*}.$$

It is easy to use direct substitution and verify the fact that the expressions obtained satisfy the membrane equations (3.6). Thus, when the shells are not too long and are loaded uniformly along the surface, we can use the solution of the membrane theory as a particular solution of the problem.

Furthermore by substituting expressions (4.9) into equalities (3.5) and carrying out the usual simplifications, we find

$$\widetilde{u}^{P}_{,m} = u^{*}_{,m} = -\frac{r_0}{Eh\lambda_m}\left(T^{*}_{1,m} - \nu T^{*}_{2,m}\right),$$

$$\widetilde{v}^{P}_{,m} = v^{*}_{,m} = \frac{r_0}{Eh\lambda_m^2}\left[\frac{dT^{*}_{,m}}{d\varphi} + (1+\nu) r_0\left(\frac{dq_{n,m}}{d\varphi} + 2q_{2,m}\right)\right], \qquad (4.10)$$

$$\widetilde{w}^{P}_{,m} = w^{*}_{,m} = \frac{r_0}{Eh\lambda_m^2}\left[-\Delta_m T^{*}_{,m} + (1+\nu) r_0\left(\lambda_m q_{1,m} - \frac{dq_{2,m}}{d\varphi}\right)\right].$$

To obtain the corresponding relationships corresponding to the homogeneous problem, we substitute $\widetilde{T}_{,m}$ (4.7) into (homogeneous) relationships (3.4). As a result and with the aid of equations (4.4) we obtain

$$\widetilde{T}_{2,m} = \frac{i}{2b^2}\sum_{k=1}^{4}\widetilde{C}_k(\gamma_k^2 - \lambda_m^2)e^{\gamma_k\varphi},$$

$$\widetilde{T}_{1,m} = \sum_{k=1}^{4}\widetilde{C}_k\left\{1 - \frac{i}{2b^2}(\gamma_k^2 - \lambda_m^2)\right\}e^{\gamma_k\varphi},$$

$$\widetilde{S}_{,m} = \frac{i}{2b^2)_m}\sum_{k=1}^{4}\widetilde{C}_k\left[\gamma_k(\gamma_k^2 - \lambda_m^2) + \gamma_k\right] =$$

$$= -\lambda_m\sum_{k=1}^{4}\widetilde{C}_k\frac{1}{\gamma_k}\left\{1 - \frac{i}{2b^2}(\gamma_k^2 - \lambda_m^2)\right\}e^{\gamma_k\varphi}.$$

If in the above equations we neglect the small terms and add the quantities (4.9) to the simplified expressions, we shall arrive at general expressions for the complex forces /252

$$\widetilde{T}_{1,m} = T^{*}_{1,m} + \sum_{k=1}^{4}\widetilde{C}_k e^{\gamma_k\varphi},$$

$$\widetilde{T}_{2,m} = T^{*}_{2,m} + \frac{i}{2b^2}\sum_{k=1}^{4}\widetilde{C}_k(\gamma_k^2 - \lambda_m^2)e^{\gamma_k\varphi}, \qquad (4.11)$$

$$\widetilde{S}_{,m} = S^{*}_{,m} - \lambda_m\sum_{k=1}^{4}\widetilde{C}_k\frac{1}{\gamma_k}e^{\gamma_k\varphi}.$$

Finally substituting expressions (4.11) into the homogeneous relationships (3.5) and carrying out similar simplifications we arrive at the general expressions for the complex displacements

$$\tilde{u}_{,m} = u^{*}_{,m} - \frac{r_0}{Eh\lambda_m}\sum_{k=1}^{4}\tilde{C}_k e^{\gamma_k\varphi},$$
$$\tilde{v}_{,m} = v^{*}_{,m} + \frac{r_0}{Eh\lambda_m^2}\sum_{k=1}^{4}\tilde{C}_k\gamma_k e^{\gamma_k\varphi}, \qquad (4.12)$$
$$\tilde{w}_{,m} = w^{*}_{,m} - \frac{r_0}{Eh\lambda_m^2}\sum_{k=1}^{4}\tilde{C}_k(\gamma_k^2 - \lambda_m^2)e^{\gamma_k\varphi}.$$

5. The Calculation of an Oval Shell Composed of Four Circular Plates

Let us consider a thin-walled cylinder whose middle surface cross section consists of four uniformly conjugated arcs. Two of these arcs have a radius r_0^{I} and the other two have a radius r_0^{II} (fig. 62). The oval shell is /253 loaded with a uniform internal pressure p and is freely supported along the curvilinear edges.

In calculating the shell we shall break it up into four plates, each of which is loaded with a uniform normal pressure and is freely supported along the curvilinear edges and is elastically joined with the adjoining plates.

To obtain a strict solution of this problem it is necessary to expand the surface load in the series

$$q_n = \frac{4}{\pi}p\sum_{m=1}^{\infty}\frac{1}{m}\sin\lambda_m\xi \qquad (5.1)$$

and make use of the relationships from the preceding section and seek solutions for m = 1, 2, 3, ... In practice the calculation is usually limited to the first term of expansion (5.1), which means that the uniform pressure is replaced with a load of type

$$q_n = \frac{4}{\pi}p\sin\lambda_1\xi. \qquad (5.2)$$

Two considerations are used as guidelines in this type of substitution.

1. Although the first term of expansion (5.1) gives a poor presentation of the actual load (fig. 63), the unknown values of the stresses and particularly

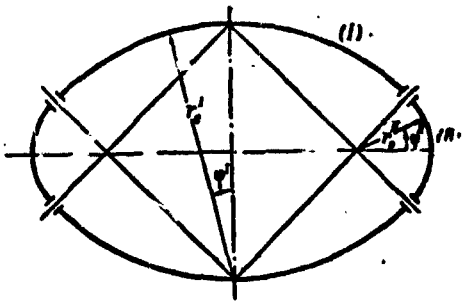

Figure 62

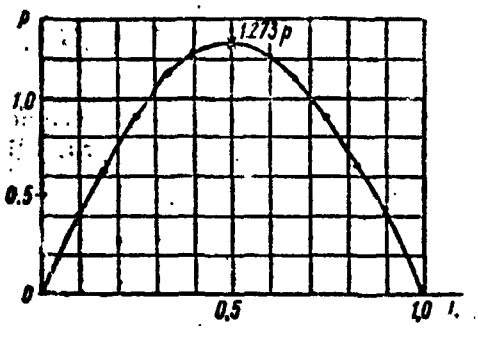

Figure 63

the fle.ure are represented by well converging series so that the first term in these series gives a good representation of the quantities themselves.

2. The fact that the first term of the expansion gives a total loading (0.81 pL), which is somewhat less than the value obtained from uniform pressure, it is compensated to a large measure by the displacement of part of the load from the ends of the shell (where it is taken up by the supports) toward the center. This redistribution of the load increases the stresses and flexures, and compensates for the decrease in the total loading. In any case, if we divide the values of the stresses and of the flexures obtained from the solution by a factor 0.81, we obtain quantities which are known to exceed /254 the values of stresses and flexures actually occurring in the shell.

It follows from what we have said that the solution corresponding to the first term of the expansion makes it possible to deduce a sufficiently careful conception concerning the actual operation of the shell.

Since the plates are deformed symmetrically with respect to their planes of symmetry, it is sufficient to consider one quarter of the composite shell. In this case due to symmetry the relationships of the preceding section assume the form ($\sigma_2 = \sigma_1$, $\sigma_4 = \sigma_3$)

$$
\begin{aligned}
\tilde{T}_{1,1} &= \tilde{C}_1(e^{\gamma_1\varphi} + e^{-\gamma_1\varphi}) + \tilde{C}_2(e^{\gamma_2\varphi} + e^{-\gamma_2\varphi}),\\
\tilde{T}_{2,1} &= T^{\circ}_{2,1} + \frac{1}{2b^2}\left\{\tilde{C}_1(\gamma_1^2 - \lambda_1^2)(e^{\gamma_1\varphi} + e^{-\gamma_1\varphi}) + \right.\\
&\quad \left. + \tilde{C}_2(\gamma_2^2 - \lambda_1^2)(e^{\gamma_2\varphi} + e^{-\gamma_2\varphi})\right\},\\
\tilde{S} &= -\lambda_1\left\{\tilde{C}_1\frac{1}{\gamma_1}(e^{\gamma_1\varphi} - e^{-\gamma_1\varphi}) + \tilde{C}_2\frac{1}{\gamma_2}(e^{\gamma_2\varphi} - e^{-\gamma_2\varphi})\right\},\\
\frac{d\tilde{T}_{,1}}{d\varphi} &= \tilde{C}_1\gamma_1(e^{\gamma_1\varphi} - e^{-\gamma_1\varphi}) + \tilde{C}_2\gamma_2(e^{\gamma_2\varphi} - e^{-\gamma_2\varphi}),\\
\tilde{w}_{,1} &= w^{\circ}_{,1} - \frac{r_0}{Eh\lambda_1^2}\left\{\tilde{C}_1(\gamma_1^2 - \lambda_1^2)(e^{\gamma_1\varphi} + e^{-\gamma_1\varphi}) + \right.\\
&\quad \left. + \tilde{C}_2(\gamma_2^2 - \lambda_1^2)(e^{\gamma_2\varphi} + e^{-\gamma_2\varphi})\right\},
\end{aligned}
\tag{5.3}
$$

where

$$
T^{\circ}_{2,1} = \frac{4}{\pi}r_0 p, \quad w^{\circ}_{,1} = \frac{4}{\pi}\frac{r_0^2 p}{Eh}. \tag{5.4}
$$

Let us consider a shell whose calculation (utilizing the real conditions of conjugation) is shown in (ref. 31, p. 191). In this example for the first plate we have

$$
\frac{r_0^1}{h} = 150, \quad \frac{r_0^1}{L} = 0.625, \quad \varphi_0^1 = -\pi/4, \quad \nu = 0.3.
$$

From this we obtain

$$
\begin{gathered}
2b^2 = 496, \quad b = 15.7; \quad \frac{1}{2b^2} = 2.02\cdot 10^{-3},\\
\lambda_1 = \frac{\pi r_0^1}{L} = 1.964, \quad \lambda_1^2 = 3.86,\\
f = \frac{\lambda_1 - \frac{1}{2}}{b\lambda_1} = 0{,}108, \quad c_1 = 6{,}37, \quad c_2 = 2{,}65,\\
d_1 = 2{,}42, \quad d_2 = 5{,}84,\\
\gamma_1 = -\gamma_3 = 6{,}37 + i\,2{,}42, \quad \gamma_1^2 - \lambda_1^2 = 30{,}9 + i\,30{,}9,\\
\gamma_2 = -\gamma_4 = 2{,}65 - i\,5{,}84, \quad \gamma_2^2 - \lambda_1^2 = -30{,}95 + i\,30{,}95,\\
T^{\circ}_{2,1} = 1{,}273 pr_0^1, \quad w^{\circ}_{,1} = 1{,}273\frac{p(r_0^1)^2}{Eh}.
\end{gathered}
$$

Substituting these values into equalities (5.3) and assuming that $\varphi = \varphi_0^I = -\pi/2$, we obtain the following expressions for the complex boundary quantities /255

$$\widetilde{T}^{I}_{2,1}=(-5.79-i11.9)\widetilde{C}^{I}_{1}+(0.425+i0.553)\widetilde{C}^{I}_{3}+1.273pr^{I}_{0},$$
$$\widetilde{T}^{I}_{1,1}=(-48.9+i142)\widetilde{C}^{I}_{1}+(-0.97+i7.86)\widetilde{C}^{I}_{3},$$
$$\widetilde{S}^{II}_{,1}=(1.34+i43.2)\widetilde{C}^{I}_{1}+(-2.38+i0.763)\widetilde{C}^{I}_{3},$$
$$\left(\frac{1}{r_0}\frac{d\widetilde{T}_{,1}}{d\varphi}\right)^{I}=\frac{1}{r^{I}_{0}}\left\{(654-i784)\widetilde{C}^{I}_{1}+(-44,8-i27,0)\widetilde{C}^{I}_{3},\right\};$$

For the second plate

$$\frac{r^{II}_{0}}{h}=40,\quad \frac{r^{II}_{0}}{L}=\frac{1}{6},\quad \varphi^{II}_{0}=\pi/4,$$

from which it follows that

$$2b^2=\sqrt{12(1-\nu^2)}\,\frac{r^{II}_{0}}{h}=132,\quad b=8.12,\quad \frac{1}{2b^2}=7.58\cdot10^{-3}.$$
$$\lambda_1=0.524,\quad \lambda_1^2=0.275,$$
$$\gamma_1=-\gamma_2=2.31+i0.915,\quad \gamma_1^2-\lambda_1^2=4,22+i4,23,$$
$$\gamma_3=-\gamma_4=0.959-i2.21,\quad \gamma_3^2-\lambda_1^2=-4.24-i4,24,$$
$$T^{*}_{2,1}=1.273pr^{II}_{0},\quad w^{*}_{,1}=1.273\frac{p(r^{II}_{0})^2}{Eh},$$
$$\widetilde{T}^{II}_{2,1}=(-0.279+i0.025)\widetilde{C}^{II}_{1}+(-0.0663-i0.0386)\widetilde{C}^{II}_{3}+1.273pr^{II}_{0},$$
$$\widetilde{T}^{II}_{1,1}=(4.75+i3.95)\widetilde{C}^{II}_{1}+(-0.432-i1.63)\widetilde{C}^{II}_{3},$$
$$\widetilde{S}^{II}_{,1}=(-1.21-i0.467)\widetilde{C}^{II}_{1}+(-0.487+i0.277)\widetilde{C}^{II}_{3},$$
$$\left(\frac{1}{r_0}\frac{d\widetilde{T}_{,1}}{d\varphi}\right)^{II}=\frac{1}{r^{II}_{0}}\left\{(6.59+i13.74)\widetilde{C}^{II}_{1}+(-5.92-i1.85)\widetilde{C}^{II}_{3}\right\}.$$

Equating the boundary conditions we find (noting that $\frac{r^{II}_{0}}{r^{I}_{0}}=0.267$) a system for determining the complex constants

$$(5,79+i11.9)\widetilde{C}^{I}_{1}+(-0,425-i0,553)\widetilde{C}^{I}_{3}+$$
$$+(-0.279+i0,025)\widetilde{C}^{II}_{1}+(-0,0663-i0.0386)\widetilde{C}^{II}_{3}=0.934pr^{I}_{0},$$
$$(48.9-i142)\widetilde{C}^{I}_{1}+(0.97-i7.86)\widetilde{C}^{I}_{3}+$$
$$+(4.75+i3,95)\widetilde{C}^{II}_{1}+(-0,432-i1.63)\widetilde{C}^{II}_{3}=0,$$
$$(-1.34-i43.2)\widetilde{C}^{I}_{1}+(2,38-i0.763)\widetilde{C}^{I}_{3}+$$

$$+(-1.21-i0.467)\widetilde{C}_1^{II}+(-0.487+i0.277)\widetilde{C}_3^{II}=0,$$
$$(-654+i784)\widetilde{C}_1^{I}+(44.8+i27.0)\widetilde{C}_3^{I}+$$
$$+(24.7+i51.5)\widetilde{C}_1^{II}+(-22.2-i6.92)\widetilde{C}_3^{II}=0.$$

Solving this system we find

$$\widetilde{C}_1^{I}=(0.0140-i0.0186)pr_0^1,\quad \widetilde{C}_1^{II}=(-0.554+i0.093)pr_0^1,$$
$$\widetilde{C}_3^{I}(-0.2865+i0.420)pr_0^1,\quad \widetilde{C}_3^{II}=(-1.425+i0.803)pr_0^1.$$

Substituting the values obtained for the constants into relationships (5.3) and (5.4) we obtain values for the stresses and flexures which are tabulated in table 4. /256

In this table we are struck by the discrepancy (small compared with the maximum value 27.5) in the normal flexure. It cannot be attributed to the error in the complex joining of the shells because in reference 31, which used the real form of the junction, almost the same discontinuity in the normal flexure was obtained. This insignificant shortcoming of this solution should be attributed to the error inherent in the solutions which are utilized. Figure 64 presents the diagram of the basic stresses, and figure 65 shows a substantially enlarged distribution of normal flexures.

6. Asymptotic Investigation of the Homogeneous Equation for a Circular Cylindrical Shell

Let us write the basic homogeneous complex equation for a circular cylindrical shell in the following form

$$\frac{\partial^4\widetilde{T}}{\partial\varphi^4}+2\frac{\partial^4\widetilde{T}}{\partial\varphi^2\partial\xi^2}+\frac{\partial^4\widetilde{T}}{\partial\xi^4}+\frac{\partial^2\widetilde{T}}{\partial\varphi^2}+ia^{-1}\frac{\partial^2\widetilde{T}}{\partial\xi^2}=0. \tag{6.1}$$

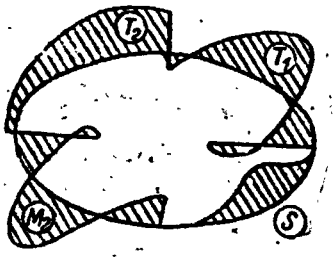

Figure 64

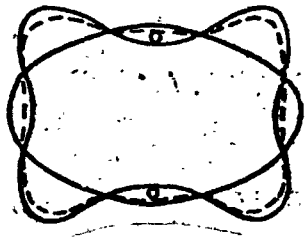

Figure 65

TABLE 4

φ^{I}	0°	−15°	−25°	−35°	−45°			
φ^{II}					45°	30°	15°	0°
$w \Big/ \frac{pr_0^4}{Eh}$	−10.6	0.5	16.8	27.5	6.16 4.97	−4.8	−10.6	−13.0
$\sigma_{T_1} \Big/ \frac{pr_0^2}{h}$	−0.545	0.72	1.87	1.50	−1.07 −1.07	−2.45	−3.48	−3.96
$\sigma_S \Big/ \frac{pr_0^2}{h}$	0	0.05	−0.42	−1.07	−1.18 −1.18	−0.94	−0.52	0
$\sigma_{T_2} \Big/ \frac{pr_0^2}{h}$	1.29	1.34	1.29	1.05	0.618 0.618	0.46	0.36	0.33
$\sigma_{M_2} \Big/ \frac{pr_0^2}{h}$	−1.46	−0.79	1.29	3.95	−0.416 −0.416	−2.93	−3.32	−3.30

Here a is used to designate the small parameter $a = \frac{c}{r_0} = \frac{1}{2b^2}$. We seek the /258 solution of the above equation in the following form

$$\tilde{T} = e^{pt+qt}. \tag{6.2}$$

Substituting this expression into (6.1), we obtain the following characteristic equations for the exponents of variability of p and q

$$p^4 + 2p^2q^2 + q^4 + p^2 + ia^{-1}q^2 = 0. \tag{6.3}$$

If we assign some specific value to p, by solving this problem we obtain a specific value for q, and vice versa. These values depend on the value of the small parameter a

$$p = p(a) = p_a, \quad q = q(a) = q_a.$$

We shall answer the following question: What is the behavior of the roots of the characteristic equation when $a \to 0$, in other words, we determine their limiting values

$$p_0 = \lim_{a \to 0} p_a, \quad q_0 = \lim_{a \to 0} q_a.$$

For this purpose we represent p and q in the form

$$p_a = n_a a^{-s}, \quad q = l_a a^{-t}, \tag{6.4}$$

where n_a and l_a are constants of the order of unity. Without belittling generality we may assume that they are equal to zero. For this purpose it is sufficient to assume that the values $n_a = 0$ and $l_a = 0$ correspond, respectively, to $s = -\infty$ and $t = -\infty$.

Substituting expressions (6.4) into the characteristic equation (6.3) we obtain the equality

$$n_a^{4(1)} a^{-4s} + 2n_a^{2(2)} l_a^{2} a^{-2(s+t)} + l_a^{4(3)} a^{-4t} + n_a^{2(4)} a^{-2s} + il_a^{2(5)} a^{-1-2t} = 0. \tag{6.5}$$

For assigned s and t in the terms of this equality, the power exponents associated with the quantity a will, in general, be different. Let us divide all

terms by a in a minimum degree (minimum in the algebraic sense). Then in all of the terms a is contained either in the zero degree or in the positive degree. Let us make a tend to zero. In this case the equation will retain only those terms which do not contain a with positive exponents. Thus we arrive at a limiting equation which determines the quantities n_0 and l_0. Because the variation in the coefficients of algebraic equation as a function of its root is continuous, the values

$$p_0 = n_0 a^{-s}, \quad q_0 = l_0 a^{-t}$$

which satisfy the limiting equation are the limiting values of the roots of equation (6.5), i.e.,

$$p_0 = \lim_{a \to 0} p_a, \quad q_0 = \lim_{a \to 0} q_a.$$

Thus, in order to determine the asymptotic (limiting) values of the roots of equation (6.5), it is necessary, first of all, to establish the values of quantity pairs $\{s, t\}$, which lead to nontrivial limiting solutions. We shall call such pairs asymptotically stable. /259

We begin by making an effort to find the asymptotically stable pair of values $\{s, t\}$ which retains all of the terms in equation (6.5). For this purpose we equate the exponents of all terms and obtain a system of equalities.

$$-\overset{(1)}{4s} = -\overset{(2)}{2(s+t)} = -\overset{(3)}{4t} = -\overset{(4)}{2s} = -1 - \overset{(5)}{2t}.$$

We can easily see that this system is incompatible. From this it follows that the entire equation (6.5) and consequently (6.1) are asymptotically unstable. This should be understood in the sense that for any selection of the quantities $\{s, t\}$ the complete equation (6.1) will certainly contain small and insignificant terms when the values of a are small.

Let us consider another case by equating the exponents of the third and fifth term $-\overset{(3)}{4t} = -1 - \overset{(5)}{2t}$. From this it follows that $t = 1/2$. The retained terms are associated with the characteristic equation $(l_0^2 + i) l_0^2 a^{-2} = 0$. Dividing the terms of the stated equality by the factor $l_0^2 a^{-2}$ (l_0 cannot be equal to zero in accordance with the rule established above), we arrive at an equation for l_0

$$l_0^2+i=0. \tag{6.6}$$

The latter is associated with the initial equations

characteristic equation

$$q^2+ia^{-1}=0 \tag{6.7}$$

and differential equation

$$\frac{\partial^2\tilde{T}}{\partial\xi^2}+ia^{-1}\tilde{T}=0. \tag{6.8}$$

When we made the transition from the complete equation to (6.8) we dropped the first, second and fourth terms. The relative error associated with this may be characterized by the order of the ratios of dropped terms to one of the retained terms (it does not matter which term, because their orders are equivalent), i.e., by the ratios

$$\frac{a^{-4s}}{a^{-4t}}=a^{4(t-s)}=a^{4\left(\frac{1}{2}-s\right)},\quad \frac{a^{-2(s+t)}}{a^{-4t}}=a^{2\left(\frac{1}{2}-s\right)},\quad \frac{a^{-2s}}{a^{-4t}}=a^{2(1-s)}.$$

We shall call these quantities asymptotic errors and abbreviate them by a. e. By examining the above ratios, it is easy to see that the requirement a. e. < 1 places the following limitation $s < 1/2$ on the interval over which s can vary. In this case the maximum asymptotic error in the considered case is equal to $(a.\ e.)_{max} = a^{2\left(\frac{1}{2}-s\right)}$. When $s \leq 0$ a. e. $\leq a^1 = \frac{1}{2b^2}$, i.e., the asymptotic error of the simplified solution does not exceed the accuracy of the equations in the theory of thin shells. In cases of this type we shall call the simplified equations the asymptotically accurate equations (with respect to the considered values of s and t). /260

In exactly the same way we can obtain six more asymptotically stable pairs of values {s, t}. These values together with the corresponding simplified equations and values of asymptotic errors are presented in table 5. Figure 66 is prepared from the information in this table and we now proceed to consider this figure.

The straight line segments shown in figure 66 make it possible to determine the quantity t from a known value of s and vice versa to determine the asymptotic (limiting) values of the characteristic equation roots. For sufficiently thin (refined) shells these limiting values will be close to the real values of the roots. Furthermore, and this is even more important, they /261

TABLE 5

$\widetilde{T} = e^{px+qt}$; $p = ha^{-s}$; $q = la^{-t}$

	(1)	(2)	(3)	(4)	(5)	s / t	a. e.$_{max}$	Limiting equation	Characteristic equation	Differential equation
	$-4s$	$-2(s+t)$	$-4t$	$-2s$	$-1-2t$	—	—	$(h_0^2+l_0^2)^2 + h_0^2 + lh_0^2 = 0$	$(p^2+q^2)^2 + p^2 + la^{-1}q^2 = 0$	$\Delta\Delta\widetilde{T} + \frac{\partial^2\widetilde{T}}{\partial x^2} + la^{-1}\frac{\partial^2\widetilde{T}}{\partial t^2} = 0$
I	$-4s$	$-2(s+t)$	$-4t$	a^s	$-1-2t$	$s=\frac{1}{2}$ $t=\frac{1}{2}$	$\boxed{\frac{a^s}{a^{\frac{1}{2}}+t}}$	$(h_0^2+l_0^2)^2 + lh_0^2 = 0$	$(p^2+q^2)^2 + la^{-1}q^2 = 0$	$\Delta\Delta\widetilde{T} + la^{-1}\frac{\partial^2\widetilde{T}}{\partial t^2} = 0$
II	$a^{2(\frac{1}{2}-s)}$	$a^{2(\frac{1}{2}-s)}$	$-4t$	$a^{2(1-s)}$	$-1-2t$	$t=\frac{1}{2}$ $s<\frac{1}{2}$	$a^{2(\frac{1}{2}-s)}$	$l_0^2 + l = 0$	$q^2 + la^{-1} = 0$	$\frac{\partial^2\widetilde{T}}{\partial t^2} + la^{-1}\widetilde{T} = 0$
III	$-4s$	$-2(s+t)$	$-4t$	a^{2s}	$a^{2(s-\frac{1}{2})}$	$s=t>\frac{1}{2}$	$a^{2(s-\frac{1}{2})} - a^{2(t-\frac{1}{2})}$	$(h_0^2+l_0^2)^2 = 0$	$(p^2+q^2)^2 = 0$	$\Delta\Delta\widetilde{T} = 0$
IV	$-4s$	$a^{2(\frac{1}{2}-s)}$	$a^{4(\frac{1}{2}-s)}$	a^{2s}	$-1-2t$	$t-2s+\frac{1}{2}=0$ $0<s<\frac{1}{2}$ $-\frac{1}{2}<t<\frac{1}{2}$	$a^{2s} = a^{\frac{1}{2}+t}$ $a^{2(\frac{1}{2}-s)} = a^{\frac{1}{2}-t}$	$h_0^4 + lh_0^2 = 0$	$p^4 + la^{-1}q^2 = 0$	$\frac{\partial^4\widetilde{T}}{\partial x^4} + la^{-1}\frac{\partial^2\widetilde{T}}{\partial t^2} = 0$
V	$-4s$	a^1	a^0	$-2s$	$-1-2t$	$t=-\frac{1}{2}$ $s=0$	$\boxed{\frac{a^1}{a^{\frac{1}{2}}-t}}^{**}$	$h_0^4 + h_0^2 + lh_0^2 = 0$	$p^4 + p^2 - la^{-1}q^2 = 0$	$\frac{\partial^4\widetilde{T}}{\partial x^4} + \frac{\partial^2\widetilde{T}}{\partial x^2} + la^{-1}\frac{\partial^2\widetilde{T}}{\partial t^2} = 0$
VI	$-4s$	a^{-2s}	a^{-4t}	$-2s$	$a^{-2(\frac{1}{2}+t)}$	$s=0$ $t<-\frac{1}{2}$	$a^{-2(\frac{1}{2}+t)}$	$h_0^2 + 1 = 0$	$p^2 + 1 = 0$	$\frac{\partial^2\widetilde{T}}{\partial x^2} + \widetilde{T} = 0$
VII	a^{-2s}	$a^{2(\frac{1}{2}-s)}$	$a^{2(1-s)}$	$-2s$	$-1-2t$	$t=-\frac{1}{2}+s$ $s<0$ $t<-\frac{1}{2}$	$a^{-2s} = a^{-2(\frac{1}{2}+t)}$	$h_0^2 + lh_0^2 = 0$	$p^2 + la^{-1}q^2 = 0$	$\frac{\partial^2\widetilde{T}}{\partial x^2} + la^{-1}\frac{\partial^2\widetilde{T}}{\partial t^2} = 0$

Remarks:

* The square block surrounds the value of asymptotic error when solution I with other values of s and t is used.

** The same for solution V.

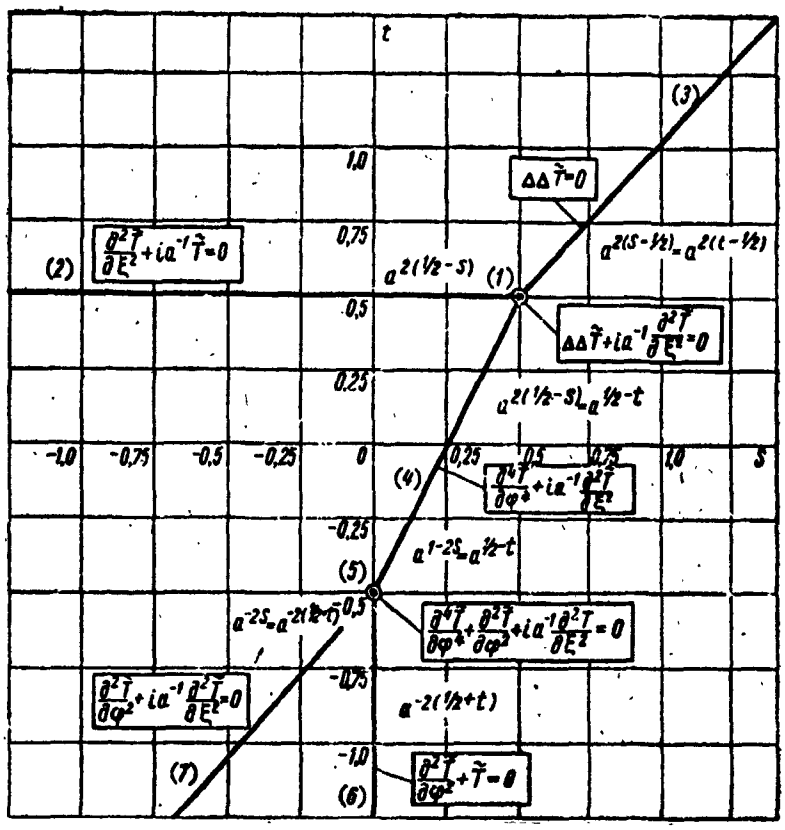

Figure 66

make it possible for us to select a suitable simplified equation and to evaluate its asymptotic error beforehand.

We shall begin the analysis of figure 66 by considering the node equation (1). For those values $\{s, t\}$ for which it is derived ($s = t = 1/2$), its asymptotic error is equal to a^1, i.e., it is asymptotically accurate. However, this equation apparently may be used for other values of $\{s, t\}$, because equations (2), (3) and (4) are its particular cases. The asymptotic error of equation (1) is associated with the absence of the fourth term in this equation which is contained in the node equation (5). Therefore the asymptotic error of the considered equation will be determined by the error of equation (4) at the point where it joins equation (5), i.e., by the value $a^{1/2+t}$. In exactly the same manner it can be shown that the asymptotic error of equation (5) is determined by the quantity $a^{1/2-t}$.

Let us consider in detail two of the most interesting classes of problems.

Circular Cylindrical Plates Closed With Respect to φ

In this case (see Chapter V, section 11) the solution is sought in the form of an expansion in $\begin{matrix}\cos k\varphi \\ \sin k\varphi\end{matrix}$. Therefore, according to (6.2) and (6.4) we should let

$$n_0=i, \quad a^{-s}=k. \tag{6.9}$$

As we know, k = 0 corresponds to the symmetric case. According to figure 66 solution (2) produces two large roots of the order of $a^{-1/2} = \sqrt{2b}$. In this case in view of (6.9) a. e. = $a^{2\,(1/2-s)} = a^1\, k^2 = 0$, i.e., this solution may be considered asymptotically accurate. From the eighth column of table 5 we have the following for the case under consideration

$$l_0^2+i=0,$$

from which

$$l_0=\pm\frac{1-i}{\sqrt{2}}, \quad q_0=\pm(1-i)b.$$

Thus the large root of the characteristic equation is associated with the boundary effect type solutions

$$e^{(1-i)bt}, \quad e^{-(1-i)bt},$$

which was obtained previously in section 11 of Chapter V. In the present case they are accurate solutions of the homogeneous equation (6.1) in the direct sense of this word. Two other roots are associated with the infinitely removed point on the straight line (7) t = $-\infty$. We can see from figure 66 that there are two zero roots of the characteristic equation in the symmetric case. /262

Let us consider k = 1, which corresponds to the antisymmetric case. According to (6.9) we must let s = 0 when we consider this case. In this case the asymptotic error of the solution (edge effect type) is equal to $a^{2\,(1/2-s)} = a^1$, i.e., the latter may be considered asymptotically exact. In regard to the slowly varying solution, if we select a solution in (6) which has the minimum asymptotic error (t = $-\infty$), we again obtain two zero roots. It

is easy to see directly that the corresponding characteristic equation (table 5, column 9) is satisfied by the zero roots p = 1, q = 0.

Now let us turn to the values k = 2, 3, ... The large roots of the order $a^{-1/2}$ are determined from equation (2) as before. In this case its asymptotic error a. e. = $a^2\left(\frac{1}{2}-s\right) = ak^2$ increases rapidly with k. The small roots, on the other hand, should be determined from equation (5) which has the same asymptotic error. We can see quite well from figure 66 that as k increases, the difference between the large and small roots is gradually removed. However, when $s > 1/4$ (which corresponds to $k > a^{-\frac{1}{4}}$), the roots should be determined from the node equation (1) whose asymptotic error is equal to $a^{2s} = \frac{1}{k^2}$.

We note that the determination of small roots from the intermediate equation (4) is associated with a large asymptotic error, because the latter is determined by two quantities $1/k^2$, ak^2. As we can see from figure 66 a. e. $\geq a^{1/2} = \sqrt{2b}$.

When $s \geq \frac{1}{2}$, their roots become quantities of the same order. When they are determined by means of the simplified equation (3), an asymptotic error $a^{2(s-1/2)} = \frac{1}{ak^2}$ is introduced. If we wish to obtain an a. e. $< a^{1/2}$, it is necessary to assume that $s > 3/4$, i.e , to assume that $k > a^{-3/4}$. Values of k which are this large are usually of little interest.

From everything we have said above it follows that for a close cylindrical shell when $k < \sqrt[4]{a}$ we should use equation (2) to determine the rapidly varying solution and use equation (5) for the slowly varying solution. When $k > a^{-\frac{1}{4}}$ it is necessary to utilize equation (1).

Let us now turn to the cylindrical plate supported along the curvilinear edges. If we introduce the reduced plate length $l = \frac{L}{m}$ (i.e., the length of one of the freely supported sections), then all of the quantities λ_m introduced in section 3 may be written in the form $\lambda_m = \lambda = \pi \frac{r_0}{l}$ for any m. /263

Table 5 and figure 66 obtained above make it possible to establish a relationship between the relative reduced length of the plate and the asymptotic properties of the characteristic equation roots. Let us assume that

$$l_0 = l, \quad a^{-t} = \lambda. \tag{6.10}$$

For large negative values of t two small roots are determined from equation (7), and two (of the order of unity) roots are determined from (6). In this case the error of both equations is equal to $a^{-2(1/2+t)} = \frac{\lambda^2}{a}$. The condition a. e. $\leq a^{1/2}$ leads to the requirement $t \leq -0.75$ and according to (6.10) places a rather severe limitation $\lambda \leq a^{3/4}$ on the quantity λ. Thus equations (6) and (7) are designed for very small values of λ, or, in other words, for shells of very large reduced length. Such shells are called thin-walled rods. Simple approximate methods (ref. 72) have been worked out for their calculation. When $0.75 < t \leq 0$ ($a^{0.75} < \lambda \leq 1$) we should use equation (5). Its asymptotic error is determined by the quantity $a^{1/2-t} = a^{1/2}\lambda$. Thus equality (5), which is known in the literature as the equation of the semimembrane theory of cylindrical shells (ref. 31), should be used to compute plates with large and moderate relative reduced length.

When $t \geq 0$ the roots of the characteristic equation should be determined from equation (1). The error associated with this is equal to $a^{1/2+t} = \frac{a^{1/2}}{\lambda}$. Equation (3), on the other hand, may be used only when the shells are very short $t > 3/4$ ($\lambda > \frac{1}{a^{3/4}}$).

We note that the utilization of equation (4) is associated with substantial asymptotic error, which is determined by the larger of the two quantitites $a^{1/2}\lambda$, $a^{1/2}\frac{1}{\lambda}$. However, since it is particularly simple, it is frequently used. In the example which was considered in the preceding section, λ had the following values: $\lambda^{II} = 1.964$; $\lambda^{I} = 0.524$, which corresponds to a region adjoining node equations (1) and (5). Therefore, in order to avoid a large error, the roots were obtained from the total equation.

The results which are presented were obtained in considering the /264
solution of a rather specific form (6.2). If we have in mind only a solution of this type, the analysis which has been presented does not yield substantial simplifications, because it is not very difficult to solve the general (biquadratic) characteristic equation in each particular case. However, the relationship which we have obtained between the value of the exponents of solution variability (p, q) and the relative role of various terms in the general differential equation is of a general nature. In problems which are more complex the presented analysis makes it possible, in conjunction with the expected

nature of the unknown solution, to select with reliability the initial simplified differential equation and to simplify the relationships associating the quantities of interest to us with the basic complex function.

In conclusion we note two circumstances:

1. The particular cases of a closed shell and a freely supported shell considered above are investigated in detail in ref. 11, Chapters 11 and 12. Since A. L. Gol'denveyzer adopted the real solution of eighth order as the basic one, his analysis is substantially more cumbersome and less illustrative.

2. By examining the tenth column of table 5 we become convinced that any term of the total equation enters into any one of the simplified equations. Thus the analysis presented confirms the well-known fact (see, for example, ref. 11, p. 230) that the complex V. V. Novozhilov equation (6.1) contains all real terms and does not contain the small insignificant ones. This cannot be said of the total real equation for cylindrical shells used by A. L. Gol'denveyzer.

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LINEAR THEORY OF SHELLS. PART II

SPECIAL PROBLEMS OF THE THEORY

K. F. Chernykh
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The publication of this book has been decreed by the Editorial and Publication Soviet of the Leningrad University.

The book presents several basic problems of the theory: thermoelastic stresses, calculation of toroidal expansion tanks and pipes with a circular axis, variational methods and general theorems, asymptotic integration and membrane theory. The complex method is further developed and detailed. The basic relationships of the theory of shells are derived using generalized coordinates. For the convenience of the reader, the principles of tensor analysis and the theory of surfaces are presented.

The book is designed for engineers and scientific workers who are concerned with the stress analysis of thin-walled structures. The book may also serve as a text for senior college students and for Fellows specializing in the theory of elasticity.

PREFACE

The Soviet reader has access to a rather large number of books which are concerned with the theory of shells in general and with the specific problems associated with this theory (I believe that it is not necessary for me to list these books, because they are sufficiently well known). Therefore, it is not very easy to say anything new in this field, either in form or in content.

I believe, however, that the book by my student K. F. Chernykh will occupy a worthy position among those of his elder colleagues.

The author bases his book on the equations of the theory of shells in a complex form. He reveals the wide possibilities included in these equations and presents a series of illustrations which show the expediency of this method as an approach which aids in the transformation of the equations of the theory of shells, in the qualitative investigation of the properties associated with their solutions and, finally, in the development of numerical solutions for a series of specific problems which are of practical interest.

The reader will encounter this in the first part of the book.

V. V. Novozhilov

FROM THE AUTHOR

In the second part of the book presented to the attention of the reader more specialized problems of the linear theory of shells are considered. The purpose of the book is to acquaint the reader with the present state of the art of certain basic problems in the theory and to help him overcome the barrier which separates the engineer who conducts calculations from the engineer who carries on research. The book illuminates a series of problems which are usually considered in the periodical literature. Some original results are also included. The author will be satisfied if the reader's acquaintance with the book will arouse some interest in the theory of shells and point the way for the calculation of specific thin-walled structures.

We shall now briefly discuss the contents of the book.

Chapter VII presents the basic information of the theory of surfaces in tensor notation.[1] For the convenience of the reader, the principles of tensor analysis are also presented in this chapter. In selecting the material the author has made an effort to convey (in a form familiar to a specialist in mechanics) the minimum information necessary to read the special literature on the theory of shells with ease. In addition, several relationships were obtained and used in the subsequent chapters for exposing the basic properties of relationships in the theory of shells.

Chapter VIII derives the basic relationships of the theory (including its complex version) in an invariant (tensor) form. Here as in the first part the initial relationships are from the general theory of elasticity. Section 8 may be looked upon as a summary of the basic equations for the orthogonal system of coordinates.

Chapter IX gives a systematic presentation of the general theorems of the theory of shells. New versions of boundary conditions are associated with the uniqueness theorem (in particular, deformation conditions). Various variational principles are then deduced from the established extremal principles. A great deal of attention is devoted to the expanded (Reissner) variational equations and also to the variational principles of the complex theory.

Chapter X considers dislocation (multiple-valued) displacements. Courses are set for the utilization of dislocation displacements for the determination of mounting stresses in thin-walled structures. The multiple-valued stress

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[1] The first six chapters are contained in the first part of the book.

functions which occur when the edge loading is unbalanced are considered in greater detail than in the first part of the book.

Chapter XI develops the method of complex displacements. It is shown in particular that the well-known equation of Meissner-Clark follows from the system of equations containing the complex displacements. This equation is generalized for the case of antisymmetric loading. A relationship is established between the equations containing complex displacements and the equations of V. Z. Vlasov for "flat shells." A more systematic form is proposed for expressing the latter. The refined Vlasov equation is derived.

Chapter XII considers a series of problems associated with the utilization of the membrane theory. Special attention is devoted to criteria which establish the membrane state. A new feature here involves the separation of the problem into two parts: the applicability of the membrane solution as a particular solution of the moment theory and as a general solution. Conditions are presented which are sufficient to guarantee a positive answer to the first question. The second question is discussed. Singular cases are investigated in detail (the exception cases).

Chapter XIII presents a short asymptotic analysis of the relationships from the theory of shells. Basic attention is given to the simple edge effect. The problem of separating the state of stress into a basic state, (slowly varying) and into the edge effect is considered in detail. The new feature here involves the corresponding separation of the boundary conditions. In this case specific boundary conditions are formulated for the basic (degenerate) problem. It is shown that the edge effect is produced by the tangential deformation of the shell's edge (by the relative elongation of the boundary contour). The latter gives us a broader understanding of the shell edge operation and presents a simple mechanical explanation of the edge effect. The coefficients of compliance (rigidity) are obtained for a shell of general form with an arbitrary nonasymptotic edge. The basic types of stress states are briefly considered.

Chapter XIV considers thermoelastic stresses in the classical formulation. The temperature terms are introduced into the relationships from the complex version of the theory. The question of temperature fields which do not produce thermoelastic stresses is discussed.

Chapter XV introduces the basic relationships for circular toroidal shells. These permit a uniform examination of expansion tanks and pipes with a circular axis loaded symmetrically and antisymmetrically. Special consideration is given to the asymptotic method (the method of standard equations).

Chapter XVI considers several basic problems associated with toroidal expansion tanks carrying symmetric and antisymmetric loads.

Chapter XVII presents a general solution to the Saint-Venant problem for a bent circular thin-walled pipe subjected to edge loading of the general form.

The list of references supplements those presented in the first part and establishes new on[illegible]. For the convenience of the reader the sources listed in the first part of the book and referred to in the second part of the book are again presented.

In sorting the material and selecting a method of presentation, we must always consider the important situation that unnecessary generalization of presentation and the use of complex mathematical methods inevitably radically reduces the number of readers. In this connection the problem of utilizing tensor notation is particularly acute. On the one hand, the author is almost in agreement with the widespread opinion that tensor notation is a type of shorthand that we can "live without." On the other hand, we have to consider the fact that the domestic and particularly the foreign literature makes wide use of the tensor method of presentation. By renouncing this method we cut ourselves off from a large source of information. In addition, the frequently invariant nature of relationships makes the latter more transparent and serves as an auxiliary control in their correct formulation.

These considerations have led the author to a compromise solution. The basic contents of the second part is given in the conventional (nontensor) presentation and does not assume familiarity with Chapters VII and VIII. We recommend that the reader who does not wish to spend his time studying these chapters become familiar only with section 8 of Chapter VIII, which presents the relationships in terms of physical components.

We note, by the way, that in presenting the material the author has striven to make the groups of chapters independent of each other for easier reading, and has to reduce to a minimum the number of references, particularly for chapters in the first part.

Unfortunately, for various reasons, the following important problems of the linear theory have not been illuminated in the book: concentrated forces, stress concentrations near holes, calculation of composite shells (T-Joints, elbows, etc.), discontinuous solutions of the general form, structurally anisotropic shells, flat and irregular coverings, dynamic problems and certain other problems.

Furthermore, in order to establish the limits for applicability of the linear theory of thin shells and its role in the general theory of shells it is necessary, even if briefly, to consider questions associated with small nonlinearity (geometric and physical) and to consider the theory of shells of medium thickness.

The author hopes to return to the enumerated problems in the future.

In preparing the book for publication, the author was assisted by the coworkers of the Laboratory for the Theory of Shells at the Leningrad University: V. A. Shamina, V. I. Kruglyakova, V. V. Ponyatovskiy, V. M. Mal'kov, M. P. Yevseyeva, L. M. Kuznetsova, N. V. Smirnova.

The manuscript for Chapter VII was examined by Yu. F. Borisov, while the manuscript for Chapter IX was examined by S. G. Mikhlin. Unfortunately the author was able to take into account only part of the valuable remarks made by them. As in the first part of the book, a whole series of ideas and recommendations by V. V. Novozhilov was utilized.

The author expresses his deepest gratitude to all these persons.

CHAPTER VII. TENSOR NOTATION IN THE THEORY OF SHELLS. PRINCIPLES OF THE THEORY OF SURFACES

In the present chapter we present the minimum amount of information on /7 the principles of tensor analysis and theory of surfaces necessary for reading the special literature on the theory of shells. As a basis for this we have adopted the monograph of V. F. Kagan (ref. 17).

In selecting the material the author made an effort, first of all, to present (in a form not too abstract) that basic information from the theory of surfaces usually used in the theory of shells. In addition, in the last three sections certain relationships necessary in the subsequent chapters of the book were specially obtained.

We note that it is not necessary to become familiar with the present chapter and with the succeeding one in order to understand the basic contents of the book. These chapters can be omitted in the reading, if the relationships in terms of orthogonal coordinates, presented in section 8 of Chapter VIII, are accepted as true.

To the reader who wishes to achieve a more solid acquaintance with the theory of surfaces and with tensor analysis we recommend the textbooks (refs. 223 and 224), in addition to reference 17.

1. Contravariant and Covariant Vector Components

Let us refer the considered part of space to curvilinear coordinates of the general form θ^1, θ^2, θ^3. The position of an arbitrary point may be determined (fig. 1) by the radius vector

$$\vec{R}=\vec{R}(\theta^1, \theta^2, \theta^3). \tag{1.1}$$

We shall assume that all points in the assumed coordinate system are ordinary, i.e., that the basic coordinate vectors

$$\vec{R}_1, \vec{R}_2, \vec{R}_3 \quad \left(\vec{R}_i=\frac{\partial \vec{R}}{\partial \theta^i}\right). \tag{1.2}$$

are tangent to the coordinate lines θ^1, θ^2, θ^3 and do not lie in a single plane, so that the mixed scalar-vector product $(\vec{R}_1, \vec{R}_2, \vec{R}_3)$ is not equal to zero. Then an arbitrary vector may be represented as an expression /8

$$\vec{U}=U^1\vec{R}_1+U^2\vec{R}_2+U^3\vec{R}_3=U^\alpha\vec{R}_\alpha \tag{1.3}$$

(here and below the Greek symbol occurring twice indicates that summation is taken over this symbol). The quantities U^1, U^2, U^3 are called the contravariant components of the vector $\vec{U}$ in the assumed system of coordinates.

In addition to the base vectors (these according to our conditions do not all lie in the same plane) we introduce the reciprocal vectors $\vec{R}^1$, $\vec{R}^2$, $\vec{R}^3$, determining them from the equalities

$$\vec{R}^i\vec{R}_j=\delta^i_j;\quad \left(\delta^i_j=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}\right). \tag{1.4}$$

It is easy to verify that the relationships associated with the reciprocal and base vectors have the following form

$$\vec{R}^1=\frac{[\vec{R}_2,\vec{R}_3]}{(\vec{R}_1,\vec{R}_2,\vec{R}_3)};\quad \vec{R}^2=\frac{[\vec{R}_3,\vec{R}_1]}{(\vec{R}_1,\vec{R}_2,\vec{R}_3)};\quad \vec{R}^3=\frac{[\vec{R}_1,\vec{R}_2]}{(\vec{R}_1,\vec{R}_2,\vec{R}_3)} \tag{1.5}$$
$$\left((\vec{R}_1,\vec{R}_2,\vec{R}_3)=\vec{R}_1[\vec{R}_2,\vec{R}_3]\right).$$

By utilizing the reciprocal vectors we may represent any arbitrary vector $\vec{U}$ in the following form

$$\vec{U}=U_1\vec{R}^1+U_2\vec{R}^2+U_3\vec{R}^3=U_\alpha\vec{R}^\alpha. \tag{1.6}$$

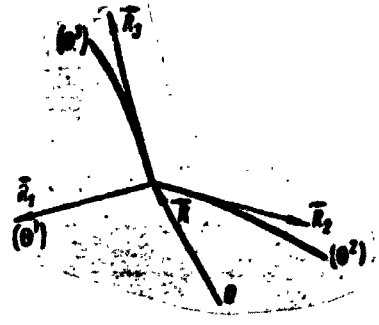

Figure 1

The quantities U_1, U_2, U_3 are called the covariant components of the vector $\vec{U}$.

By using equality (1.5) we can obtain the following two relationships

$$\vec{U}\vec{R}_i = U_\alpha(\vec{R}^\alpha\vec{R}_i) = U_\alpha\delta_i^\alpha = U_i; \tag{1.7}$$

$$\vec{U}\vec{R}^i = U^\alpha(\vec{R}_\alpha\vec{R}^i) = U^\alpha\delta_\alpha^i = U^i, \tag{1.8}$$

which shows that the covariant component of a vector is equal to the scalar product of the considered vector and the corresponding base coordinate vector, while a contravariant component is equal to the scalar product of the vector and the corresponding reciprocal vector.

In the case of Cartesian coordinates

$$\theta^1 = x^1;\ \theta^2 = x^2;\ \theta^3 = x^3;$$

$$\vec{R}_1 = \frac{\partial\vec{R}}{\partial x^1} = \vec{i}_x;\ \vec{R}_2 = \frac{\partial\vec{R}}{\partial x^2} = \vec{i}_y;\ \vec{R}_3 = \frac{\partial\vec{R}}{\partial x^3} = \vec{i}_z.$$

Here $\vec{i}_x$, $\vec{i}_y$, $\vec{i}_z$ are the constant coordinate orthogonal unit vectors. It /9
follows from equality (1.4) that

$$\vec{R}_1 = \vec{R}^1 = \vec{i}_x;\ \vec{R}_2 = \vec{R}^2 = \vec{i}_y;\ \vec{R}_3 = \vec{R}^3 = \vec{i}_z. \tag{1.9}$$

From this and from (1.7) and (1.8) it follows that in a Cartesian system of coordinates the covariant and contravariant components of a vector coincide with its orthogonal projections on the corresponding coordinate axes. Let us establish the law for the variation of the contravariant components of a vector when transforming from coordinates θ^1, θ^2, θ^3 to some other system $\theta^{1'}$, $\theta^{2'}$, $\theta^{3'}$. First of all, by introducing the designation

$$D_j^{i'} = \frac{\partial\theta^{i'}}{\partial\theta^j},\ D_{j'}^i = \frac{\partial\theta^i}{\partial\theta^{j'}}, \tag{1.10}$$

we obtain, remembering the rule for the differentiation of complicated functions,

$$\vec{R}_{i'} = \frac{\partial\vec{R}}{\partial\theta^{i'}} = \frac{\partial\vec{R}}{\partial\theta^1}\frac{\partial\theta^1}{\partial\theta^{i'}} + \frac{\partial\vec{R}}{\partial\theta^2}\frac{\partial\theta^2}{\partial\theta^{i'}} + \frac{\partial\vec{R}}{\partial\theta^3}\frac{\partial\theta^3}{\partial\theta^{i'}} = \vec{R}_\alpha D_{i'}^\alpha. \tag{1.11}$$

From this and from (1.7) we obtain the equalities

$$U_{i'} = \vec{U}\vec{R}_{i'} = \left(\vec{U}\vec{R}_{\alpha}\right) D^{\alpha}_{i'} = U_{\alpha} D^{\alpha}_{i'}, \tag{1.12}$$

which establish a rule for the transformation of covariant vector components when transforming to a new system of coordinates.

The reciprocal vectors are transformed by equations

$$\vec{R}^{j'} = \vec{R}^{\beta} D^{j'}_{\beta}. \tag{1.13}$$

This equality cannot be established in a manner analogous to (1.11), because the reciprocal vector are not directly associated with differentiation. However, we can pursue a different path and show that the vectors $\vec{R}^{1'}$, $\vec{R}^{2'}$, $\vec{R}^{3'}$ determined by means of (1.13) are reciprocal to $\vec{R}_{1'}$, $\vec{R}_{2'}$, $\vec{R}_{3'}$. Let us demonstrate this. Indeed, according to (1.13), (1.11) and (1.5)

$$\vec{R}^{i'}\vec{R}_{j'} = \left(\vec{R}^{\beta}\vec{R}_{\alpha}\right) D^{i'}_{\beta} D^{\alpha}_{j'} = \delta^{\beta}_{\alpha} D^{i'}_{\beta} D^{\alpha}_{j'} = D^{i'}_{\alpha} D^{\alpha}_{j'} = \delta^{i'}_{j'}, \tag{1.14}$$

because the terms corresponding to $\alpha \neq \beta$ are equal to zero, while

$$D^{i'}_{\alpha} D^{\alpha}_{j'} = \frac{\partial\theta^{i'}}{\partial\theta^{1}}\frac{\partial\theta^{1}}{\partial\theta^{j'}} + \frac{\partial\theta^{i'}}{\partial\theta^{2}}\frac{\partial\theta^{2}}{\partial\theta^{j'}} + \frac{\partial\theta^{i'}}{\partial\theta^{3}}\frac{\partial\theta^{3}}{\partial\theta^{j'}} = \frac{\partial\theta^{i'}}{\partial\theta^{j'}} = \delta^{i'}_{j'}. \tag{1.15}$$

It is now easy to obtain the equalities

$$U^{i'} = \vec{U}\vec{R}^{i'} = \left(\vec{U}\vec{R}^{\beta}\right) D^{i'}_{\beta} = U^{\beta} D^{i'}_{\beta}, \tag{1.16}$$

which establish the law for the transformation of contravariant vector components. The vector $\vec{U}$ is usually called contravariant or covariant, depending on the components used to assign it. This is not entirely correct, because in both cases we speak of the same geometric object--a vector, which is independent of the form in which it is assigned. However, for the sake of brevity we shall make use of this terminology, both with respect to vectors and with respect to tensors. /10

Equalities (1.12) and (1.16) are conditions which make it possible to consider a triplet of values U_1, U_2, U_3 (U^1, U^2, U^3) as the covariant

(contravariant) components of a vector. In other words, these equalities may be looked upon as an analytic determination of the vector.

Let us consider two important examples.

Let us assume that the point M (x^1, x^2, x^3) in space, which is referred to a Cartesian system of coordinates, proceeds to a neighboring point M_1 $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$. This displacement is characterized by the displacement vector

$$\vec{dR}\,(dx^1, dx^2, dx^3).$$

In transforming to the coordinates $\theta^1, \theta^2, \theta^3$ we obtain

$$\begin{gathered} d\theta^1 = \frac{\partial\theta^1}{\partial x^1}dx^1 + \frac{\partial\theta^1}{\partial x^2}dx^2 + \frac{\partial\theta^1}{\partial x^3}dx^3 = dx^\alpha D^1_\alpha; \\ d\theta^2 = dx^\alpha D^2_\alpha; \quad d\theta^3 = dx^\alpha D^3_\alpha \\ \left(D^i_j = \frac{\partial\theta^i}{\partial x^j}\right). \end{gathered} \tag{1.17}$$

Taking into account equality (1.16) we arrive at the conclusion that an infinitely small displacement of the point is expressed by a contravariant vector whose components in any coordinate system are the differentials of the coordinate. It is actually this situation which explains the odd, at first glance, use of the superscripts for the coordinates.

We now consider the scalar field $\varphi\ (\theta^1, \theta^2, \theta^3)$. A vector $\vec{\nabla}\varphi$, called the gradient of the field, may be compared with this field. As we know, its components in the Cartesian system of coordinates are equal to $\frac{\partial\varphi}{\partial x^1}, \frac{\partial\varphi}{\partial x^2}, \frac{\partial\varphi}{\partial x^3}$. Let us now consider the three quantities $\frac{\partial\varphi}{\partial\theta^1}, \frac{\partial\varphi}{\partial\theta^2}, \frac{\partial\varphi}{\partial\theta^3}$ in an arbitrary coordinate system. According to the rule of differentiation for a multiple function

$$\frac{\partial\varphi}{\partial\theta^i} = \frac{\partial\varphi}{\partial x^1}\frac{\partial x^1}{\partial\theta^i} + \frac{\partial\varphi}{\partial x^2}\frac{\partial x^2}{\partial\theta^i} + \frac{\partial\varphi}{\partial x^3}\frac{\partial x^3}{\partial\theta^i} = \frac{\partial\varphi}{\partial x^\alpha}D^\alpha_i.$$

From this we can see that in any coordinate system the derivatives $\frac{\partial\varphi}{\partial\theta^1}$, /11

$\frac{\partial\varphi}{\partial\theta^2}, \frac{\partial\varphi}{\partial\theta^3}$ determine a covariant vector which is called the gradient of a scalar field $\varphi\ (\varphi^1, \theta^2, \theta 3)$

$$\operatorname{grad}\varphi=\vec{\nabla}\varphi=\frac{\partial\varphi}{\partial\theta^{\alpha}}\vec{R}^{\alpha}. \qquad (1.18)$$

2. Tensors. Tensor Algebra

The concepts introduced in the preceding section concerning contravariant and covariant vector components permit far-reaching generalizations. Let us assume that some object (geometric, mechanical or of some other form) is characterized in each system of coordinates by nine quantities U_{ij} (i, j = 1, 2, 3).

It is a practice to call such an object an extensive variable of the second valence (rank), depending upon the number of variable signs (indices). The quantities U_{ij} are called components of the extensive variable. In transforming to another system of coordinates the components U_{ij}, generally speaking, vary in accordance with some law.

From the standpoint of application the most important are the extensive variables whose components vary according to the law

$$U_{i'j'}=U_{\alpha\beta}D^{\alpha}_{i'}D^{\beta}_{j'}, \qquad (2.1)$$

which in its structure resembles the law for the transformation of the covariant components of a vector (1.12). Such extensive variables are called doubly covariant tensors.

In exactly the same way we can introduce the concepts of covariant and mixed tensors. Thus extensive variables, whose components vary in accordance with the law

$$U^{i'j'}=U^{\alpha\beta}D^{i'}_{\alpha}D^{j'}_{\beta}, \qquad (2.2)$$

$$U^{i'}_{\cdot j'}=U^{\alpha}_{\cdot\beta}D^{i'}_{\alpha}D^{\beta}_{j'}, \qquad (2.3)$$

are, respectively, the doubly contravariant and mixed tensors.[1] It is easy to introduce a tensor of higher rank. As an example, an extensive variable for which

$$U^{i'j'}_{\cdot\cdot k'}=U^{\alpha\beta}_{\cdot\cdot\gamma}D^{i'}_{\alpha}D^{j'}_{\beta}D^{\gamma}_{k'}, \qquad (2.4)$$

[1] For nonsymmetric tensors (the determination of symmetry is presented below) the order in which the indices follow is important. Therefore dots are used for unlike indices. For symmetric tensors where the order of indices is not important the points are dropped.

is a tensor of the third rank, doubly contravariant and singly covariant. In the light of this determination the vector should be considered a tensor of the first rank.

Various geometric, static and other considerations lead to tensors. /12 However, we can also indicate a simple analytical approach to the construction of tensors. Let us assume, for example, that the following quantities are assigned in some coordinate system

$$U_{ij} = U_{ij}(\theta^1, \theta^2, \theta^3).$$

We assign to the considered object, in any other coordinate system, components given by equalities (2.1). In this way the equalities which establish the tensor nature of the object will be satisfied automatically, and we have constructed a doubly covariant tensor. In the same way we can construct tensors of other types and ranks.

As an example let us construct a tensor on the basis of the so-called Kronecker symbol, which is determined in some initial system of coordinates θ^1, θ^2, θ^3 by the equalities

$$U^i_{\cdot j} = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Let us construct the corresponding mixed tensor. In view of what has been stated, the unknown tensor will have the following components in the new system of coordinates

$$U^{i'}_{\cdot j'} = \delta^{\alpha}_{\beta} D^{i'}_{\alpha} D^{\beta}_{j'}.$$

However, when $\alpha \neq \beta$ the respective terms drop out. Therefore, according to (1.15)

$$U^{i'}_{\cdot j'} = D^{i'}_{\alpha} D^{\alpha}_{j'} = \delta^{i'}_{j'}.$$

The constructed mixed tensor of the second rank thus has the Kronecker symbols as its components in any system of coordinates.

Let us briefly consider some of the basic properties of tensors. First of all the following is valid.

THEOREM 1. If the components of several monotypic tensors[1] (for example, $U^{ij}_{..k}$, $V^{ij}_{..k}$, $W^{ij}_{..k}$) are multiplied, respectively, by the scalars u, v and w and then added, the resulting extensive variable

$$uU^{ij}_{..k}+vV^{ij}_{..k}+wW^{ij}_{..k}$$

will be a tensor of the same type.

The proof of this obvious proposition consists of using equality (2.4) to establish the law for the variation of the components associated with the constructed extensive variables during conversion to another system of coordinates.

The following obvious property is associated with the preceding.

THEOREM 2. If a tensor quantity is equal to zero in some coordinate system, then it is equal to zero in any other coordinate system. /13

Thus, if in some system of coordinates the tensor components, illustrating the preceding theorem

$$uU^{ij}_{..k}+vV^{ij}_{..k}+wW^{ij}_{..k}=0, \tag{2.5}$$

are equal to zero, then the above equality remains valid in any system of coordinates. The latter properties are frequently utilized to formulate various laws (geometric, physical, etc.) in an invariant form (the form which does not depend on the selection of the coordinate system). To achieve this, a relationship is established between the tensor components in some very simple system (most frequently in the Cartesian system). In view of the formulated theorem, this equality remains valid in any other system of coordinates, thus possessing an invariant nature.

In applications, tensors are frequently encountered whose components do not vary when some of the like indices are interchanged (upper or lower). Such tensors are called symmetric with respect to these indices. Thus, for example, a tensor for which $U_{ijk} = U_{jik}$ for any value i and j is symmetric with respect to this pair of signs. A tensor is called simply symmetric, if its components have only like indices and do not interchange their values during any interchange of indices.

A tensor which is skew-symmetric with respect to a given pair of indices is one whose components change sign when these indices are interchanged. For

[1] Tensors with identical quantity and index distribution are homogeneous.

example, a tensor for which $V_{ijk} = -V_{jik}$ is skew-symmetric with respect to the first two indices. A tensor is simply skew-symmetric if it contains only like indices and changes its sign during each interchange. In particular it follows from this determination that in a skew-symmetric tensor only the components with different indices differ from zero.

The formulated property of symmetry and skew-symmetry is invariant (it does not depend on the selection of the coordinates). This is easy to see if we apply theorem 2, respectively, to tensors

$$(U_{ijk} - U_{jik}) \quad \text{and} \quad (V_{ijk} + V_{jik}).$$

Let us assume, furthermore, that we have two tensors of any type and rank (for example U_{ij} and V^k). Constructing all products of each component of the first tensor and each component of the second tensor, we let

$$W_{ij}^{\cdot\cdot k} = U_{ij} V^k.$$

THEOREM 3. The resulting extensive variable ($W_{ij}^{\cdot\cdot k}$) is a tensor whose /14 rank is equal to the sum of the ranks of the multiplied tensors, while its type is determined by their types.

Indeed,

$$W_{i'j'}^{\cdot\cdot k'} = U_{i'j'} V^{k'} = U_{\alpha\beta} V^{\gamma} D_{i'}^{\alpha} D_{j'}^{\beta} D_{\gamma}^{k'} = W_{\alpha\beta}^{\cdot\cdot\gamma} D_{i'}^{\alpha} D_{j'}^{\beta} D_{\gamma}^{k'}.$$

The last operation is called the multiplication of two tensors. It is clear that it can be extended to any number of tensors.

The inverse operation of contraction in the certain sense corresponds to the operation of multiplication. Let us consider, for example, the tensor $U_{ij}^{\cdot\cdot k l}$. We isolate the components for which j = k and add them. The sums corresponding to the arbitrary values of the three (not summed) indices

$$V_i^{\cdot l} = U_{i1}^{\cdot\cdot 1 l} + U_{i2}^{\cdot\cdot 2 l} + U_{i3}^{\cdot\cdot 3 l} = U_{i\alpha}^{\cdot\cdot\alpha l},$$

form an extensive variable whose rank is reduced by two (in the considered case it is the second one).

THEOREM 4. The extensive variable which we obtained by contracting the tensor of the k-th rank with respect to like indices represents a tensor of the (k-2)-th rank in which the other indices retain their place.

The proof of this theorem is very simple. Indeed,

$$V_i^{i'} = U_{i\cdot\cdot\cdot\alpha}^{\cdot\cdot\cdot\alpha\,i'} = U_{\lambda\mu}^{\cdot\cdot\cdot\nu\beta} D_{i'}^{\lambda} \left(D_{\nu}^{\mu} \cdot D_{\alpha}^{\alpha'}\right) D_{\beta}^{i'} =$$
$$= \left(U_{\lambda\mu}^{\cdot\cdot\cdot\nu\beta}\, \delta_{\nu}^{\mu}\right) D_{i'}^{\lambda} \cdot D_{\beta}^{i'} = U_{\lambda\mu}^{\cdot\cdot\cdot\mu\beta} D_{i'}^{\lambda} \cdot D_{\beta}^{i'} = V_{\lambda}^{\beta} D_{i'}^{\lambda} \cdot D_{\beta}^{i'}.$$

The resulting equality shows that the constructed extensive variable is a tensor of the corresponding type (in the considered case it is a tensor of the second rank, covariant once and contravariant once). If the contracted tensor contains unlike indices, the operation of contraction may be continued. In particular, for the tensor considered above contraction leads to a tensor of zero rank--to an invariant scalar $V = V_{\cdot\alpha}^{\cdot\alpha}$ independent of the coordinate system.

The contraction of two tensors over given indices is understood to mean the contraction of the product of these tensors. Thus, for example, by contracting the tensors U_{ij} and U^k over the indices $i = k$ we obtain a tensor (covariant vector)

$$V_j = U_{\alpha j} U^{\alpha}. \qquad (2.6)$$

Thus the contraction of two tensors over any pair of signs again leads to a tensor.

In a certain sense the converse also takes place.

THEOREM 5. If the result of contracting an extensive variable with /15 any tensor of a definite type leads to a tensor, the considered extensive variable is the tensor of the corresponding type.

We shall present the proof of the theorem by using example (2.6). Let us assume that in this example V_j is a covariant vector (tensor of the first rank), while U^i is an arbitrary contravariant vector. We shall show that in this case the extensive variable U_{ij} must always be a covariant tensor of the second rank. As proof of this we use any coordinate system $(\theta^1, \theta^2, \theta^3)$ to construct the tensor V_{ij}, which coincides with the extensive variable U_{ij} in the initial coordinate system (see the beginning of the section). If we contract the resulting tensor with the contravariant vector U^j, we obtain the covariant vector

$$W_j = V_{\alpha j} U^{\alpha}. \tag{2.7}$$

From their construction the vectors V_j (2.6) and W_j coincide in the initial coordinate system. Then according to theorem 2 they coincide in any coordinate system. Therefore it follows from (2.6) and (2.7) that in any system of coordinates and for any U^k

$$(U_{\alpha j} - V_{\alpha j}) U^{\alpha} = 0.$$

The last equality can be satisfied only when $U_{ij} = V_{ij}$. Thus, U_{ij} is a tensor.

Theorem 5 presents a simple approach used rather frequently to determine the tensor nature of various extensive variables.

In concluding the section we shall establish a simple but frequently useful property of tensors. Let us assume that we have two tensors, one of which (for example U_{ij}) is symmetric, while the other (V^{ij}) is skew-symmetric. The result of contracting these tensors over all indices is equal to zero

$$U_{\alpha\beta} V^{\alpha\beta} = 0. \tag{2.8}$$

The proof is simple. Indeed, along with products $U_{ij} V^{ij}$ the term $U_{ji} V^{ji}$ is always present in the sum. However, since the first tensor is symmetric, while the second one is skew-symmetric

$$U_{ij} V^{ij} + U_{ji} V^{ji} = U_{ij} (V^{ij} + V^{ji}) = 0.$$

Since the entire sum may be represented as the sum of pairs of this form, the total contraction is equal to zero.

In the present section the components of the tensors were scalar quantities. However, everything we stated is valid if the components of the tensors are vector quantities. The quantities (1.11) and (1.13) show that the vector quantities $\vec{R}_i$ and $\vec{R}^i$ may be considered the respective components of the covariant and contravariant tensors of rank 1 with vector components. Then expansions (1.3) and (1.6) may be viewed as the contractions of two tensors of rank 1 with scalar and vector components. In this case contraction leads to an invariant vector. /16

What we have stated is sufficient to note the difference between the invariant vector and tensors with vector components.

3. Basic Metric and Discriminant Tensors

Let us consider the point M (θ^i) and the point M_1 $(\theta^i + d\theta^i)$ close to it. The vector which joins them is equal to

$$d\vec{R} = \vec{R}_\alpha d\theta^\alpha. \tag{3.1}$$

We compute its length

$$ds^2 = (d\vec{R})^2 = (\vec{R}_\alpha d\theta^\alpha)(\vec{R}_\beta d\theta^\beta) = \vec{R}_\alpha \vec{R}_\beta d\theta^\alpha d\theta^\beta. \tag{3.2}$$

Furthermore, we introduce the designations

$$g_{ij} = \vec{R}_i \vec{R}_j. \tag{3.3}$$

As we can see from the last equality the quantities g_{ij} determine the covariant tensor of the second rank. The same equality provides for the symmetric nature of the introduced tensor. We can also introduce the corresponding contravariant tensor

$$g^{ij} = \vec{R}^i \vec{R}^j. \tag{3.4}$$

Finally the mixed tensor according to (1.5) coincides with the Kronecker tensor

$$g^i_j = \vec{R}_i \vec{R}^j = \delta^i_j.$$

For reasons to be clarified later, the introduced tensor is called the basic metric space tensor (sometimes known as the fundamental tensor).

The introduced tensors make it easy to transform the contravariant components into mixed and covariant ones, and vice versa. Indeed, according to (1.7) and (1.8)

$$U_l=\vec{U}\vec{R}_l=\left(U^s\vec{R}_s\right)\vec{R}_l=U^s\left(\vec{R}_s\vec{R}_l\right)=U^s g_{sl};$$
$$U^l=\vec{U}\vec{R}^l=\left(U_s\vec{R}^s\right)\vec{R}^l=U_s\left(\vec{R}^s\vec{R}^l\right)=U_s g^{sl}. \tag{3.5}$$

It is interesting to note that the basic and reciprocal vectors are transformed into each other by means of the same equations. Indeed,

$$U_l=\vec{U}\vec{R}_l,\ U^l=\vec{U}\vec{R}^l. \tag{3.6}$$

On the other hand (see (3.5))

$$U^l=U_s g^{sl}=\vec{U}\vec{R}_s g^{sl}.$$

Subtracting $(3.6)_2$ from this equality we obtain

$$\left(\vec{R}^l-\vec{R}_s g^{sl}\right)\vec{U}=0. \tag{3.7}$$

Since the latter relationship must be satisfied for any vectors $\vec{U}$, it follows from it that /47

$$\vec{R}^l=\vec{R}_s g^{sl}. \tag{3.8}$$

In exactly the same way we obtain the inverse relationship

$$\vec{R}_l=\vec{R}^s g_{sl}. \tag{3.9}$$

We shall obtain an interesting corollary from the last equality. To do this we multiply the left and right sides by $\vec{R}^j$ and utilize (1.4) and (3.3)-(3.4) to find

$$g_{ls}g^{js}=\delta_l^j. \tag{3.10}$$

We introduce the determinant

$$|g_{ij}|=\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}=g, \tag{3.11}$$

which is called the discriminant of the basic metric tensor. Then equality (3.10) shows that the quantity g^{ij} is the reduced minor (algebraic supplement) of the element g_{ij} in the determinant $|g_{ij}|$ divided by g. Conversely, g_{ij} is the reduced minor in the determinant $|g^{ij}| = \bar{g}$ divided by $\bar{g}$. As we know (for example, ref. 40, Vol. III Part 1)

$$\bar{g} = \frac{1}{g}. \tag{3.12}$$

We note that according to (3.3) and (3.4) in the orthogonal system of coordinates

$$\begin{matrix} g_{12}=g_{13}=g_{23}=0, \\ g^{12}=g^{13}=g^{23}=0, \end{matrix} \text{ i.e. } \begin{matrix} g_{ij}=0 \\ g^{ij}=0 \end{matrix} \quad \text{when } i \neq j. \tag{3.13}$$

If, on the other hand, the orthogonal system is also a Cartesian system, then (see (1.9)).

$$\begin{matrix} g_{11}=g_{22}=g_{33}=1, \\ g^{11}=g^{22}=g^{33}=1, \end{matrix} \text{ i.e. } g^{ii}=g_{ii}=1. \tag{3.14}$$

We can see from equality (3.5) that by using tensors g_{ij} and g^{ij} we can transform the vector type by raising and lowering the indices. This approach is used with success for tensors of a higher rank. Thus, for example, by multiplying tensors U_{ij} and g^{kl} and contracting them we obtain mixed tensors

$$U_i^l = U_{ik}g^{kl}, \quad U^l_j = U_{kj}g^{kl}. \tag{3.15}$$

(The resulting tensors may differ from each other, if U_{ij} is nonsymmetric. /18 Therefore, points are used in (3.15) which show the order of the indices. In the case when U_{ij} is symmetric the order of the indices becomes insignificant and the points are dropped so that we write U_i^l, U_j^l.) Furthermore, by using the same approach a covariant tensor is transformed to a contravariant one

$$U^{ij} = U_{\alpha\beta} g^{\alpha i} g^{\beta j}. \tag{3.16}$$

The approach described above may be applied to tensors of arbitrary type and rank. In this case we have a property known as the inertia of contraction.

THEOREM 6. The result of contracting a tensor or a pair of tensors over two indices does not change when the indices of contraction are interchanged (dropping one of them and raising the other simultaneously).

We shall prove this proposition by considering an example in which two tensors of rank two are contracted

$$U_{i\alpha} V^{j\alpha}.$$

We have

$$U_{i\alpha} = U_i^{\beta} g_{\beta\alpha}, \quad V^{j\alpha} = V^j_{\cdot\gamma} g^{\gamma\alpha}.$$

From this according to (3.10) it follows that

$$U_{i\alpha} V^{j\alpha} = U_i^{\beta} V^j_{\cdot\gamma} \left(g_{\beta\alpha} g^{\gamma\alpha} \right) = U_i^{\beta} V^j_{\cdot\gamma} \delta^{\gamma}_{\beta} = U_i^{\beta} V^j_{\cdot\beta}.$$

Replacing the summation sign (β by α) we arrive at the relationship

$$U_{i\alpha} V^{j\alpha} = U_i^{\alpha} V^j_{\cdot\alpha}, \tag{3.17}$$

which proves the theorem. As an example showing the application of this theorem we shall consider the scalar product of two vectors $\vec{U}$ and $\vec{V}$

$$\vec{U} \cdot \vec{V} = U^{\alpha} \vec{R}_{\alpha} V^{\beta} \vec{R}_{\beta} = \left(\vec{R}_{\alpha} \vec{R}_{\beta} \right) U^{\alpha} V^{\beta} = g_{\alpha\beta} U^{\alpha} V^{\beta}.$$

(This equality, by the way, also provides for the tensor nature of the extensive variable g_{ij}, because its contraction with the arbitrary contravariant tensor $U^i V^j$ leads to an invariant scalar.) By utilizing theorem 6 we can now write the scalar product in the form

$$g^{\alpha}_{\beta}U_{\alpha}V^{\beta} = \delta^{\alpha}_{\beta}U_{\alpha}V^{\beta} = U_{\alpha}V^{\alpha}.$$

Raising and lowering the indices β we obtain a third form for the scalar product $g^{\alpha\beta}U_{\alpha}V_{\beta}$

Thus

$$\vec{U}\vec{V} = g_{\alpha\beta}U^{\alpha}V^{\beta} = U_{\alpha}V^{\alpha} = U^{\alpha}V_{\alpha} = g^{\alpha\beta}U_{\alpha}V_{\beta}. \tag{3.18}$$

In particular, the square of the length of vector $\vec{U}$ is determined by the following expressions /19

$$|\vec{U}|^2 = \vec{U}\vec{U} = g_{\alpha\beta}U^{\alpha}U^{\beta} = U_{\alpha}U^{\alpha} = g^{\alpha\beta}U_{\alpha}U_{\beta}. \tag{3.19}$$

Since

$$\vec{U}\vec{V} = |\vec{U}||\vec{V}|\cos(\vec{U}, \vec{V}),$$

the cosine of the angle between the two vectors $\vec{U}$ and $\vec{V}$ is computed by means of equation

$$\cos(\vec{U}, \vec{V}) = \frac{\vec{U}\vec{V}}{|\vec{U}||\vec{V}|} = \frac{g_{\alpha\beta}U^{\alpha}V^{\beta}}{\sqrt{g_{\alpha\beta}U^{\alpha}U^{\beta}}\sqrt{g_{\alpha\beta}V^{\alpha}V^{\beta}}}. \tag{3.20}$$

Let us return to the equality (3.2), which can now be represented in the form

$$ds^2 = g_{\alpha\beta}d\theta^{\alpha}d\theta^{\beta}. \tag{3.21}$$

It is stated that the last expression "determines the metrics of space." This underlines the situation that, if we have knowledge of the quantities $g_{\alpha\beta}$, which are called the components of the basic metric tensor, we can carry out all metric computations in space: determine the lengths of arcs, compute areas and volumes, find the angles between the lines.

Let us assume that the curve in space is assigned the parametric equation $\theta^i = \theta^i(\theta)$. Then, according to (3.21) the length of the arc element of the curve is equal to

$$ds = d\theta \sqrt{g_{\alpha\beta} \frac{d\theta^\alpha}{d\theta} \frac{d\theta^\beta}{d\theta}}, \qquad (3.22)$$

while the length of the finite part of the considered curve is determined by the integral

$$s = \int_{(\theta)_1}^{(\theta)_2} d\theta \sqrt{g_{\alpha\beta} \frac{d\theta^\alpha}{d\theta} \frac{d\theta^\beta}{d\theta}}. \qquad (3.23)$$

If we let $(d\theta^1, d\theta^2, d\theta 3)$ and $(\delta\theta^1, \delta\theta^2, \delta\theta 3)$ be the components of elements tangent to two intersecting curves, the angle between the curves χ according to (3.20) is determined from the relationship

$$\cos\chi = \frac{g_{\alpha\beta} d\theta^\alpha \delta\theta^\beta}{\sqrt{g_{\alpha\beta} d\theta^\alpha d\theta^\beta} \sqrt{g_{\alpha\beta} \delta\theta^\alpha \delta\theta^\beta}}. \qquad (3.24)$$

The resulting equalities make it possible to determine the geometric meaning of the components associated with the basic metric tensor. Thus, by considering the element of the first coordinate line $(d\theta^1, 0, 0)$, we find from (3.21)

$$ds_1 = \sqrt{g_{11}}\, d\theta^1. \qquad (3.25)_1$$

In the same way /20

$$ds_2 = \sqrt{g_{22}}\, d\theta^2, \; ds_3 = \sqrt{g_{33}}\, d\theta^3. \qquad (3.25)_{2,\,3}$$

From this we can see that the quantities $\sqrt{g_{ii}}$ are scale factors between the increment of the corresponding curvilinear coordinate and the resulting increment in the arc of the coordinate line.

Further, by computing the angle between the two coordinate lines $(d\theta^1, 0, 0)$ and $(0, \delta\theta^2, 0)$, we obtain from (3.24)

$$\cos\chi^{(3)} = \frac{g_{12}}{\sqrt{g_{11}} \sqrt{g_{22}}}. \qquad (3.26)_1$$

In the same way we find

$$\cos\chi^{(2)}=\frac{g_{13}}{\sqrt{g_{11}}\sqrt{g_{33}}},\quad \cos\chi^{(1)}=\frac{g_{23}}{\sqrt{g_{22}}\sqrt{g_{33}}}. \qquad (3.26)_{2,\ 3}$$

The components with different indices of the basic metric tensor therefore determine the cosines of the respective coordinate angles. In particular, when the coordinate angle is a right angle the corresponding components become equal to zero.

We further introduce the discriminant tensor and determine its covariant and contravariant components by means of equalities

$$\varepsilon_{ijk}=(\vec{R}_i,\ \vec{R}_j,\ \vec{R}_k); \qquad (3.27)$$

$$\varepsilon^{ijk}=(\vec{R}^i,\ \vec{R}^j,\ \vec{R}^k). \qquad (3.28)$$

By using equalities (1.11) and (1.13) it is easy to see that the quantities ε_{ijk} and ε^{ijk} are, indeed, the components of a tensor of rank 3. In view of the well-known property of scalar-vector product the resulting tensor is skew-symmetric over all indices. It follows from this that only those components are different from zero for which all three indices are different. In this case they either coincide with ε_{123} (ε^{123}) or differ from it in sign depending on whether the rearrangement which converts the indices i, j, k into 1, 2, 3 is even or odd.

Let us compute ε_{123}. From the equation well known in algebra (ref. 40), Vol. III, Part 1)

$$(\vec{R}_1,\ \vec{R}_2,\ \vec{R}_3)^2=\begin{vmatrix}\vec{R}_1\cdot\vec{R}_1 & \vec{R}_1\cdot\vec{R}_2 & \vec{R}_1\cdot\vec{R}_3\\ \vec{R}_2\cdot\vec{R}_1 & \vec{R}_2\cdot\vec{R}_2 & \vec{R}_2\cdot\vec{R}_3\\ \vec{R}_3\cdot\vec{R}_1 & \vec{R}_3\cdot\vec{R}_2 & \vec{R}_3\cdot\vec{R}_3\end{vmatrix}=\begin{vmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{vmatrix}=g. \qquad (3.29)$$

From this it follows that

$$\varepsilon_{123}=\sqrt{g}. \qquad (3.30)$$

In the same way we find

$$e^{123}=\sqrt{\bar{g}}=\frac{1}{\sqrt{g}}. \tag{3.31}$$

By using relationships (3.27), (3.28) and (1.5) it is easy to verify the following equalities which are frequently used /21

$$[\vec{R}_i, \vec{R}_j]=\varepsilon_{ijs}\vec{R}^s; \tag{3.32}$$

$$[\vec{R}^i, \vec{R}^j]=e^{ijs}\vec{R}_s. \tag{3.33}$$

The latter makes it particularly possible to compute the area of the coordinate element. Thus,

$$dS_1=|[\vec{R}_2 d\theta^2, \vec{R}_3 d\theta^3]|=|[\vec{R}_2, \vec{R}_3]|\,d\theta^2 d\theta^3=\varepsilon_{23s}|\vec{R}^s|\,d\theta^2 d\theta^3=$$
$$=\varepsilon_{231}|\vec{R}^1|\,d\theta^2 d\theta^3=\sqrt{g}\sqrt{g^{11}}\,d\theta^2 d\theta^3.$$

In the same way two analogous relationships are obtained, and as a result the area of the basic area element θ^i = const is computed by means of equation

$$dS_i=\sqrt{gg^{ii}}\,d\theta^j d\theta^k \quad (i\neq j\neq k). \tag{3.34}$$

The volume of the parallelopiped element, whose ribs are the elements of the tangential and coordinate lines, is equal to

$$d\Omega=(\vec{R}_1 d\theta^1, \vec{R}_2 d\theta^2, \vec{R}_3 d\theta^3)=(\vec{R}_1, \vec{R}_2, \vec{R}_3)\,d\theta^1 d\theta^2 d\theta^3=$$
$$=\sqrt{g}\,d\theta^1 d\theta^2 d\theta^3. \tag{3.35}$$

In concluding the section we obtain an equality which is necessary for considering the theory of stresses. Let us consider a tetrahedron element $OP_1P_2P_3$ (fig. 2) with ribs $\vec{R}_1 d\theta^1$, $\vec{R}_2 d\theta^2$, $\vec{R}_3 d\theta^3$. It is easy to see that

$$\overrightarrow{P_1P_2}=\vec{R}_2 d\theta^2-\vec{R}_1 d\theta^1, \quad \overrightarrow{P_1P_3}=\vec{R}_3 d\theta^3-\vec{R}_1 d\theta^1.$$

Furthermore,

$$[\overrightarrow{P_1P_2},\ \overrightarrow{P_1P_3}]=[\vec{R}_2d\theta^2-\vec{R}_1d\theta^1,\ \vec{R}_3d\theta^3-\vec{R}_1d\theta^1]=$$
$$=[\vec{R}_2,\ \vec{R}_3]\,d\theta^2d\theta^3+[\vec{R}_3,\ \vec{R}_1]\,d\theta^3d\theta^1+[\vec{R}_1,\ \vec{R}_2]\,d\theta^1d\theta^2.$$

Each term in this equality represents a product of the area corresponding to the face of the tetrahedron and the unit vector normal to it. Therefore, by letting $\vec{\nu}$ be the unit vector of the normal to the face $P_1P_2P_3$ we obtain

$$\vec{\nu}dS_\nu=\left(\frac{\vec{R}^1}{\sqrt{g^{11}}}\right)dS_1+\left(\frac{\vec{R}^2}{\sqrt{g^{22}}}\right)dS_2+\left(\frac{\vec{R}^3}{\sqrt{g^{33}}}\right)dS_3, \tag{3.36}$$

from which it follows that /22

$$\frac{1}{\sqrt{g^{ii}}}\frac{dS_i}{dS_\nu}=\nu_i. \tag{3.37}$$

4. Covariant Differentiation. Principles of Tensor Analysis

In the construction of tensor analysis we have to take into account the situation that the coordinate vectors are variable quantities when we have non-Cartesian coordinate systems. Let

$$\vec{U}=U^\alpha\vec{R}_\alpha. \tag{4.1}$$

We have

$$\frac{\partial\vec{U}}{\partial\theta^j}=\frac{\partial U^\alpha}{\partial\theta^j}\vec{R}_\alpha+U^\alpha\vec{R}_{\alpha j}\qquad\left(\vec{R}_{ij}=\frac{\partial^2\vec{R}}{\partial\theta^i\partial\theta^j}\right). \tag{4.2}$$

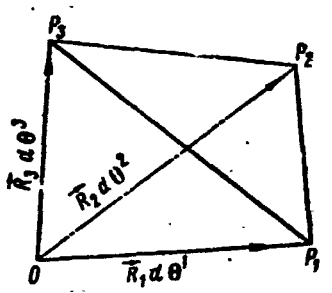

Figure 2

Let us resolve the vector $\vec{R}_{ij}$ in terms of the base vectors. As we have established in section 1, its components along these directions are equal to the scalar products of the vector and the corresponding reciprocal vectors. Therefore, by introducing the designations

$$G_{ij}^{k}=\vec{R}_{ij}\cdot\vec{R}^{k}, \tag{4.3}$$

we have

$$\vec{R}_{\alpha j}=G_{\alpha j}^{\beta}\vec{R}_{\beta}. \tag{4.4}$$

Substituting the resulting expression into (4.2) and changing the summation signs we find

$$\frac{\partial\vec{U}}{\partial\theta^{j}}=\left(\frac{\partial U^{\alpha}}{\partial\theta^{j}}+G_{\beta j}^{\alpha}U^{\beta}\right)\vec{R}_{\alpha}. \tag{4.5}$$

By using the designation

$$U_{ij}^{i}=\frac{\partial U^{i}}{\partial\theta^{j}}+G_{\beta j}^{i}U^{\beta} \tag{4.6}$$

the last equality can be rewritten in the form

$$\frac{\partial\vec{U}}{\partial\theta^{j}}=U_{ij}^{\alpha}\vec{R}_{\alpha}. \tag{4.7}$$

It is easy to obtain analogous equations for the covariant vector. Indeed, let

$$\vec{U}=U_{\alpha}\vec{R}^{\alpha}. \tag{4.8}$$

Then

$$\frac{\partial\vec{U}}{\partial\theta^{j}}=\frac{\partial U_{\alpha}}{\partial\theta^{j}}\vec{R}^{\alpha}+U_{\alpha}\frac{\partial\vec{R}^{\alpha}}{\partial\theta^{j}}. \tag{4.9}$$

In order to resolve the vector $\frac{\partial \vec{R}^{\alpha}}{\partial \theta^{j}}$ along the reciprocal vectors, it is /23
necessary to compute its scalar product with the base vectors. However, according to (1.4) and (4.3)

$$\frac{\partial \vec{R}^{k}}{\partial \theta^{j}}\vec{R}_{i}=\frac{\partial (\vec{R}^{k}\vec{R}_{i})}{\partial \theta^{j}}-\vec{R}^{k}\vec{R}_{ij}=-\vec{R}_{ij}\vec{R}^{k}=-G^{k}_{ij}, \tag{4.10}$$

therefore

$$\frac{\partial \vec{R}^{\alpha}}{\partial \theta^{j}}=-G^{\alpha}_{\beta j}\vec{R}^{\beta}. \tag{4.11}$$

Substituting this expression into (4.9) and changing the summation signs, we obtain

$$\frac{\partial \vec{U}}{\partial \theta^{j}}=\left(\frac{\partial U_{\alpha}}{\partial \theta^{j}}-G^{\beta}_{\alpha j}U_{\beta}\right)\vec{R}^{\alpha}. \tag{4.12}$$

By introducing the designation

$$U_{i|j}=\frac{\partial U_{i}}{\partial \theta^{j}}-G^{\beta}_{ij}U_{\beta}, \tag{4.13}$$

we write the preceding equality in the form

$$\frac{\partial \vec{U}}{\partial \theta^{j}}=U_{\alpha|j}\vec{R}^{\alpha}. \tag{4.14}$$

Thus, in order to differentiate a vector assigned in terms of the general coordinates, it is necessary to compute the quantities G^{k}_{ij} which are called Christoffels of the second kind. However, it is more convenient to use the Christoffels of the first kind for the calculation

$$G_{h,ij}=\vec{R}_{h}\vec{R}_{ij}. \tag{4.15}$$

It is easy to establish a relationship between the two quantities. Indeed,

$$\vec{R}_{ij}\vec{R}^{l}=\vec{R}_{ij}(g^{l\alpha}\vec{R}_{\alpha})=g^{l\alpha}\vec{R}_{ij}\vec{R}_{\alpha};$$
$$\vec{R}_{ij}\vec{R}_{k}=\vec{R}_{ij}(g_{\alpha k}\vec{R}^{\alpha})=g_{\alpha k}\vec{R}_{ij}\vec{R}^{\alpha}.$$

From this we obtain a relationship between Christoffels of the first and second kind

$$G_{ij}^{l}=g^{l\alpha}G_{\alpha,ij}; \tag{4.16}$$
$$G_{k,ij}=g_{\alpha k}G_{ij}^{\alpha}. \tag{4.17}$$

Christoffels of the first kind are computed rather simply by using the components of the basic metric tensor. Indeed, by differentiating equality (3.3) with respect to θ^h we obtain $\vec{R}_i\vec{R}_{jh} + \vec{R}_j\vec{R}_{ih} = \dfrac{\partial g_{ij}}{\partial\theta^h}$.

In the same way we obtain two analogous relationships /24

$$\vec{R}_j\vec{R}_{hl}+\vec{R}_h\vec{R}_{jl}=\frac{\partial g_{jh}}{\partial\theta^l},$$
$$\vec{R}_h\vec{R}_{lj}+\vec{R}_l\vec{R}_{hj}=\frac{\partial g_{hl}}{\partial\theta^j}.$$

Subtracting the first expression from the sum consisting of the second and the third and taking into account the symmetric property of the quantities $\vec{R}_{ij}$, $\vec{R}_{ih}$, $\vec{R}_{jh}$ we obtain the following expression for $\vec{R}_h\vec{R}_{ij} = G_{h,ij}$

$$G_{h,ij}=\frac{1}{2}\left(\frac{\partial g_{jh}}{\partial\theta^i}+\frac{\partial g_{hi}}{\partial\theta^j}-\frac{\partial g_{ij}}{\partial\theta^h}\right). \tag{4.18}$$

To begin with, the symmetric property of the Christoffels with respect to the indices i and j follows from this equation and from (4.16).

Let us return to equality (4.13). It determines the so-called covariant derivative of the covariant vector. The variability of the base vectors is associated with the situation that the components of the vector derivative in the generalized coordinate system cease to be equal derivatives of the vector components. A comparison of equalities (4.8) and (4.14), however, shows that the covariant derivative does have this property. Furthermore, the covariant derivative is clearly of a tensor nature. This is easy to see if by using equality (4.14) we note that its contraction with the reciprocal vector leads to a covariant vector. Then a slightly modified theorem 5 (it should be formulated!) shows that the covariant derivative of a covariant vector represents

a doubly covariant tensor. In the same way equality (4.7) shows that the covariant derivative of a contravariant vector leads to a mixed tensor of rank two.

This also explains the great deal of attention devoted in tensor analysis to covariant differentiation.

The following property of a covariant derivative is obvious (A and B are constants)

$$(AU_i+BV_i)_{ij}=AU_{ij}+BV_{ij}. \tag{4.19}$$

Furthermore let φ be some invariant quantity. Then

$$(\varphi U_i)_{ij}=\frac{\partial \varphi U_i}{\partial \theta^j}-G_{ij}^{\beta}\varphi U_{\beta}=\varphi\left(\frac{\partial U_i}{\partial \theta^j}-G_{ij}^{\beta}U_{\beta}\right)+\frac{\partial \varphi}{\partial \theta^j}U_i.$$

If we agree that the covariant derivative of an invariant (scalar or vector) is its partial derivative with respect to the corresponding coordinate

$$\varphi_{ij}=\frac{\partial \varphi}{\partial \theta^j}, \tag{4.20}$$

then

$$(\varphi U_i)_{ij}=\varphi U_{ij}+\varphi_{ij}U_i, \tag{4.21}$$

i.e., the covariant derivative of the product of a vector and an invariant /25 is constructed using the same law as in the case of the ordinary derivative of a product of two functions.

In the preceding discussions it did not matter whether the components of the tensor were scalar or vector quantities. Only the law which governed their transformation when transforming to a new system of coordinates was important. Therefore the basic and reciprocal coordinate vectors, in view of (1.11) and (1.13), should be considered as tensors of rank one, which are respectively covariant and contravariant. As we have already stated, such tensors are called tensors with vector components. There are tensors with vector components with a more complex structure. The theorems formulated above as well as the relationships derived above are fully applicable to these.

In particular, the quantity $\vec{R}_{i/j}$ determines a doubly covariant tensor with vector components. In the Cartesian system of coordinates, as we have already stated, all components of a basic metric tensor are constants. From equalities (4.18), (4.16) and (4.13) it follows that all Christoffels become equal to zero, and the covariant derivative coincides with the ordinary partial derivative. Furthermore, since the coordinate vectors are constant in the Cartesian system of coordinates, the introduced quantities become equal to zero. In view of their tensor nature they are also equal to zero in any coordinate system, i.e.,

$$\vec{R}_{ij/l}=0. \tag{4.22}$$

Similarly

$$\vec{R}^{l}_{ij}=0, \tag{4.23}$$

i.e., during the covariant differentiation the coordinate vectors should be looked upon as constants. The same conclusion can be reached by comparing equations (4.1) with (4.7) and (4.8) with (4.14).

By drawing an analogy with the differential of invariant functions the absolute differential of a vector (tensor of rank 1) is understood to be a vector having the component

$$U^{i}_{/}=U^{i}_{/\alpha}d\theta^{\alpha}=U^{i}_{/1}d\theta^{1}+U^{i}_{/2}d\theta^{2}+U^{i}_{/3}d\theta^{3}. \tag{4.24}$$

The vector itself is equal to

$$\vec{U}_{/}=U^{i}_{/}\vec{R}_{i}=U^{i}_{/\alpha}d\theta^{\alpha}\vec{R}_{i}. \tag{4.25}$$

In applications a derivative of a vector in a specific direction is frequently used. The unit vector along this direction is equal to /26

$$\vec{t}=\frac{d\vec{R}}{ds}=\frac{\partial\vec{R}}{\partial\theta^{\alpha}}\frac{d\theta^{\alpha}}{ds}=\frac{d\theta^{\alpha}}{ds}\vec{R}_{\alpha}.$$

From this we obtain an expression for its contravariant components

$$t^l = \frac{d\theta^l}{ds} \tag{4.26}$$

(s is the arc length along some curve in a given direction).

Let us contract this vector with a mixed tensor (4.6)

$$U^l_{|\alpha} t^\alpha = \frac{\partial U^l}{\partial \theta^\alpha} \frac{\partial \theta^\alpha}{ds} + G^l_{\beta\alpha} U^\beta \frac{d\theta^\alpha}{ds} = \frac{dU^l}{ds} + G^l_{\beta\alpha} U^\beta t^\alpha. \tag{4.27}$$

In the Cartesian system of coordinates the Christoffels become equal to zero, while the quantities $\frac{dU^l}{ds}$ represent the components of the vector $\frac{d\vec{U}}{ds} = \frac{d\vec{U}}{dx^\alpha} t^\alpha$ -- derivative of the vector taken in the given direction. In view of its tensor nature, expressions (4.27) determine its components in the general system of coordinates.

Let us introduce the concept of the covariant derivative of a tensor by considering a specific type tensor $W_{ij} = U_i V_j$, formed by the multiplication of two covariant vectors U_i and V_j.

In this case it is natural to require that the law for the differentiation of a product remain valid, i.e.

$$W_{ij|k} = V_j U_{i|k} + U_i V_{j|k}. \tag{4.28}$$

Utilizing the law for the covariant differentiation of a vector (4.13), we find

$$W_{ij|k} = V_j \left(\frac{dU_i}{d\theta^k} - G^\alpha_{ik} U_\alpha \right) + U_i \left(\frac{dV_j}{d\theta^k} - G^\alpha_{jk} V_\alpha \right) =$$
$$= \left(U_i \frac{\partial V_j}{\partial \theta^k} + \frac{\partial U_i}{\partial \theta^k} V_j \right) - G^\alpha_{ik} U_\alpha V_j - G^\alpha_{jk} U_i V_\alpha.$$

Thus

$$W_{ij|k} = \frac{dW_{ij}}{d\theta^k} - G^\alpha_{ik} W_{\alpha j} - G^\alpha_{jk} W_{i\alpha}. \tag{4.29}$$

We shall use this equation to determine the covariant derivative of an arbitrary covariant tensor of rank two. In the same way we establish the equalities

$$W^{i}_{\cdot j|k}=\frac{\partial W^{i}_{\cdot j}}{d\theta^{k}}+G^{i}_{\alpha k}W^{\alpha}_{\cdot j}-G^{\alpha}_{jk}W^{i}_{\cdot\alpha}; \tag{4.30}$$

$$W^{ij}_{|k}=\frac{\partial W^{ij}}{\partial\theta^{k}}+G^{i}_{\alpha k}W^{\alpha j}+G^{j}_{\alpha k}W^{i\alpha}. \tag{4.31}$$

From these relationships we can easily establish the law for obtaining /27 the covariant derivative of an arbitrary tensor. Having adopted it, it is not difficult, as an example, to write

$$W^{ij}_{\cdot\cdot k|k}=\frac{\partial W^{ij}_{\cdot\cdot k}}{\partial\theta^{k}}+G^{i}_{\alpha k}W^{\alpha j}_{\cdot\cdot k}+G^{j}_{\alpha k}W^{i\alpha}_{\cdot\cdot k}-G^{\alpha}_{kk}W^{ij}_{\cdot\cdot\alpha}. \tag{4.32}$$

An interesting property is possessed by the components of a metric and a discriminant tensor. For these

$$\begin{aligned} g_{ij|k}=g^{i}_{j|k}=g^{ij}_{|k}=0;\\ \varepsilon_{ijk|k}=\ldots=\varepsilon^{ijk}_{|k}=0. \end{aligned} \tag{4.33}$$

The validity of these follows from the fact that being components of tensors, the quantities which are in the left sides of the equalities become equal to zero in the Cartesian system of coordinates. Thus, during covariant differentiation the components of a metric and a discriminant tensor may be considered as constants. Utilizing this fact we can easily obtain a whole series of useful relationships. For example

$$V_{j|k}=(g_{\alpha j}V^{\alpha})_{|k}=g_{\alpha j}V^{\alpha}_{|k}. \tag{4.34}$$

By carrying out a covariant differentiation of a vector we obtain a tensor of rank two. The latter may be subjected to covariant differentiation once again. In this way we arrive at the concept of the second covariant derivative. The question arises as to whether the value of the second covariant derivative depends on the order of differentiation. The answer to this question is given by the Ricci identity, which we present without proof (ref. 17)

$$U_{i|jk}-U_{i|kj}=R^{\cdot\cdot\cdot\alpha}_{kj,i}U_{\alpha}; \tag{4.35}$$

since the vector U_i is arbitrary, the extensive variable $R^{\cdot\cdot\cdot h}_{kj,i}$ is a mixed tensor of rank four. Dropping the upper sign in this tensor we obtain the tensor, which is four times covariant

$$R_{kj,\,ih}=g_{h\alpha}R^{\alpha}_{kj,\,i}\ (R^{h}_{kj,\,i}=g^{h\alpha}R_{kj,\,i\alpha}). \tag{4.36}$$

It is usually called the Rieman-Christoffel tensor. It can be shown (ref. 17) that it is associated with the components of the basic metric tensor by the following relationships

$$\begin{aligned} R_{jk,\,ih}=&\frac{1}{2}\left\{\frac{\partial^2 g_{jh}}{\partial\theta^i\partial\theta^k}-\frac{\partial^2 g_{ij}}{\partial\theta^h\partial\theta^k}-\frac{\partial^2 g_{kh}}{\partial\theta^i\partial\theta^j}+\frac{\partial^2 g_{ik}}{\partial\theta^h\partial\theta^j}\right\}+\\ &+g^{\alpha\beta}\left(G_{\alpha,\,hj}G_{\beta,\,ik}-G_{\alpha,\,hk}G_{\beta,\,ij}\right). \end{aligned} \tag{4.37}$$

From these relationships the following properties of the introduced tensor can be established

$$\begin{aligned} R_{jk,\,ih}&=-R_{kj,\,ih};\\ R_{jk,\,hi}&=-R_{jk,\,ih};\\ R_{jk,\,ih}&=R_{ih,\,jk}, \end{aligned} \tag{4.38}$$

and these show that only six of the components are substantially different /28

$$R_{31,\,31};\ R_{32,\,32};\ R_{12,\,12};\ R_{31,\,32};\ R_{32,\,12};\ R_{31,\,12}. \tag{4.39}$$

The following relationships (ref. 17) are applicable to tensors of rank two

$$\begin{aligned} U_{ij|kl}-U_{ij|lk}&=R^{\alpha}_{lk,\,i}U_{\alpha j}+R^{\alpha}_{lk,\,j}U_{i\alpha};\\ U^{ij}_{|kl}-U^{ij}_{|lk}&=-R^{i}_{lk,\,\alpha}U^{\alpha j}-R^{j}_{lk,\,\alpha}U^{i\alpha}. \end{aligned} \tag{4.40}$$

We can see from relationships (4.37) that in the Cartesian system of coordinates all components of the Rieman-Christoffel tensor are equal to zero. Then, according to Theorem 2, they are equal to zero in any other coordinate system. In this case it follows from (4.35) and (4.40) that

$$\begin{aligned} U_{ij|kl}&=U_{ij|kj};\\ U_{ij|kl}&=U_{ij|lk}.\\ &\cdots\cdots \end{aligned} \tag{4.41}$$

Thus, during the covariant differentiation of arbitrary tensors the order of differentiation may be changed.

In concluding this section we consider the skew-symmetric tensor

$$V_{ij}=V_{i|j}-V_{j|i}, \tag{4.42}$$

in which V_i are the components of some vector $\vec{V}$. Replacing the covariant derivatives by their expressions (4.13), we find

$$V_{ij}=\frac{\partial V_i}{\partial\theta^j}-\frac{\partial V_j}{\partial\theta^i}. \tag{4.43}$$

Contracting it with the contravariant discriminant tensor, we obtain a vector called the rotation of vector $\vec{V}$

$$\left(\operatorname{rot}\vec{V}\right)^i=\varepsilon^{i\alpha\beta}V_{\alpha\beta}=\varepsilon^{i\alpha\beta}\left(\frac{\partial V_\alpha}{\partial\theta^\beta}-\frac{\partial V_\beta}{\partial\theta^\alpha}\right). \tag{4.44}$$

5. Coordinates Normally Fixed With Respect to the Surface. Shell Geometry

A special coordinate system fixed with respect to the middle surface of the shell is usually used in the theory of shells. In this system the radius vector of a point is equal to

$$\vec{R}=\vec{r}(\alpha^1,\ \alpha^2)+\zeta\vec{n}, \tag{5.1}$$

where $\vec{r}$ is the radius vector of the orthogonal projection of the considered point on the middle surface; ζ is the distance from the point to the middle surface; $\vec{n}$ is the unit normal vector; α^1, α^2 are the curvilinear coordinates of the middle surface. All relationships may be obtained in the selected system of coordinates by replacing the radius vector in the equalities obtained above with expression (5.1) and assuming that /29

$$\theta^1=\alpha^1;\ \theta^2=\alpha^2;\ \theta^3=\zeta. \tag{5.2}$$

We note that below we shall consider that all indices assume the values 1 and 2 (unless otherwise stated). The indices corresponding to the third coordinate ζ will be shown in explicit form.

First of all, from equality (5.1) we find

$$\begin{aligned}&\vec{R}_1=\vec{r}_1+\zeta\vec{n}_1;\\&\vec{R}_2=\vec{r}_2+\zeta\vec{n}_2;\\&\vec{R}_3=\vec{n};\end{aligned}\qquad\left(\vec{r}_i=\frac{\partial\vec{r}}{\partial\alpha^i},\ \vec{n}_i=\frac{\partial\vec{n}}{\partial\alpha^i}\right)\tag{5.3}$$

On the middle surface ($\zeta = 0$)

$$\vec{R}_1=\vec{r}_1,\ \vec{R}_2=\vec{r}_2,\ \vec{R}_3=\vec{n}.\tag{5.4}$$

Further, by taking into account the obvious equalities (since the vectors $\vec{r}_i$ and $\vec{n}$ are orthogonal)

$$\begin{aligned}&\vec{r}_i\vec{n}_j=\frac{\partial(\vec{r}_i\vec{n})}{\partial\alpha^j}-\vec{n}\vec{r}_{ij}=-\vec{n}\vec{r}_{ij};\\&\vec{r}_j\vec{n}_i=\frac{\partial(\vec{r}_j\vec{n})}{\partial\alpha^i}-\vec{n}\vec{r}_{ji}=-\vec{n}\vec{r}_{ij},\end{aligned}\tag{5.5}$$

we introduce the quantities

$$\begin{aligned}&a_{ij}=\vec{r}_i\vec{r}_j;\\&b_{ij}=-\vec{r}_i\vec{n}_j=\vec{n}\vec{r}_{ij}=-\vec{r}_j\vec{n}_i;\\&\nu_{ij}=\vec{n}_i\vec{n}_j.\end{aligned}\tag{5.6}$$

As we can easily see, all of these are components of the doubly covariant symmetric tensors assigned on the middle surface.

Let us now compute the components of the basic metric tensor. We obtain

$$\begin{aligned}&g_{ij}=R_iR_j=a_{ij}-2\zeta b_{ij}+\zeta^2\nu_{ij};\\&g_{13}=g_{23}=0;\ g_{33}=1.\end{aligned}\tag{5.7}$$

In section 3 it was stated that the quantities g^{ij} are algebraic additions in the determinant g, which in our case take the form

$$g=\begin{vmatrix}g_{11}&g_{12}&0\\g_{12}&g_{22}&0\\0&0&1\end{vmatrix}.$$

Therefore

$$g^{11}=\frac{g_{22}}{g};\ g^{12}=g^{21}=-\frac{g_{12}}{g};\ g^{22}=\frac{g_{11}}{g};\ g^{13}=g^{23}=0;\ g^{33}=1. \tag{5.8}$$

Thus, for the shell

$$ds^2=(a_{\alpha\beta}-2\zeta b_{\alpha\beta}+\zeta^2\nu_{\alpha\beta})\,d\alpha^\alpha d\alpha^\beta+d\zeta^2, \tag{5.9}$$

we also have, in accordance with (3.34)

$$\begin{aligned} dS_1&=\sqrt{g}g^{11}d\alpha^2d\zeta=\sqrt{g_{22}}d\alpha^2d\zeta;\\ dS_2&=\sqrt{g_{11}}d\alpha^1d\zeta;\qquad dS_3=\sqrt{g}d\alpha^1d\alpha^2. \end{aligned} \tag{5.10}$$

It follows from (5.7) that

$$a_{ij}=g_{ij}|_{\zeta=0}. \tag{5.11}$$

The quantities a_{ij} are therefore components of the basic metric tensor of /30
the middle surface. The last equality makes it possible to utilize the relationships obtained for g_{ij} in the preceding section. Assuming that $\zeta = 0$ in equality (3.10) we obtain

$$a_{ik}\cdot a^{kj}=a_i^j=\delta_i^j. \tag{5.12}$$

It follows from this that the components a^{ij} are the reduced minors of the components a_{ij} in the determinant

$$a=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-(a_{12})^2. \tag{5.13}$$

From this as well as from (5.8), it follows that

$$a^{11}=\frac{a_{22}}{a};\ a^{12}=a^{21}=-\frac{a_{12}}{a};\ a^{22}=\frac{a_{11}}{a}. \tag{5.14}$$

The following relationships also take place

$$ds_k = \sqrt{a_{kk}}da^k = \sqrt{aa^{ii}}da^k;\ ds_3 = d\zeta\ (i \neq k); \tag{5.15}$$

$$\cos\chi = (\cos\chi^{(3)})|_{\zeta=0} = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}}; \tag{5.16}$$

$$dS = (dS_3)|_{\zeta=0} = \sqrt{a}da^1da^2, \tag{5.17}$$

where ds_k is now the arc element of the middle surface coordinate line; χ is the angle between the coordinate line; dS is the area of an element on the middle surface.

We introduce further a discriminant tensor on the surface and determine it from the equalities

$$\begin{aligned} c_{ij} &= \varepsilon_{ij3}|_{\zeta=0} = \{\vec{r}_i, \vec{r}_j, \vec{n}\}; \\ c^{ij} &= \varepsilon^{ij3}|_{\zeta=0} = \{\vec{r}^i, \vec{r}^j, \vec{n}\}. \end{aligned} \tag{5.18}$$

From these it is easy to see that the quantities c_{ij}, c^{ij} determine the skew-symmetric tensor of rank two. In this connection, since $g|_{\zeta=0} = a$, it follows from (3.30) and (3.31) that

$$\begin{gathered} c_{12} = \sqrt{a};\ c_{21} = -\sqrt{a};\ \ c^{12} = \frac{1}{\sqrt{a}};\ c^{21} = -\frac{1}{\sqrt{a}}; \\ c_{11} = c_{22} = c^{11} = c^{22} = 0. \end{gathered} \tag{5.19}$$

By direct verification we can establish the validity of the relationships

$$\begin{aligned} c^{ij}c_{kl} &= \delta^i_k\delta^j_l - \delta^i_l\delta^j_k; \\ c^{ij}c^{kl} &= a^{ik}a^{jl} - a^{il}a^{jk}; \\ c_{ij}c_{kl} &= a_{ik}a_{jl} - a_{il}a_{jk}. \end{aligned} \tag{5.20}$$

(Since the tensors which are equated are skew-symmetric only, the equality /31 $c^{12}c_{12} = \delta^1_1\delta^2_2 - \delta^1_2\delta^2_1$ depends on all signs.) By carrying out a successive contraction we obtain two other useful equalities

$$c^{ik}c_{ik} = \delta^i_i;\ c^{\beta\alpha}c_{\beta\alpha} = 2. \tag{5.21}$$

A comparison of relationships (5.19) with (3.30)-(3.31) shows that

$$\varepsilon_{ijk}=\sqrt{\frac{g}{a}}c_{ij};\ \varepsilon^{ijk}=\sqrt{\frac{a}{g}}c^{ij}. \tag{5.22}$$

Finally, by means of (5.18) we find from (3.32) and (3.33)

$$\begin{aligned}&[\vec{r}_i,\ \vec{r}_j]=c_{ij}\vec{n};\ [\vec{r}^i,\ \vec{r}^j]=c^{ij}\vec{n};\\ &[\vec{n},\ \vec{r}_j]=c_{j\alpha}\vec{r}^{\alpha};\ [\vec{n},\ \vec{r}^j]=c^{j\alpha}\vec{r}_{\alpha}.\end{aligned} \tag{5.23}$$

Let us return to equality (5.9). It may be written in the form

$$ds^2=\mathrm{I}-2\zeta\mathrm{II}+\zeta^2\mathrm{III}+d\zeta^2, \tag{5.24}$$

where

$$\begin{aligned}\mathrm{I}&=a_{\alpha\beta}d\alpha^{\alpha}d\alpha^{\beta};\\ \mathrm{II}&=b_{\alpha\beta}d\alpha^{\alpha}d\alpha^{\beta};\\ \mathrm{III}&=\nu_{\alpha\beta}d\alpha^{\alpha}d\alpha^{\beta}\end{aligned} \tag{5.25}$$

are three quadratic forms which determine the metrics of the shell. The first of these, as we have established, determines the metrics of the middle surface; the second one determines its curvature. In essence we have already dealt with it in Chapter I. It is considered in more detail in the next section. Here we shall limit ourselves to the introduction of two invariants associated with it

$$\begin{aligned}2K&=c^{\alpha\beta}c^{\gamma\delta}b_{\alpha\gamma}b_{\beta\delta};\\ 2H&=a^{\alpha\beta}b_{\alpha\beta}=a^{\alpha}_{\beta}b^{\beta}_{\alpha}=\delta^{\alpha}_{\beta}b^{\beta}_{\alpha}=b^{\alpha}_{\alpha}.\end{aligned} \tag{5.26}$$

The first of these (K) is called the Gaussian curvature, while the second (H) is called the middle curvature of the surface. The expression for the Gaussian curvature may be given another form. Expanding the sum (5.26) for this purpose, we obtain

$$\begin{aligned}2K&=c^{12}c^{12}b_{11}b_{22}+c^{12}c^{21}b_{12}b_{21}+c^{21}c^{12}b_{21}b_{12}+c^{21}c^{21}b_{22}b_{11}=\\ &=\frac{2}{a}(b_{11}b_{22}-b_{12}^2).\end{aligned}$$

Introducing the designation

$$b=\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix},$$

we find

$$K=\frac{b}{a}. \tag{5.27}$$

As for the third quadratic form it is not independent. We shall show /32 that it can be expressed in terms of the coefficients for the first two forms. To do this we expand the vectors $\vec{n}_i$ in terms of the reciprocal coordinate vectors. First of all,

$$\vec{n}_i\vec{n}=\frac{1}{2}\frac{\partial(\vec{n}\vec{n})}{\partial a}=\frac{1}{2}\frac{\partial 1}{\partial a^i}=0.$$

Further, according to (5.6)

$$(\vec{n}_i)_j=\vec{n}_i\vec{r}_j=-b_{ij}.$$

Therefore

$$\vec{n}_i=-b_{i\alpha}\vec{r}^{\alpha}=-b_i^{\alpha}\vec{r}_{\alpha}. \tag{5.28}$$

From this and from $(5.6)_3$ it follows that

$$\nu_{ij}=b_{i\alpha}b_{j\beta}(\vec{r}^{\alpha}\vec{r}^{\beta})=a^{\alpha\beta}b_{i\alpha}b_{j\beta}=b_{i\alpha}b_j^{\beta}a_{\beta}^{\alpha}=b_{i\alpha}b_j^{\beta}\delta_{\beta}^{\alpha}=b_{i\alpha}b_j^{\alpha}. \tag{5.29}$$

By considering the components of the Rieman-Christoffel tensor (see the end of this section) we see that it is possible to express this quantity in a more compact form

$$\nu_{ij}=2Hb_{ij}-Ka_{ij}. \tag{5.30}$$

We can now represent expressions (5.3) and (5.7) in the form

$$\vec{R}_i = \vec{r}_i - \zeta b_i^\alpha \vec{r}_\alpha = (\delta_i^\alpha - \zeta b_i^\alpha)\vec{r}_\alpha,$$
$$\vec{R}_3 = \vec{n}; \tag{5.31}$$

$$g_{ij} = (1 - \zeta^2 K) a_{ij} - 2\zeta (1 - \zeta H) b_{ij},$$
$$g_{i3} = g_{33} = 0, \quad g_{33} = 1. \tag{5.32}$$

Furthermore, according to (5.22), (5.31) and (3.27) we find

$$\sqrt{\frac{g}{a}} c_{ij} = e_{ij3} = (\vec{R}_i, \vec{R}_j, \vec{n}) =$$
$$= (\delta_i^\alpha - \zeta b_i^\alpha)(\delta_j^\beta - \zeta b_j^\beta)(\vec{r}_\alpha, \vec{r}_\beta, \vec{n}) = (\delta_i^\alpha - \zeta b_i^\alpha)(\delta_j^\beta - \zeta b_j^\beta) c_{\alpha\beta}.$$

Multiplying the left and right sides of the resulting equality by c^{ij} and contracting with respect to both signs, we obtain

$$\sqrt{\frac{g}{a}} (c_{\gamma\delta} c^{\gamma\delta}) = (\delta_\gamma^\alpha - \zeta b_\gamma^\alpha)(\delta_\delta^\beta - \zeta b_\delta^\beta) c_{\alpha\beta} c^{\gamma\delta} =$$
$$= (c_{\alpha\beta} c^{\alpha\beta}) - \zeta [b_\delta^\beta (c_{\alpha\beta} c^{\alpha\delta}) + b_\gamma^\alpha (c_{\alpha\beta} c^{\gamma\beta})] +$$
$$+ \zeta^2 (b_\gamma^\alpha b_\delta^\beta c_{\alpha\beta} c^{\gamma\delta}).$$

Now taking into account relationships (5.21) and (5.26), we obtain the equality /33

$$\sqrt{\frac{g}{a}} = 1 - 2H\zeta + K\zeta^2, \tag{5.33}$$

which shows that the quantity $\sqrt{\frac{g}{a}}$ is invariant with respect to the transformation of the coordinates on the surface.

Let us now consider covariant differentiation. First of all, by using expressions (5.32) it is easy to compute the Christoffels by means of equations in the preceding section

$$\Gamma_{h,\,ij} = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial a^i} + \frac{\partial g_{hi}}{\partial a^j} - \frac{\partial g_{ij}}{\partial a^h}\right);$$
$$\Gamma_{3,\,ij} = -\Gamma_{i,\,j3} = \zeta K a_{ij} + (1 - 2H\zeta) b_{ij}; \tag{5.34}$$
$$\Gamma_{i,\,33} = \Gamma_{3,\,\alpha 3} = \Gamma_{3,\,33} = 0;$$

$$\Gamma_{ij}^h = \frac{1}{2}\left(\frac{\partial g_{j\alpha}}{\partial a^i} + \frac{\partial g_{\alpha i}}{\partial a^j} - \frac{\partial g_{ij}}{\partial a^\alpha}\right) g^{\alpha h};$$
$$\Gamma_{ij}^3 = \zeta K a_{ij} + (1 - 2H\zeta) b_{ij}; \tag{5.35}$$
$$\Gamma_{i3}^h = -[\zeta K a_{\alpha i} + (1 - 2H\zeta) b_{\alpha i}] g^{\alpha h};$$
$$\Gamma_{33}^h = \Gamma_{i3}^3 = \Gamma_{33}^3 = 0.$$

We introduce the following designations for the Christoffels on the middle surface

$$\Gamma^h_{ij} = G^h_{ij}|_{\zeta=0};\ \Gamma_{h,\,ij} = G_{h,\,ij}|_{\zeta=0}. \tag{5.36}$$

According to (5.34) and (5.35)

$$\begin{aligned} &\Gamma_{h,\,ij} = \frac{1}{2}\left(\frac{\partial a_{jh}}{\partial \alpha} + \frac{\partial a_{hi}}{\partial \alpha^j} - \frac{\partial a_{ij}}{\partial \alpha^h}\right);\\ &\Gamma_{3,\,ij} = -\Gamma_{i,\,j3} = b_{ij};\ \Gamma_{i,\,33} = \Gamma_{3,\,i3} = \Gamma_{3,\,33} = 0; \end{aligned} \tag{5.37}$$

$$\begin{aligned} &\Gamma^h_{ij} = \frac{1}{2}\left(\frac{\partial a_{js}}{\partial \alpha^i} + \frac{\partial a_{si}}{\partial \alpha^j} - \frac{\partial a_{ij}}{\partial \alpha^s}\right)a^{sh};\\ &\Gamma^3_{ij} = b_{ij};\ \Gamma^h_{i3} = -b^h_i;\\ &\Gamma^h_{33} = \Gamma^3_{i3} = \Gamma^3_{33} = 0. \end{aligned} \tag{5.38}$$

We shall now establish two other relationships which will be required in the future. First of all it follows from (5.37) that

$$\frac{\partial a_{ij}}{\partial \alpha^h} = \Gamma_{i,\,jh} + \Gamma_{j,\,ih}. \tag{5.39}$$

Further we compute the quantity $\frac{\partial a}{\partial \alpha^k}$ by means of (5.14)

$$\begin{aligned} \frac{\partial a}{\partial \alpha^k} &= \frac{\partial}{\partial \alpha^k}\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = \begin{vmatrix} \frac{\partial a_{11}}{\partial \alpha^k} & \frac{\partial a_{12}}{\partial \alpha^k}\\ a_{21} & a_{22}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{12}\\ \frac{\partial a_{21}}{\partial \alpha^k} & \frac{\partial a_{22}}{\partial \alpha^k}\end{vmatrix} =\\ &= a\left\{a^{11}\frac{\partial a_{11}}{\partial \alpha^k} + a^{12}\frac{\partial a_{12}}{\partial \alpha^k} + a^{21}\frac{\partial a_{21}}{\partial \alpha^k} + a^{22}\frac{\partial a_{22}}{\partial \alpha^k}\right\} =\\ &= aa^{\alpha\beta}\frac{\partial a_{\alpha\beta}}{\partial \alpha^k} = aa^{\alpha\beta}\left(\Gamma_{\alpha,\,\beta k} + \Gamma_{\beta,\,\alpha k}\right) = a\left(\Gamma^\beta_{\beta k} + \Gamma^\alpha_{\alpha k}\right) = 2a\Gamma^\alpha_{\alpha k}. \end{aligned}$$

Thus the following relationship takes place

$$\Gamma^\alpha_{\alpha k} = \frac{1}{2a}\frac{\partial a}{\partial \alpha^k} = \frac{1}{\sqrt{a}}\frac{\partial \sqrt{a}}{\partial \alpha^k}. \tag{5.40}$$

We consider further the covariant derivative of the covariant vector /34

$$V_{i|j} = \frac{\partial V_i}{\partial \alpha^j} - G^\alpha_{ij}V_\alpha - G^3_{ij}V_n.$$

Utilizing equalities (5.36) and (5.38) we obtain

$$(V_{i/j})_0 = \frac{\partial (V_i)_0}{\partial \alpha^j} - \Gamma^{\alpha}_{ij} (V_\alpha)_0 - b_{ij} (V_n)_0 \tag{5.41}$$

(The zero sign means that the quantity has been computed on the middle surface.)

The quantity $V_{i/j}$ is a component of the doubly covariant tensor for any value of ζ, in particular when $\zeta = 0$, i.e., the quantity $(V_{i/j})_0$ is a component of the tensor assigned in the middle surface. The last term in (5.41) is the same. It follows immediately that the covariant surface tensor is given by the expression

$$\nabla_j U_i = \frac{\partial U_i}{\partial \alpha^j} - \Gamma^{\alpha}_{ij} U_\alpha \quad (i = 1 \quad 2), \tag{5.42}$$

which is called the covariant derivative on the surface of the covariant surface vector.

From the meaning of the derivation we should assume that

$$U_1 = (V_1)_0, \; U_2 = (V_2)_0. \tag{5.43}$$

However, the expression which has been obtained can be applied to an arbitrary surface vector. This is easy to see by applying the above transformations to a vector whose value in the shell is the same as its value in the middle surface.

In the same way we can introduce the concept of surface covariant tensor derivatives of any rank and type. Thus

$$\begin{aligned} \nabla_j U^i &= \frac{\partial U^i}{\partial \alpha^j} + \Gamma^i_{\alpha j} U^\alpha; \\ \nabla_k W_{ij} &= \frac{\partial W_{ij}}{\partial \alpha^k} - \Gamma^{\alpha}_{ik} W_{\alpha j} - \Gamma^{\alpha}_{jk} W_{i\alpha}; \end{aligned} \tag{5.44}$$

$$\begin{aligned} \nabla_k W^i{}_{\cdot j} &= \frac{\partial W^i{}_{\cdot j}}{\partial \alpha^k} + \Gamma^i_{\alpha k} W^\alpha{}_{\cdot j} - \Gamma^{\alpha}_{jk} W^i{}_{\cdot \alpha}; \\ \nabla_k W^{ij} &= \frac{\partial W^{ij}}{\partial \alpha^k} + \Gamma^i_{\alpha k} W^{\alpha j} + \Gamma^j_{\alpha k} W^{i\alpha}. \end{aligned} \tag{5.45}$$

We shall show that the covariant derivatives (on the surface) of tensors a_{ij} and c_{ij} are equal to zero. First of all, according to (4.33) and (4.29) /35

$$0 = e_{i/3|k} = \frac{\partial e_{i/3}}{\partial \alpha^k} - (\omega_{ik} e_{\alpha/3} + \underline{G^3_{ik} e_{3/3}}) - (G^\alpha_{jk} e_{i\alpha 3} + \underline{G^3_{jk} e_{i33}}) - \\ - (\underline{G^\alpha_{3k} e_{i/\alpha}} + G^3_{3k} e_{i/3})$$

(We recall that i, j, k, α = 1, 2.) The underlined terms are equal to zero, because they correspond to the components of the skew-symmetric tensor with the same indices. Dropping them, letting ζ tend to zero and using relationships (5.18) and (5.38), we obtain the equality

$$0 = \frac{\partial c_{ij}}{\partial \alpha^k} - \Gamma^\alpha_{ik} c_{\alpha j} - \Gamma^\alpha_{jk} c_{i\alpha},$$

i.e.,

$$\nabla_k c_{ij} = 0. \tag{5.46}$$

In the same way it is shown that

$$\nabla_k a_{ij} = 0. \tag{5.47}$$

Thus, during covariant differentiation on the surface a_{ij} and c_{ij} may be looked upon as constants.

In deriving the relationships of the theory of shells from the equalities of the three-dimensional theory of elasticity it will be necessary for us to convert from the covariant differentiation of a vector in space to the covariant differentiation on the surface. One of the necessary relationships is furnished by equality (5.41). Rewriting it in terms of the introduced designations and adding the expressions obtained in the same manner, we find

$$\begin{aligned} &(V_{i/j})_0 = \nabla_j (V_i)_0 - b_{ij} (V_n)_0; \quad (V_{n/j})_0 = \frac{\partial (V_n)_0}{\partial \alpha^j} + b^\alpha_j (V_\alpha)_0; \\ &(V_{i/\zeta})_0 = \left(\frac{\partial V_i}{\partial \zeta}\right)_0 + b^\beta_i (V_\beta)_0; \quad (V_{n/\zeta})_0 = \left(\frac{\partial V_n}{\partial \zeta}\right)_0; \end{aligned} \tag{5.48}$$

$$\begin{aligned} &(V^i_{/j})_0 = \nabla_j (V^i)_0 - b^i_j (V_n)_0; \quad (V_{n/j})_0 = \frac{\partial (V_n)_0}{\partial \alpha^j} + b_{\alpha j} (V^\alpha)_0; \\ &(V^i_{/\zeta})_0 = \left(\frac{\partial V^i}{\partial \zeta}\right)_0 - b^i_\alpha (V^\alpha)_0; \quad (V_{n/\zeta})_0 = \left(\frac{\partial V_n}{\partial \zeta}\right)_0. \end{aligned} \tag{5.49}$$

By using the same relationships, the derivative of the vector

$$\vec{U}=U_\alpha\vec{r}^\alpha+U_n\vec{n}=U^\alpha\vec{r}_\alpha+U_n\vec{n} \tag{5.50}$$

(assuming that the value of the latter in the shell is equal to its value at the middle surface) may be represented as

$$\begin{aligned}\frac{\partial\vec{U}}{\partial\alpha^j}=\nabla_j\vec{U}=(\nabla_jU_\alpha-\underline{b_{\alpha j}U_n})\vec{r}^\alpha+\left(\frac{\partial U_n}{\partial\alpha^j}+\underline{b_j^\alpha U_\alpha}\right)\vec{n}=\\ =(\nabla_jU^\alpha-\underline{b_j^\alpha U_n})\vec{r}_\alpha+\left(\frac{\partial U_n}{\partial\alpha^j}+\underline{b_{j\alpha}U^\alpha}\right)\vec{n}.\end{aligned} \tag{5.51}$$

Comparing this expression with the preceding one, we can see that the underlined terms owe their origin to the variability of the coordinate vectors. /36 In this case

$$\begin{aligned}&\nabla_j\vec{r}_i=b_{ij}\vec{n};\quad\nabla_j\vec{n}=\frac{\partial\vec{n}}{\partial\alpha^j}=-b_j^\alpha\vec{r}_\alpha;\\ &\nabla_j\vec{r}^i=b_j^i\vec{n};\quad\nabla_j\vec{n}=-b_{j\alpha}\vec{r}^\alpha.\end{aligned} \tag{5.52}$$

The last equality makes it possible to establish equations for the covariant differentiation of tensors with vector components. Thus, for

$$\vec{U}_i=U_{i\alpha}\vec{r}^\alpha+U_{in}\vec{n};\ \vec{U}^i=U^{i\alpha}\vec{r}_\alpha+U^{in}\vec{n} \tag{5.53}$$

the following expressions are valid

$$\begin{aligned}&\nabla_j\vec{U}_i=(\nabla_jU_{i\alpha}-b_{j\alpha}U_{in})\vec{r}^\alpha+(\nabla_jU_{in}+b_j^\alpha U_{i\alpha})\vec{n};\\ &\nabla_j\vec{U}^i=(\nabla_jU^{i\alpha}-b_j^\alpha U^{i\alpha})\vec{r}_\alpha+(\nabla_jU^{in}+b_{j\alpha}U^{i\alpha})\vec{n}.\end{aligned} \tag{5.54}$$

The Codazzi-Gauss relationships, which are widely used in the theory of shells, can be obtained by writing down the components of the Rieman-Christoffel tensor (4.37) in the adopted system of coordinates assuming that $\zeta = 0$ and setting the resulting expressions to zero. As a result from equality $R_{12,12}|_{\zeta=0} = 0$ we obtain the Gauss relationship

$$\begin{aligned}\frac{\partial^2a_{12}}{\partial\alpha^1\partial\alpha^2}-\frac{1}{2}\frac{\partial^2a_{11}}{(\partial\alpha^2)^2}-\frac{1}{2}\frac{\partial^2a_{22}}{(\partial\alpha^1)^2}+a^{\alpha\beta}\{\Gamma_{\alpha,21}\Gamma_{\beta,12}-\Gamma_{\alpha,22}\Gamma_{\beta,11}\}=\\ =b_{11}b_{22}-(b_{12})^2.\end{aligned} \tag{5.55}$$

The equalities $R_{31,31}|_{\zeta=0} = R_{32,32}|_{\zeta=0} = 0$ lead to the Codazzi relationships

$$\nabla_1 b_{12} = \nabla_2 b_{11}. \tag{5.56}$$

The remaining three components give us equalities (5.30) which were utilized above.

By using (5.27), relationships (5.55) may be represented in the form

$$\frac{\partial^2 a_{12}}{\partial\alpha^1\partial\alpha^2} - \frac{1}{2}\frac{\partial^2 a_{11}}{(\partial\alpha^2)^2} - \frac{1}{2}\frac{\partial^2 a_{22}}{(\partial\alpha^1)^2} + a^{\alpha\beta}\{\Gamma_{\alpha,21}\Gamma_{\beta,12} - \Gamma_{\alpha,22}\Gamma_{\beta,11}\} = aK. \tag{5.57}$$

We can see from this that the Gaussian curvature, which we first introduced as a common invariant of the coefficients for the first and second quadratic form of the middle surface (5.27), is completely determined by assigning coefficients to the first (metric) form of the surface.

Let us consider the determinant of second order made up from the components of the covariant tensor φ_{ij} /37

$$\varphi = \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{vmatrix}. \tag{5.58}$$

By using equalities (5.19) and conducting a direct check it is easy to see that the following relationship is valid

$$\varphi = \frac{a}{2} c^{\alpha\lambda} c^{\beta\mu} \varphi_{\alpha\beta} \varphi_{\lambda\mu}. \tag{5.59}$$

Thus the discriminant tensor makes it possible to compute the determinant (discriminant) of the tensor. This is really what explains the name assigned to the tensor c_{ij}. Let us designate by $\tilde{\varphi}^{ij}$ the reduced minor (algebraic addition) of the element φ_{ij} in the determinant (5.58). Then it is easy to verify that

$$\tilde{\varphi}^{ij} = \frac{a}{\varphi} c^{i\mu} c^{j\lambda} \varphi_{\mu\lambda}. \tag{5.60}$$

If, however, we define φ as the determinant

$$\varphi = \begin{vmatrix} \varphi^{11} & \varphi^{12} \\ \varphi^{21} & \varphi^{22} \end{vmatrix}, \tag{5.61}$$

then the following equations are valid

$$\varphi = \frac{1}{2a} c_{\alpha\lambda} c_{\beta\mu} \varphi^{\alpha\beta} \varphi^{\lambda\mu}; \tag{5.62}$$

$$\tilde{\varphi}_{ij} = \frac{1}{a\varphi} c_{i\mu} c_{j\lambda} \varphi^{\mu\lambda}. \tag{5.63}$$

At the end of section 1 we introduced the concept of the gradient of a scalar function assigned in space. We can also determine the gradient of the function $\varphi(\alpha^1, \alpha^2)$ assigned on a surface, if we interpret this (1.18) as the following invariant vector quantity

$$\operatorname{grad} \varphi = \vec{r}^{\alpha} \nabla_{\alpha} \varphi. \tag{5.64}$$

Let us introduce one other quantity

$$a^{\alpha\beta} \nabla_{\alpha} U_{\beta}. \tag{5.65}$$

In the Cartesian system of coordinates $a^{11} = a^{22} = 1$, $a^{12} = 0$ and it assumes the form

$$\frac{\partial U_1}{\partial x^1} + \frac{\partial U_2}{\partial x^2}.$$

This is the well-known expression for the divergence of vector $\vec{U}$, div $\vec{U}$. Since according to its structure the quantity (5.65) is invariant (the contraction of the covariant $\nabla_{\alpha} U_{\beta}$ and contravariant $a^{\alpha\beta}$ tensors), we can look upon it as the determination of the operation div(...) on an arbitrary surface, assuming thereby that

$$\operatorname{div} \vec{U} = a^{\alpha\beta} \nabla_{\alpha} U_{\beta}. \tag{5.66}$$

Applying this operation to the vector grad φ, we obtain, in accordance with (5.64) /38

$$\text{div grad}\,\varphi = a^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi. \tag{5.67}$$

In the Cartesian system of coordinates it coincides with the Laplace operation on the function φ

$$\Delta\varphi = \frac{\partial^2\varphi}{(\partial x^1)^2} + \frac{\partial^2\varphi}{(\partial x^2)^2}.$$

From this and from the invariance of expression (5.67) it follows that we can regard it as the generalized Laplace operation for the case of an arbitrary surface

$$\Delta\varphi = a^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi. \tag{5.68}$$

The resulting expression may be transformed slightly. Indeed, according to (5.47), (5.44) and (5.40)

$$a^{\alpha\beta}\nabla_\alpha U_\beta = \nabla_\alpha U^\alpha = \frac{\partial U^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\lambda}U^\lambda =$$
$$= \frac{\partial U^\alpha}{\partial x^\alpha} + \frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}}{\partial x^\lambda}U^\lambda = \frac{\partial U^\alpha}{\partial x^\alpha} + \frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}}{\partial x^\alpha}U^\alpha = \frac{1}{\sqrt{a}}\frac{\partial(\sqrt{a}U^\alpha)}{\partial x^\alpha}.$$

Therefore

$$\text{div}\,\vec{U} = \frac{1}{\sqrt{a}}\frac{\partial(\sqrt{a}U^\alpha)}{\partial x^\alpha}. \tag{5.69}$$

Similarly we find

$$\Delta\varphi = \frac{1}{\sqrt{a}}\left\{\frac{\partial\sqrt{a}\varphi^2}{\partial x^\alpha}\right\} = \frac{1}{\sqrt{a}}\left\{\frac{\partial\sqrt{a}\varphi^1}{\partial x^1} + \frac{\partial\sqrt{a}\varphi^2}{\partial x^2}\right\}, \tag{5.70}$$

where φ^1, φ^2 are the covariant components of the vector grad φ.

6. Orthogonal Coordinates. Physical Components of Tensors. Oblique Angle Coordinates

Most frequently in the theory of shells the middle surface is referred to an orthogonal system of coordinates. In this system the relationships presented above are substantially simplified. We introduce, for the coefficients of the first quadratic form, the following designations

$$a_{11}=A^2;\ a_{22}=B^2\quad (a_{12}=0) \tag{6.1}$$

(the quantities A, B are usually called the Lamé parameters).

By using the relationships of the preceding section it is now easy to compute

$$\begin{gathered} a=A^2B^2; \\ a^{11}=\frac{1}{A^2};\quad a^{22}=\frac{1}{B^2};\quad a^{12}=0; \end{gathered} \tag{6.2}$$

$$\begin{gathered} \Gamma_{1,11}=A\frac{\partial A}{\partial\alpha};\quad \Gamma_{1,12}=\Gamma_{1,21}=A\frac{\partial A}{\partial\beta};\quad \Gamma_{1,22}=-B\frac{\partial B}{\partial\alpha}; \\ \Gamma_{2,22}=B\frac{\partial B}{\partial\beta};\quad \Gamma_{2,12}=\Gamma_{2,21}=B\frac{\partial B}{\partial\alpha};\quad \Gamma_{2,11}=-A\frac{\partial A}{\partial\beta}; \end{gathered} \tag{6.3}$$

$$\begin{gathered} \Gamma_{11}^1=\frac{1}{A}\frac{\partial A}{\partial\alpha};\quad \Gamma_{12}^1=\Gamma_{21}^1=\frac{1}{A}\frac{\partial A}{\partial\beta};\quad \Gamma_{22}^1=-\frac{B}{A^2}\frac{\partial B}{\partial\alpha}; \\ \Gamma_{22}^2=\frac{1}{B}\frac{\partial B}{\partial\beta};\quad \Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{B}\frac{\partial B}{\partial\alpha};\quad \Gamma_{11}^2=-\frac{A}{B^2}\frac{\partial A}{\partial\beta} \end{gathered} \tag{6.4}$$

(here and below in using the orthogonal system of coordinates we shall replace α^1, α^2 by the designations α, β, which are usually adopted in the nontensor presentation). /39

We rewrite the vector

$$\vec{U}=U_\alpha\vec{r}^{\,\alpha}+U_n\vec{n}=U^\alpha\vec{r}_\alpha+U_n\vec{n}$$

in the form

$$\vec{U}=(U_\alpha\sqrt{a^{\alpha\alpha}})\left(\frac{\vec{r}^{\,\alpha}}{\sqrt{a^{\alpha\alpha}}}\right)+U_n\vec{n}=(U^\alpha\sqrt{a_{\alpha\alpha}})\left(\frac{\vec{r}_\alpha}{\sqrt{a_{\alpha\alpha}}}\right)+U_n\vec{n}.$$

The quantities $\frac{\vec{r}^i}{\sqrt{a^{ii}}}$, $\frac{\vec{r}_i}{\sqrt{a_{ii}}}$ are equal (because the system of coordinates (1.4)) uses unit orthogonal vectors. The components of the vector in terms of these unit orthogonal vectors are called the physical components of the vector and are usually designated by $U_{(i)}$

$$U_{(i)}=U_i\sqrt{a^{ii}}=U^i\sqrt{a_{ii}}. \tag{6.5}$$

Since $\vec{n}$ is also a unit vector, the component U_n may be looked upon as the physical component of the unit orthogonal vector $\vec{n}$.

Utilizing the Lamé parameters we rewrite (6.5) in the form

$$U_{(i)}=\frac{1}{A_i}U_i=A_iU^i \qquad (A_1=A,\ A_2=B). \tag{6.6}$$

Conversely

$$U_i=A_iU_{(i)};\quad U^i=\frac{1}{A_i}U_{(i)}. \tag{6.7}$$

The concept of the physical components is extended to tensors of arbitrary type and rank. Thus,

$$U_{(ij)}=\frac{1}{A_iA_j}U_{ij}=\frac{A_j}{A_i}U_i^{\ j}=\frac{A_i}{A_j}U^i_{\ j}=A_iA_jU^{ij}; \tag{6.8}$$

$$U_{ij}=A_iA_jU_{(ij)};\quad U_i^{\ j}=\frac{A_i}{A_j}U_{(ij)};$$

$$U^i_{\ j}=\frac{A_j}{A_i}U_{(ij)};\quad U^{ij}=\frac{1}{A_iA_j}U_{(ij)} \qquad (A_1=A,\ A_2=B). \tag{6.9}$$

In particular,

$$a_{(11)}=a_{(22)}=1;\quad a_{(12)}=0; \tag{6.10}$$

$$c_{(12)}=1;\quad c_{(21)}=-1;\quad c_{(11)}=c_{(22)}=0. \tag{6.11}$$

It is easy to show that in terms of the physical components

$$\nabla_{(1)}U_{(1)}=\frac{1}{A}\frac{\partial U_{(1)}}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}U_{(2)},$$
$$\nabla_{(1)}U_{(2)}=\frac{1}{A}\frac{\partial U_{(2)}}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}U_{(1)};$$
$$\nabla_{(2)}U_{(1)}=\frac{1}{B}\frac{\partial U_{(1)}}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}U_{(2)};$$
$$\nabla_{(2)}U_{(2)}=\frac{1}{B}\frac{\partial U_{(2)}}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}U_{(1)};$$
(6.12)

$$\nabla_{(1)}W_{(11)}=\frac{1}{A}\frac{\partial W_{(11)}}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}(W_{(12)}+W_{(21)});$$
$$\nabla_{(1)}W_{(12)}=\frac{1}{A}\frac{\partial W_{(12)}}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}(W_{(22)}-W_{(11)});$$
$$\nabla_{(1)}W_{(21)}=\frac{1}{A}\frac{\partial W_{(21)}}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}(W_{(22)}-W_{(11)});$$
$$\nabla_{(1)}W_{(22)}=\frac{1}{A}\frac{\partial W_{(22)}}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}(W_{(12)}+W_{(21)});$$
$$\nabla_{(2)}W_{(11)}=\frac{\partial W_{(11)}}{B\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}(W_{(12)}+W_{(21)});$$
$$\nabla_{(2)}W_{(12)}=\frac{\partial W_{(12)}}{B\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}(W_{(11)}-W_{(22)});$$
$$\nabla_{(2)}W_{(21)}=\frac{\partial W_{(21)}}{B\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}(W_{(11)}-W_{(22)});$$
$$\nabla_{(2)}W_{(22)}=\frac{\partial W_{(22)}}{B\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}(W_{(12)}+W_{(21)}).$$
(6.13)

Physical components may be introduced in the entire shell, if we refer the latter to a triorthogonal system of coordinates. In this case it is necessary to replace A_i and $A_i^{\zeta}=\sqrt{g_{ii}}$ in relationships (6.6)-(6.9). Below, many examples will be used to show the convenience of utilizing the physical components which permit physical interpretation. /40

Sometimes the physical components of a tensor are introduced even when the coordinate system is not orthogonal. Let, as before

$$a_{11}=A^2,\quad a_{22}=B^2. \tag{6.14}_{1,2}$$

Then from (5.6) we obtain

$$a_{12}=AB\cos\chi, \tag{6.14}_3$$

where χ (α, β) is the angle between the coordinate lines. Utilizing designations (6.14), we thereby represent the first quadratic form of the surface in a form proposed by Darby /41

$$ds^2 = A^2 d\alpha^2 + 2AB\cos\chi\, d\alpha d\beta + B^2 d\beta^2. \tag{6.15}$$

By using equations of the preceding section we compute

$$\begin{gathered}
a = A^2B^2\sin^2\chi;\quad a^{11} = \frac{1}{A^2\sin^2\chi};\quad a^{12} = -\frac{\cos\chi}{AB\sin^2\chi};\quad a^{22} = \frac{1}{B^2\sin^2\chi};\\
\Gamma^1_{11} = \frac{B\frac{\partial A}{\partial\alpha} + A\cos\chi\frac{\partial A}{\partial\beta} - \cos\chi\frac{\partial AB\cos\chi}{\partial\alpha}}{AB\sin^2\chi};\\
\Gamma^2_{22} = \frac{A\frac{\partial B}{\partial\beta} + B\cos\chi\frac{\partial B}{\partial\alpha} - \cos\chi\frac{\partial AB\cos\chi}{\partial\beta}}{AB\sin^2\chi};\\
\Gamma^1_{12} = \Gamma^1_{21} = \frac{\frac{\partial A}{\partial\beta} - \cos\chi\frac{\partial B}{\partial\alpha}}{A\sin^2\chi};\quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{\frac{\partial B}{\partial\alpha} - \cos\chi\frac{\partial A}{\partial\beta}}{B\sin^2\chi};\\
\Gamma^1_{22} = \frac{-B\frac{\partial B}{\partial\alpha} - A\cos\chi\frac{\partial B}{\partial\beta} + \frac{\partial AB\cos\chi}{\partial\beta}}{A^2\sin^2\chi};\\
\Gamma^2_{11} = \frac{-A\frac{\partial A}{\partial\beta} - B\cos\chi\frac{\partial A}{\partial\alpha} + \frac{\partial AB\cos\chi}{\partial\alpha}}{B^2\sin^2\chi}.
\end{gathered} \tag{6.16}$$

If the coordinate system is not orthogonal the unit vectors $\frac{\vec{r}_i}{\sqrt{a_{ii}}}$ and $\frac{\vec{r}^i}{\sqrt{a^{ii}}}$ no longer coincide. Therefore, in introducing the physical components of a vector it is necessary to specify clearly which system of unit orthogonal vectors is chosen as the basic one. The system $\frac{\vec{r}_i}{\sqrt{a_{ii}}}$ is chosen as such a basic system. We shall call the physical components of the vector its components (not projections!) along the basic system of orthogonal unit vectors. In this connection it is easy to see that

$$U_{(i)} = \sqrt{a_{ii}}\,U^i = A_i U^i;\quad U^i = \frac{1}{A_i}U_{(i)}. \tag{6.17}$$

Utilizing equation (3.5) we find that

$$U_{(i)} = (U_\alpha a^{\alpha i}\sqrt{a_{ii}});\quad U_i = a_{i\beta}\left(\frac{1}{\sqrt{a_{\beta\beta}}!}U_{(\beta)}\right) \tag{6.18}$$

(! means that summation need not be taken over the specified signs).

In the same way we introduce the physical components of tensors. For example

$$U^{ij}=\frac{1}{\sqrt{a_{ii}a_{jj}}}U_{(ij)}=\frac{1}{A_iA_j}U_{(ij)};$$
$$U_{ij}=\frac{a_{i\beta}}{\sqrt{a_{\beta\beta}}}\frac{a_{j\gamma}}{\sqrt{a_{\gamma\gamma}}}U_{(\beta\gamma)}. \tag{6.19}$$

The oblique-angled system of coordinates are sometimes convenient in the solution of specific problems, for example, when considering transport shells. Due to the lack of space we do not write down all relationships in terms of these coordinates, assuming that the reader will do this himself if necessary. In particular it is proposed to verify the validity of the following expressions /42

$$a_{(11)}=\frac{1}{\sin^2\chi};\ a_{(12)}=-\frac{\cos\chi}{\sin^2\chi};\ a_{(22)}=\frac{1}{\sin^2\chi}; \tag{6.20}$$

$$c_{(12)}=-c_{(21)}=\frac{1}{\sin\chi};\quad c_{(11)}=c_{(22)}=0. \tag{6.21}$$

7. Coordinates Fixed With Respect to Lines on the Surface

Let us assume that curve Γ is assigned on the middle surface (fig. 3). With this curve we can associate a trihedron of unit orthogonal vectors $\vec{\nu}$, $\vec{t}$, $\vec{n}$, where $\vec{\nu}$ is the unit orthogonal vector of the so-called tangential normal, $\vec{t}$ is the unit vector of the tangent, and $\vec{n}$, as before, is the unit orthogonal vector of the normal to the middle surface. The unit orthogonal vectors which have been introduced satisfy the obvious relationships

$$\vec{\nu}^2=\vec{t}^2=\vec{n}^2=1;\ \vec{\nu}\cdot\vec{t}=\vec{\nu}\cdot\vec{n}=\vec{t}\cdot\vec{n}=0; \tag{7.1}$$

$$[\vec{\nu},\vec{t}]=\vec{n};\ [\vec{t},\vec{n}]=\vec{\nu};\ [\vec{n},\vec{\nu}]=\vec{t};$$
$$\{\vec{\nu},\vec{t},\vec{n}\}=1. \tag{7.2}$$

Let

$$\vec{\nu}=\nu_\alpha\vec{r}^{\,\alpha}=\nu^\alpha\vec{r}_\alpha;$$
$$\vec{t}=t_\alpha\vec{r}^{\,\alpha}=t^\alpha\vec{r}_\alpha. \tag{7.3}$$

It is easy to establish a relationship between the components of both vectors. Indeed, by using equalities (5.23) we find from (7.2)

$$\vec{\nu}=\left[t_\alpha\vec{r}^{\,\alpha},\ \vec{n}\right]=t_\alpha\left[\vec{r}^{\,\alpha},\ \vec{n}\right]=c^{\beta\alpha}t_\alpha\vec{r}_\beta. \tag{7.4}$$

From this it follows that

$$\nu^l=c^{l\alpha}t_\alpha. \tag{7.5}$$

In a similar manner we obtain

$$t^l=c^{\alpha l}\nu_\alpha. \tag{7.6}$$

Furthermore,

$$\vec{t}=\frac{d\vec{r}}{ds_t};\quad \vec{\nu}=\frac{d\vec{r}}{ds_\nu}. \tag{7.7}$$

Here s_t is the arc length of the considered curve Γ, s_ν is the arc length /43
of the line from some family of lines orthogonal to Γ, which passes through M.

For $\vec{t}$ we have

$$\vec{t}=\frac{\partial\vec{r}}{\partial\alpha^\beta}\frac{d\alpha^\beta}{ds_t}=\frac{d\alpha^\beta}{ds_t}\vec{r}_\beta, \tag{7.8}$$

from which

$$t^l=\frac{d\alpha^l}{ds_t}. \tag{7.9}$$

In the same way we find

$$\nu^l=\frac{d\alpha^l}{ds_\nu}. \tag{7.10}$$

By using the relationships obtained it is convenient to compute the components of vectors and tensors when converting from $\vec{r}_1$, $\vec{r}_2$, $\vec{n}$ to $\vec{\nu}$, $\vec{t}$, $\vec{n}$. Thus

$$U_t = \vec{U}\vec{t} = \left(U_\alpha \vec{r}^{\,\alpha}\right)\left(t^\beta \vec{r}_\beta\right) = t^\beta U_\alpha \delta^\alpha_\beta = t^\alpha U_\alpha; \tag{7.11}$$

$$U_\nu = \nu^\alpha U_\alpha; \tag{7.12}$$

$$U_{tt} = t^\alpha t^\beta U_{\alpha\beta}; \quad U_{t\nu} = t^\alpha \nu^\beta U_{\alpha\beta};$$
$$U_{\nu t} = \nu^\alpha t^\beta U_{\alpha\beta}; \quad U_{\nu\nu} = \nu^\alpha \nu^\beta U_{\alpha\beta}. \tag{7.13}$$

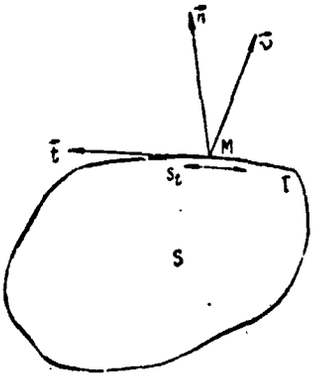

Figure 3

We note that the resulting components are the physical components of the respective tensors (section 6).

By using these relationships it is easy to verify the validity of the converse relationships

$$\vec{r}_i = \nu_i \vec{\nu} + t_i \vec{t}; \tag{7.14}$$

$$U_i = \nu_i U_\nu + t_i U_t; \tag{7.15}$$

$$U_{ij} = t_i t_j U_{tt} + t_i \nu_j U_{t\nu} + \nu_i t_j U_{\nu t} + \nu_i \nu_j U_{\nu\nu}. \tag{7.16}$$

Thus, for example, if we multiply (7.15) by ν^i and take the sum over i, we arrive at the expression (7.12)

$$\nu^\alpha U_\alpha = (\nu^\alpha \nu_\alpha) U_\nu + (\nu^\alpha t_\alpha) U_t = U_\nu$$
$$\left(\nu^\alpha \nu_\alpha = \vec{\nu}\vec{\nu} = 1; \ \nu^\alpha t_\alpha = \vec{\nu}\vec{t} = 0\right).$$

Let us establish the rule for the differentiation of unit orthogonal vectors. Utilizing condition (7.1) for this purpose we find (section 2, Chapter I) /44

$$\begin{aligned}
\frac{d\vec{t}}{ds_t} &= \sigma_t \vec{n} - \varkappa_t \vec{\nu};\\
\frac{d\vec{\nu}}{ds_t} &= \varkappa_t \vec{t} - \tau_t \vec{n};\\
\frac{d\vec{n}}{ds_t} &= \tau_t \vec{\nu} - \sigma_t \vec{t},
\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}
\sigma_t &= \vec{n}\frac{d\vec{t}}{ds_t} = -\vec{t}\frac{d\vec{n}}{ds_t};\\
\tau_t &= \vec{\nu}\frac{d\vec{n}}{ds_t} = -\vec{n}\frac{d\vec{\nu}}{ds_t};\\
\varkappa_t &= \vec{t}\frac{d\vec{\nu}}{ds_t} = -\vec{\nu}\frac{d\vec{t}}{ds_t}.
\end{aligned} \tag{7.18}$$

For reasons which will be explained below the quantity σ_t is called the normal curvature of the surface. Utilizing equations (5.28) we find

$$\begin{aligned}
\sigma_t = -\vec{t}\cdot\frac{d\vec{n}}{d\alpha^\beta}\frac{d\alpha^\beta}{ds_t} = -\left(t_\gamma \vec{r}^{\,\gamma}\right)\left(-b^\alpha_\beta \vec{r}_\alpha\right) t^\beta = a^{\gamma\alpha} b_{\alpha\beta} t_\gamma t^\beta = \\
= \delta^\gamma_\alpha b^\alpha_\beta t_\gamma t^\beta = b_{\alpha\beta} t^\alpha t^\beta.
\end{aligned} \tag{7.19}$$

By using expressions (7.9) we may give σ_t a slightly different form

$$\sigma_t = b_{\alpha\beta}\frac{d\alpha^\alpha}{ds_t}\frac{d\alpha^\beta}{ds_t} = \frac{b_{\alpha\beta}d\alpha^\alpha d\alpha^\beta}{a_{\alpha\beta}d\alpha^\alpha d\alpha^\beta} = \frac{II}{I}. \tag{7.20}$$

The quantity τ_t is called the geodetic torsion. We find the following expression for it in exactly the same way as for σ_t

$$\tau_t = -b_{\alpha\beta} t^\alpha \nu^\beta. \tag{7.21}$$

Let us transform this expression. According to (7.5)

$$\tau_t = -b_{\alpha\beta}\cdot t^\alpha c^{\beta\gamma} t_\gamma = -b_{\alpha\beta} c^\beta_{\cdot\gamma} t^\alpha t^\gamma = \left(\lambda^\beta_{\cdot\gamma\beta}\right) t^\alpha t^\gamma. \tag{7.22}$$

Let us break up the expression contained in the brackets into a symmetric part and skew-symmetric part

$$b^{\beta}_{\alpha}c_{\gamma\beta}=\frac{b^{\beta}_{\alpha}c_{\gamma\beta}+b^{\beta}_{\gamma}c_{\alpha\beta}}{2}-+\frac{b^{\beta}_{\alpha}c_{\gamma\beta}-b^{\beta}_{\gamma}c_{\alpha\beta}}{2}.$$

If we replace $b^{\beta}_{\alpha}c_{\gamma\beta}$ with the resulting expression, the second term (skew-symmetric tensor) when contracted with the symmetric tensor gives us zero, and (7.22) takes the form

$$\varkappa_t=\rho_{\alpha\beta}t^{\alpha}t^{\beta}=\frac{\rho_{\alpha\beta}d\alpha^{\alpha}d\alpha^{\beta}}{a_{\alpha\beta}d\alpha^{\alpha}d\alpha^{\beta}}=\frac{\mathrm{IV}}{\mathrm{I}}, \tag{7.23}$$

where

$$\rho_{ij}=\frac{c_{i\gamma}b^{\gamma}_{j}+c_{j\gamma}b^{\gamma}_{i}}{2} \tag{7.24}$$

are the coefficients of the so-called fourth quadratic form of the surface. /45 In this case (see (5.14))

$$\begin{aligned}\rho_{11}&=c_{1\gamma}b^{\gamma}_{1}=c_{12}b^{2}_{1}=\sqrt{a}\,b^{2}_{1}=\sqrt{a}b_{1\alpha}a^{\alpha 2}=\\&=\sqrt{a}(b_{11}a^{12}+b_{12}a^{22})=\frac{1}{\sqrt{a}}(b_{12}a_{11}-b_{11}a_{12}).\end{aligned} \tag{7.25}_1$$

In the same way we find the remaining components

$$\begin{aligned}\rho_{12}&=\rho_{21}=\frac{1}{2\sqrt{a}}(b_{22}a_{11}-b_{11}a_{22});\\\rho_{22}&=\frac{1}{\sqrt{a}}(b_{22}a_{12}-b_{12}a_{21}).\end{aligned} \tag{7.25}_{2,3}$$

Finally, the quantity $\varkappa_t$ is called the geodetic curvature. Taking into account equalities (5.51) and (7.9), we find from (7.18)

$$\begin{aligned}\varkappa_t&=-\vec{\nu}\frac{d\vec{t}}{ds_t}=-\nu_{\gamma}\vec{r}^{\gamma}t^{\beta}\frac{\partial(t^{\alpha}\vec{r}_{\alpha})}{\partial\alpha^{\beta}}=\\&=-\nu_{\gamma}t^{\beta}\vec{r}^{\gamma}\left(\vec{r}_{\alpha}\nabla_{\beta}t^{\alpha}+b_{\alpha\beta}t^{\alpha}\vec{n}\right)=-\nu_{\alpha}t^{\beta}\nabla_{\beta}t^{\alpha}\end{aligned}$$

or (7.5)

$$\varkappa_t = -\nu_\alpha \Gamma^\alpha_{(t)} = c_{\beta\alpha} t^\beta \Gamma^\alpha_{(t)}, \tag{7.26}$$

where

$$\Gamma^i_{(t)} = t^\beta \nabla_\beta t^i = \frac{dt^i}{ds_t} + \Gamma^i_{\beta\gamma} t^\beta t^\gamma \tag{7.27}$$

are the contravariant components of the so-called geodetic curvature vector.

The rules for the differentiation of unit orthogonal vectors in the direction of the normal to the contour are derived in a similar fashion. In this connection, as we have stated above, we assume that some family of orthogonal Γ curves is assigned so that the symbol $\frac{d}{ds_\nu}$ means that differentiation is carried out along the line of this family. /46

The differentiation rules have the form

$$\begin{aligned}
&\frac{d\vec{t}}{ds_\nu} = \varkappa_\nu \vec{\nu} - \tau_\nu \vec{n}; \quad \sigma_\nu = \vec{n}\frac{d\vec{\nu}}{ds_\nu} = -\vec{\nu}\frac{d\vec{n}}{ds_\nu};\\
&\frac{d\vec{\nu}}{ds_\nu} = \sigma_\nu \vec{n} - \varkappa_\nu \vec{t}; \quad \tau_\nu = \vec{t}\frac{d\vec{n}}{ds_\nu} = -\vec{n}\frac{d\vec{t}}{ds_\nu};\\
&\frac{d\vec{n}}{ds_\nu} = \tau_\nu \vec{t} - \sigma_\nu \vec{\nu}; \quad \varkappa_\nu = \vec{\nu}\frac{d\vec{t}}{ds_\nu} = -\vec{t}\frac{d\vec{\nu}}{ds_\nu}.
\end{aligned} \tag{7.28}$$

As above, it is shown that

$$\sigma_\nu = b_{\alpha\beta}\nu^\alpha \nu^\beta; \tag{7.29}$$

$$\tau_\nu = -b_{\alpha\beta}t^\alpha \nu^\beta = \tau_t; \tag{7.30}$$

$$\varkappa_\nu = -t_\alpha \Gamma^\alpha_{(\nu)} = c_{\alpha\beta}\nu^\beta \Gamma^\alpha_{(\nu)}, \tag{7.31}$$

where

$$\Gamma^i_{(\nu)} = \nu^\beta \nabla_\beta \nu^i = \frac{d\nu^i}{ds_\nu} + \Gamma^i_{\beta\gamma}\nu^\beta \nu^\gamma. \tag{7.32}$$

In order to establish the geometric meaning of the introduced designations, we recall several basic relationships associated with a spatial curve. As we know the curve in space is assigned by the equation $\vec{R} = \vec{R}(s_t)$, where s_t is the arc length from some initial point. The vector $\vec{\tau} = \frac{d\vec{R}}{ds_t}$ is a unit vector and is tangent to the curve. Differentiating it along the arc we obtain, noting that $\vec{\tau}\frac{d\vec{\tau}}{ds_t} = \frac{1}{2}\frac{d|\vec{\tau}|^2}{ds_t} = 0$,

$$\frac{d\vec{\tau}}{ds_t} = \hat{\sigma}\vec{m}. \tag{7.33}$$

Here $\vec{m}$ is the unit vector of the principal normal (fig. 4), lying in the plane which touches the curve at the considered point and orthogonal with respect to $\vec{\tau}$; $\hat{\sigma} = \hat{\sigma}(s_t)$ is the spatial curvature of the curve. Since the vector of the principal normal is always directed inside the curve, $\hat{\sigma} \geq 0$. The points at which $\hat{\sigma} = 0$ are called the lineal points, because for a straight line $\vec{\tau} = \text{const}$, and according to (7.33), $\hat{\sigma} = 0$. At the lineal points the direction of the principal normal is not determined.

The curve is also associated with the unit vector of the binormal which is given by the equality

$$\vec{b} = [\vec{\tau}, \vec{m}] \quad (\vec{\tau} = [\vec{m}, \vec{b}], \ \vec{m} = [\vec{b}, \vec{\tau}]). \tag{7.34}$$

For this vector according to (7.33)

$$\frac{d\vec{b}}{ds_t} = \left[\frac{d\vec{\tau}}{ds_t}, \vec{m}\right] + \left[\vec{\tau}, \frac{d\vec{m}}{ds_t}\right] = \left[\vec{\tau}, \frac{d\vec{m}}{ds_t}\right].$$

From this we can see that the vector $\frac{d\vec{b}}{ds_t}$ is orthogonal with respect to $\vec{\tau}$. In addition, it is orthogonal with respect to $\vec{b}$ ($|\vec{b}| = 1$). It can therefore be represented in the form /47

$$\frac{d\vec{b}}{ds_t} = -\hat{\tau}\vec{m}.$$

The quantity $\hat{\tau}$ is called the torsion (spatial) of the curve. Furthermore, by differentiating expression $(7.34)_3$ along the arc and utilizing the equalities obtained above, we have

$$\frac{d\vec{m}}{ds_t}=\left[\frac{d\vec{b}}{ds_t},\vec{t}\right]+\left[\vec{b},\frac{d\vec{t}}{ds_t}\right]=-\hat{\tau}\,[\vec{m},\vec{t}]+$$
$$+\hat{\sigma}\,[\vec{b},\vec{m}]=\hat{\tau}\vec{b}-\hat{\sigma}\vec{t}.$$

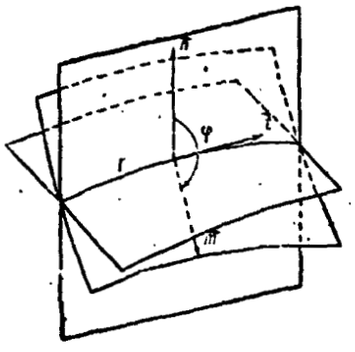

Figure 4

Collecting the resulting relationships we arrive at the well-known Serret-Frenet equations

$$\begin{aligned}
&\frac{d\vec{t}}{ds_t}=\hat{\sigma}\vec{m};\\
&\frac{d\vec{m}}{ds_t}=\hat{\tau}\vec{b}-\hat{\sigma}\vec{t};\quad \hat{\sigma}=\vec{m}\frac{d\vec{t}}{ds_t}=-\vec{t}\frac{d\vec{m}}{ds_t};\\
&\frac{d\vec{b}}{ds_t}=-\hat{\tau}\vec{m};\qquad \hat{\tau}=\vec{b}\frac{d\vec{m}}{ds_t}=-\vec{m}\frac{d\vec{b}}{ds_t}.
\end{aligned}\tag{7.35}$$

Comparing relationships $(7.17)_1$ and $(7.35)_1$ we obtain the equality

$$\hat{\sigma}\vec{m}=\sigma_t\vec{n}-\varkappa_t\vec{\nu},\tag{7.36}$$

from which it follows that the quantity $\varkappa_t$, which we called the geodetic curvature above, characterizes the deviation of the principal normal of the curve lying on the surface from the normal to the surface. The points of the curve at which $\varkappa_t = 0$ are called geodetic. Correspondingly, the curves on the surface whose points are all geodetic are called the geodetic lines. As we can see from (7.36), at the geodetic points of the curve

$$\sigma_t=\pm\hat{\sigma}.\tag{7.37}$$

In this case the sign (+) refers to the case when the positive directions of the normal to the surface and of the principal normal coincide. The sign (-) is used in the opposite case.

The lineal points are also geodetic, since according to (7.36) the equality $\hat{\sigma} = 0$ involves $\sigma_t = 0$, $\varkappa_t = 0$. It follows from this that the straight line is a geodetic line on any surface to which it belongs. A more detailed examination of the problem concerning geodetic lines will be carried out in the next section.

The normal cross section at the considered point of the middle surface M is a curve, which is obtained when the middle surface is intersected by a plane passing through the normal to the surface and tangent to the considered curve. Since the normal cross section is a two-dimensional curve, its principal normal coincides with the normal to the surface. Therefore the point M is geodetic for any normal cross section which passes through it. In addition, it is obvious that equality (7.37) is valid for the normal cross section. The term normal curvature adopted for the quantity σ_t is associated with this fact. /48

From equality (7.36) and from figure 5 it follows that

$$\sigma_t = \hat{\sigma}\cos\varphi, \quad -\varkappa_t = \hat{\sigma}\sin\varphi. \tag{7.38}$$

Thus in the general case the quantity σ_t is the projection of the spatial curvature vector $(\vec{\hat{\sigma}m})$ in the direction of the normal to the surface, while $(-\varkappa_t)$ is its projection on the middle surface. In addition, the following useful equalities follow from (7.38)

$$\begin{aligned} &\cos\varphi = \frac{\sigma_t}{\hat{\sigma}}; \\ \hat{\sigma} = \sqrt{\sigma_t^2 + \varkappa_t^2}; \quad &\sin\varphi = -\frac{\varkappa_t}{\hat{\sigma}}. \end{aligned} \tag{7.39}$$

Furthermore, the following equations follow from figure 5

$$\begin{aligned} &\vec{m} = \vec{n}\cos\varphi + \vec{\nu}\sin\varphi; \quad \vec{n} = \vec{m}\cos\varphi - \vec{b}\sin\varphi; \\ &\vec{b} = -\vec{n}\sin\varphi + \vec{\nu}\cos\varphi; \quad \vec{\nu} = \vec{m}\sin\varphi + \vec{b}\cos\varphi. \end{aligned} \tag{7.40}$$

Differentiating the first of these along the arc we obtain, utilizing the rule for the differentiation of unit orthogonal vectors (7.17)

$$\frac{d\vec{m}}{ds_t} = \cos\varphi \frac{d\vec{n}}{ds_t} + \sin\varphi \frac{d\vec{\nu}}{ds_t} + (-\vec{n}\sin\varphi + \vec{\nu}\cos\varphi)\frac{d\varphi}{ds_t} =$$
$$= \vec{b}\left(\tau_t + \frac{d\varphi}{ds_t}\right) + (-\sigma_t\cos\varphi + \varkappa_t\sin\varphi)\vec{t}.$$

A useful relationship follows from this and from (7.35)

$$\hat{\tau} = \tau_t + \frac{d\varphi}{ds_t}. \tag{7.41}$$

Relationships (7.20) and (7.23) show that the quantities σ_t and τ_t depend actually not on the specific form of the curve drawn through the considered point, but only on the direction of the curve (t^1, t^2). Therefore it is more accurate to speak of the normal curvature and geodetic torsion of a surface in the given direction rather than of the curve. /49

The quantities σ_t and τ_t generally speaking change with direction. Let us find the principal directions of the surface, i.e., the directions for which the normal curvature achieves extremal values. For this purpose it is necessary to find the extremum of expression (7.19) under the condition that

$$a_{\alpha\beta}t^\alpha t^\beta = \frac{a_{\alpha\beta}d\alpha^\alpha d\alpha^\beta}{ds_t^2} = 1. \tag{7.42}$$

As we know, a problem of this type is equivalent to finding the extremal values of the function

$$\Phi(t^1, t^2) = b_{\alpha\beta}t^\alpha t^\beta - \lambda[a_{\alpha\beta}t^\alpha t^\beta - 1] =$$
$$= b_{11}(t^1)^2 + 2b_{12}t^1t^2 + b_{22}(t^2)^2 - \lambda[a_{11}(t^1)^2 + 2a_{12}t^1t^2 + a_{22}(t^2)^2 - 1], \tag{7.43}$$

where λ is a Lagrange multiplier which must be determined. Setting to zero the partial derivatives of the constructed function, we obtain a system of equations

$$\frac{1}{2}\frac{\partial\Phi}{\partial t^1} = 0 = b_{11}t^1 + b_{12}t^2 - \lambda(a_{11}t^1 + a_{12}t^2);$$
$$\frac{1}{2}\frac{\partial\Phi}{\partial t^2} = 0 = b_{21}t^1 + b_{22}t^2 - \lambda(a_{21}t^1 + a_{22}t^2). \tag{7.44}$$

A condition for the existence of a nonzero solution

$$\begin{vmatrix} b_{11}-\lambda a_{11} & b_{12}-\lambda a_{12} \\ b_{21}-\lambda a_{21} & b_{22}-\lambda a_{22} \end{vmatrix}=0$$

is written in the following form with the aid of equalities (5.26) and (5.27)

$$\lambda^2-2H\lambda+K=0. \tag{7.45}$$

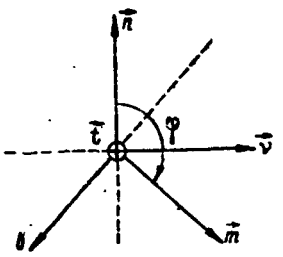

Figure 5

Let us clarify the geometric meaning of the Lagrange multiplier. For this purpose we multiply the first equation of (7.44) by t^1, the second by t^2 and add them. As a result of taking into account equality (7.42) we obtain

$$\sigma_t=b_{\beta\alpha}t^\alpha t^\beta=\lambda(a_{\alpha\beta}t^\alpha t^\beta)=\lambda. \tag{7.46}$$

Thus the values λ obtained during the solution of equation (7.44) are equal to the unknown extremal values σ_t.

From (7.45) we find

$$\sigma_{1,2}=H\pm\sqrt{H^2-K}. \tag{7.47}$$

In order that σ_1 and σ_2 be real, it is necessary that the equality $H^2-K\geq 0$ be satisfied. We shall show that it does indeed take place. Since the quantities contained in the equality have an invariant nature, the calculation steps may be carried out using arbitrary tensor components. Using the mixed ones we obtain the following with the aid of equalities (5.26)-(5.27) and (5.12) /50

$$2H = b_1^1 + b_2^2;$$
$$K = \frac{b}{a} = \frac{b_1^1 b_2^2 - (b_1^2)^2}{a_1^1 a_2^2 - (a_1^2)^2} = b_1^1 b_2^2 - (b_1^2)^2. \tag{7.48}$$

From this

$$4(H^2 - K) = (b_1^1 - b_2^2)^2 + 4(b_1^2)^2 > 0.$$

In this case the equality sign exists only at those points of the surface where

$$b_1^1 = b_2^2, \quad b_1^2 = b_2^1 = 0. \tag{7.49}$$

If we use the covariant components this condition is written in the following manner (verify!)

$$\frac{b_{11}}{a_{11}} = \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}}. \tag{7.50}$$

The points at which conditions (7.49) and (7.50) are satisfied are called spherical (umbilical). According to (7.19) and (7.49), at the spherical points

$$\begin{aligned} \sigma_t = b_{\alpha\beta} t^\alpha t^\beta = b_\beta^\alpha t^\beta t_\alpha = b_1^1 t^1 t_1 + b_2^2 t^2 t_2 = \\ = \frac{b_1^1 + b_2^2}{2}(t^1 t_1 + t^2 t_2) = H \end{aligned} \tag{7.51}$$

(since $t^1 t_1 + t^2 t_2 = \vec{t}\,\vec{t} = 1$). Thus, in the spherical points the normal curvature is constant in all directions, and all of the directions may be considered as the principal direction. We note that the sphere is a surface all of whose points are spherical. Indeed for a sphere (fig. 6)

$$\vec{r} = \vec{a} + R_0 \vec{n}. \tag{7.52}$$

From this

$$a_{ij} = \vec{r}_i \vec{r}_j = R_0 (\vec{r}_i \vec{n}_j) = -R_0 b_{ij}$$

and

$$\frac{b_{ij}}{a_{ij}} = -\frac{1}{R_0}.$$

Therefore, according to criterion (7.50) all points of a sphere are spherical. This justifies the introduced term "spherical point." We note in passing that for a sphere it follows from equations (5.26) and (5.27) that

$$K=\frac{1}{R_0^2};\quad H=-\frac{1}{R_0}. \tag{7.53}$$

The degeneration of a sphere--the plane--also consists of spherical points. Indeed, for it $\vec{n}$ = const. From this it follows that $b_{ij} = 0$ and /51

$$K=0,\quad H=0. \tag{7.54}$$

Let us return to expression (7.47). We have established that the extremal values of curvatures σ_1, σ_2 (principal curvatures) are always real and are different, if the point is not spherical. It follows from (7.47) that

$$\sigma_1\sigma_2=K,\quad \frac{\sigma_1+\sigma_2}{2}=H. \tag{7.55}$$

These equalities establish the geometric meaning of the quantities K and H, which were introduced formally earlier.

Depending on the sign of the Gaussian curvature, the points of the middle surface are referred to three types: elliptical (K > 0), parabolic (K = 0)

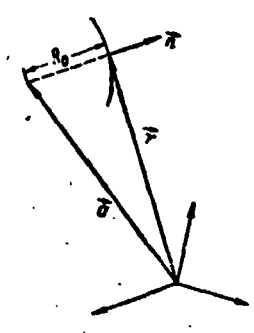

Figure 6

and hyperbolic ($K < 0$). The form of these surface types in the neighborhood of the considered point is shown in figure 7. The surface which has only elliptic points is called the surface of positive Gaussian curvature, the surface with parabolic points is called a surface of zero Gaussian curvature and the surface with only hyperbolic points is called the surface of negative Gaussian curvature.

Equality (5.57) shows that the Gaussian curvature of a surface does not depend on the coefficient of the second quadratic form, and it is determined entirely by the assignment of a_{ij} (by the metrics of the surface). Therefore all surfaces which are converted into each other by bending (without extension) have the same Gaussian curvature. In particular, all of the surfaces which unfold into a plane have zero Gaussian curvature.

It is easy to show that the principal directions are orthogonal with respect to one another. Indeed let

$$\begin{matrix} \sigma_1, & t^1_{(1)}, & t^2_{(1)}, \\ \sigma_2, & t^1_{(2)}, & t^2_{(2)} \end{matrix}$$

be the roots of equation (7.44) and components of unit vectors corresponding to the principal directions. Let us substitute the first quantities into system (7.44). Then we multiply the first identity which is obtained by $t^1_{(2)}$ and the second by $t^2_{(2)}$ and add them. As a result we obtain /52

$$b_{\alpha\beta}t^{\alpha}_{(2)}t^{\beta}_{(1)} - \sigma_1 a_{\alpha\beta}t^{\alpha}_{(2)}t^{\beta}_{(1)} = 0.$$

In the same way we introduce the equality

$$b_{\alpha\beta}t_{(1)}t^{\beta}_{(2)} - \sigma_2 a_{\alpha\beta}t^{\alpha}_{(1)}t^{\beta}_{(2)} = 0.$$

Subtracting the first from the second and utilizing the symmetry of tensors a_{ij} and b_{ij}, we obtain

$$(\sigma_2 - \sigma_1)\, a_{\alpha\beta}t^{\alpha}_{(2)}t^{\beta}_{(1)} = (\sigma_2 - \sigma_1)\, \vec{t}_{(2)}\vec{t}_{(1)} = 0.$$

Since $\sigma_2 - \sigma_1 \neq 0$ (we have already considered spherical points separately),

$$\vec{t}_{(2)}\vec{t}_{(1)} = 0,$$

which is what we were required to show.

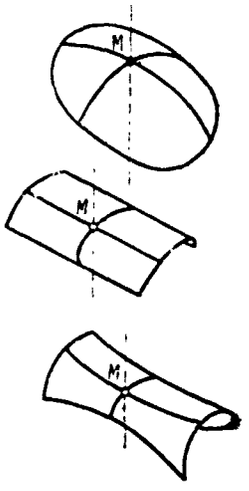

Figure 7

Furthermore, let us multiply equations $(7.44)_{1,2}$, respectively, by ν^1 and ν^2 and add them. As a result of this we obtain ($\lambda = \sigma_t$)

$$b_{\alpha\beta}\nu^\beta t^\alpha = \sigma_t a_{\alpha\beta}\nu^\beta t^\alpha = 0, \tag{7.56}$$

because

$$a_{\alpha\beta}t^\alpha\nu^\beta = \vec{t}\,\vec{\nu} = 0.$$

Comparing the resulting equality with (7.21), we see that the geodetic torsion is absent in the principal directions ($\tau_t = \tau_\nu = \tau$)

$$\tau = 0. \tag{7.57}$$

From $(7.17)_3$ we obtain an easily verified criterion, which makes it possible for us to establish whether a given direction is a principal direction

$$\frac{d\vec{n}}{ds_t} = -\sigma_t\vec{t}. \tag{7.58}$$

This proves the so-called Rodriguez theorem: if a vector is along a principal direction, the increment of the normal vector to the surface is colinear with respect to it.

Applying equality $(7.13)_2$ to tensor b_{ij} and taking into account equality (7.21) which determines τ, we can write condition (7.57) in the following form

$$b_{\alpha\nu} = 0. \tag{7.59}$$

If, however, the coordinate directions are chosen so that they are the principal directions, then conditions (7.59) in terms of these directions will be written in the form $b_{12} = 0$. Adding to this the conditions of orthogonality obtained earlier, we obtain the following for the coordinates constructed in the principal directions[1]

$$b_{12} = 0;\ a_{12} = 0. \tag{7.60}$$

Equality (5.32) shows that these conditions are equivalent to the following /53

$$g_{12} = 0. \tag{7.61}$$

From this we obtain one other convenient geometric criterion: if the shell is referred to a triorthogonal system of coordinates, the coordinate surfaces $\alpha^1 = \text{const}$ intersect the middle surface along the lines of curvature.

In the first part of the book we utilized exclusively the coordinate lines such that tangents to these lines at each point are directed in the principal directions. Such curves are called the lines of principal curvature or more briefly the lines of curvature. As we have already stated, equality (7.57) is satisfied in the principal directions. Expanding it by means of (7.23), (7.25), we obtain the relationship

$$\begin{aligned}(b_{12}a_{11} - b_{11}a_{12})(d\alpha^1)^2 + (b_{22}a_{11} - b_{11}a_{22})\,d\alpha^1 d\alpha^2 + \\ + (b_{22}a_{12} - b_{12}a_{22})(d\alpha^2)^2 = 0,\end{aligned} \tag{7.62}$$

which determines the two principal directions $\left(\frac{d\alpha^1}{d\alpha^2}\right)$. It transpires that this works only when all of its coefficients are equal to 0. However, according

[1] The coordinate lines for which the condition $b_{12} = 0$ is satisfied are usually called conjugated, so that the lines of curvature are both orthogonal and conjugate.

to (7.50) this takes place only at the spherical point where all directions are principal directions. Relationships (7.62) may be looked upon as the differential equations of the lines of curvature.

Let α^1, α^2 be orthogonal coordinates. By converting to physical components (section 6), we first establish by means of equalities (7.9) and (7.10)

$$\begin{aligned} t_{(i)} &= A_i t^i = \frac{A_i d\alpha^i}{ds_t} = \frac{ds_i}{ds_t}; \\ \nu_{(i)} &= A_i \nu^i = \frac{A_i d\alpha^i}{ds_\nu} = \frac{ds_i}{ds_\nu} \end{aligned} \qquad (7.63)$$

$$\left(\begin{matrix} \alpha^1 = \alpha, & \alpha^2 = \beta \\ A_1 = A, & A_2 = B \end{matrix}\right).$$

From this, in accordance with figure 8, we have

$$\begin{aligned} t_{(1)} &= -\sin\gamma; \quad t_{(2)} = \cos\gamma; \\ \nu_{(1)} &= \cos\gamma; \quad \nu_{(2)} = \sin\gamma. \end{aligned} \qquad (7.64)$$

Equations (7.19), (7.21) and (7.29) are now written in terms of the physical components

$$\begin{aligned} \sigma_\nu &= b_{(\mu\delta)} \nu_{(\mu)} \nu_{(\delta)}; \\ \sigma_t &= b_{(\mu\delta)} t_{(\mu)} t_{(\delta)}; \\ \tau_\nu &= \tau_t = -b_{(\mu\delta)} t_{(\mu)} \nu_{(\delta)}. \end{aligned} \qquad (7.65)$$

Assuming successively that $\gamma = 0$ and $\gamma = \frac{\pi}{2}$, we obtain /54

$$b_{(11)} = \sigma_\alpha; \quad b_{(22)} = \sigma_\beta. \qquad (7.66)$$

By taking into account equality (7.37) we can introduce the designations

$$b_{(11)} = -\frac{1}{R_\alpha}; \quad b_{(12)} = \frac{1}{R_{\alpha\beta}}; \quad b_{(22)} = -\frac{1}{R_\beta} \qquad (7.67)$$

(R_α and R_β are radii of curvature of the normal cross sections in the direction of the coordinate lines). On the other hand, the occurrence of the sign (-) associated with the minus in equality (7.37) is explained by the fact that the normal to the middle surface has been directed towards its convexity, so that its positive direction is opposite to the direction of the principal normal to the curve.

By using an analogy with (7.67) we introduce the designations

$$\sigma_\nu = -\frac{1}{R_\nu}; \quad \sigma_t = -\frac{1}{R_t}; \quad \tau_t = -\frac{1}{R_{\nu t}}. \qquad (7.68)$$

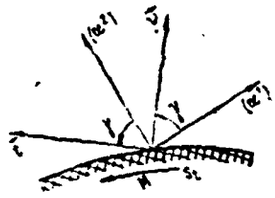

Figure 8

By taking into account (7.68), (7.67) and (7.64), equalities (7.65) give us

$$\begin{aligned}
\frac{1}{R_\nu} &= \frac{\cos^2\gamma}{R_\alpha} - \frac{2\sin\gamma\cos\gamma}{R_{\alpha\beta}} + \frac{\sin^2\gamma}{R_\beta};\\
\frac{1}{R_{\nu t}} &= \sin\gamma\cos\gamma\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right) + \frac{\cos^2\gamma - \sin^2\gamma}{R_{\alpha\beta}};\\
\frac{1}{R_t} &= \frac{\sin^2\gamma}{R_\alpha} + \frac{2\sin\gamma\cos\gamma}{R_{\alpha\beta}} + \frac{\cos^2\gamma}{R_\beta}.
\end{aligned} \tag{7.69}$$

In the lines of curvature

$$b_{(11)} = \sigma_1 = -\frac{1}{R_1};\quad b_{(12)} = \frac{1}{R_{12}} = 0;\quad b_{(22)} = \sigma_2 = -\frac{1}{R_2}. \tag{7.70}$$

Let I and II in figure 9 designate the principal directions, and let α and β be the orthogonal coordinate directions rotated with respect to the principal directions by an angle γ_0. Utilizing equalities (7.64), (7.67) and (7.70),

we obtain the well-known Euler relationships

$$\begin{aligned}
\frac{1}{R_\alpha} &= \frac{\cos^2\gamma_0}{R_1} + \frac{\sin^2\gamma_0}{R_2};\\
\frac{1}{R_{\alpha\beta}} &= \sin\gamma_0\cos\gamma_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right);\\
\frac{1}{R_\beta} &= \frac{\sin^2\gamma_0}{R_1} + \frac{\cos^2\gamma_0}{R_2}.
\end{aligned} \tag{7.71}$$

In the membrane theory and during consideration of asymptotic methods the directions characterized by equality

$$b_{\alpha\beta}t^\alpha t^\beta = 0 \quad (b_{\alpha\beta}d\alpha^\alpha d\alpha^\beta = 0). \tag{7.72}$$

are very important. Such directions are called asymptotic. Taking into account (7.19) we are convinced that in the asymptotic directions

$$\sigma_t = -\frac{1}{R_t} = 0, \tag{7.73}$$

i.e., the normal curvature of the surface is equal to zero. A line with its tangent at every point an asymptotic direction is called asymptotic. Relationship (7.72) may be considered the differential equation of asymptotic lines. Assuming that the asymptotic direction coincides with the line α, we obtain from $(7.71)_1$ /55

$$\operatorname{tg}^2\gamma_0 = -\frac{R_2}{R_1}. \tag{7.74}$$

First of all, it follows from this that the asymptotic directions exist when R_1 and R_2 have different signs, i.e., at the hyperbolic points of a surface.

In this case the following correspond to two asymptotic directions

$$\operatorname{tg}\gamma_0^{(1)} = +\sqrt{-\frac{R_2}{R_1}}; \quad \operatorname{tg}\gamma_0^{(2)} = -\sqrt{-\frac{R_2}{R_1}}. \tag{7.75}$$

From these relationships, which determine the position of the asymptotic directions with respect to the principal directions, it follows that the latter divide into half the angle between the asymptotic directions. At the parabolic points one of the principal curvatures becomes equal to zero. Let us assume for the sake of definitness that $\frac{1}{R_1} = 0$. Then it follows from (7.75) that $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$, i.e., both asymptotic directions coincide with the principal direction which has zero curvature.

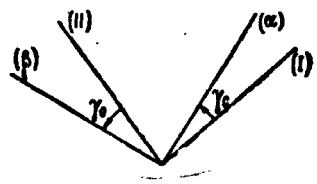

Figure 9

One other useful corollary follows from equalities (7.73) and (7.36): a straight line is an asymptotic line on any surface to which it belongs. Furthermore we obtain a condition under which the coordinate line α^1, = const is an asymptotic line. Assuming for this purpose that $d\alpha^1 = 0$ in equality (7.72), we obtain $b_{22} (d\alpha^2)^2 = 0$, from which it follows that

$$b_{22} = 0. \tag{7.76}$$

Similarly, the condition for the asymptoticity of the line α^2 = const has the form

$$b_{11} = 0. \tag{7.77}$$

In concluding the section we write down the Codazzi-Gauss relationships in terms of physical components. We shall require these in the future. We obtain these relationships by converting to physical components in equalities (5.55) and (5.56) by means of relationships presented in section 6. As a result we obtain the Codazzi-Gauss relationships /56

$$\frac{\partial}{\partial\beta}\left(\frac{A}{R_\alpha}\right)+\frac{1}{B}\frac{\partial}{\partial\alpha}\left(\frac{B^2}{R_{\alpha\beta}}\right)=\frac{1}{R_\beta}\frac{\partial A}{\partial\beta};$$
$$\frac{\partial}{\partial\alpha}\left(\frac{B}{R_\beta}\right)+\frac{1}{A}\frac{\partial}{\partial\beta}\left(\frac{A^2}{R_{\alpha\beta}}\right)=\frac{1}{R_\alpha}\frac{\partial B}{\partial\alpha}; \tag{7.78}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right)=-ABK, \tag{7.79}$$

where

$$K=\frac{1}{R_\alpha R_\beta}-\frac{1}{R_{\alpha\beta}^2}. \tag{7.80}$$

8. Polar (Semigeodetic), Parallel and Cartesian Coordinates on the Surface

Polar coordinates may be introduced on an arbitrary surface just as on a plane surface. In this case the role of straight lines on the surface is played by the geodetic line, which passed through the initial point (pole), while the circles replaced the lines which are orthogonal to the geodetic lines (the so-called geodetic circles). Let us consider in detail the question of geodetic lines on a surface. According to relationships (5.51) and (7.9) for a vector belonging to the surface

$$\vec{U}=U_\alpha\vec{r}^{\alpha}=U^\alpha\vec{r}_\alpha \quad (U_n=0), \tag{8.1}$$

the derivative in the direction established by the unit vector

$$\vec{t}=t^\alpha\vec{r}_\alpha=t_\alpha\vec{r}^{\alpha},$$

is equal to

$$\frac{d\vec{U}}{ds_t}=\frac{\partial\vec{U}}{\partial x^\beta}\frac{dx^\beta}{ds_t}=t^\beta\frac{\partial\vec{U}}{\partial x^\beta}=t^\beta\nabla_\beta U_\alpha\vec{r}^{\alpha}+t^\beta b_{\alpha\beta}U^\alpha\vec{n}. \tag{8.2}$$

The quantity

$$\frac{D\vec{U}}{ds_t}=t^\beta\nabla_\beta U_\alpha\vec{r}^{\alpha}=t^\beta\nabla_\beta U^\alpha\vec{r}_\alpha \tag{8.3}$$

is called the absolute derivative of the surface vector taken with respect to the surface. From equality (8.2) we can see that the absolute derivative with respect to the surface is equal to the projection on the surface of the conventional spatial derivative of a vector belonging to the surface.

Applying equation (8.2) to the unit vector of the tangent to the line, we obtain

$$\frac{d\vec{t}}{ds_t}=\frac{D\vec{t}}{ds_t}+b_{\alpha\beta}t^\alpha t^\beta\vec{n}, \tag{8.4}$$

where

$$\frac{D\vec{t}}{ds_t}=t^\beta\nabla_\beta t^\alpha\vec{r}_\alpha. \tag{8.5}$$

Comparing the last relationship with (7.27), we see that the vector /57 of geodetic curvature $\vec{\Gamma}_{(t)}$ (7.27) is an absolute derivative of the tangent unit orthogonal vector with respect to the surface

$$\vec{\Gamma}_{(t)} = \frac{D\vec{t}}{ds_t} = t^\beta \nabla_\beta t^\alpha \vec{r}_\alpha = \left(\frac{dt^\alpha}{ds_t} + \Gamma^\alpha_{\gamma\beta} t^\beta t^\gamma\right)\vec{r}_\alpha = \left(\frac{d^2\alpha^\alpha}{ds_t^2} + \Gamma^\alpha_{\gamma\beta}\frac{d\alpha^\beta}{ds_t}\frac{d\alpha^\gamma}{ds_t}\right)\vec{r}_\alpha. \tag{8.6}$$

In addition, a comparison of equality (8.4) with $(7.17)_1$ gives us

$$-\varkappa_t \vec{\nu} = \vec{\Gamma}_{(t)}. \tag{8.7}$$

Now expression (7.36), which associates the curvatures of the surface with the curve situated on it, may be written in the following manner

$$\vec{\overset{\frown}{om}} = \sigma_t \vec{n} + \vec{\Gamma}_{(t)}. \tag{8.8}$$

In a particular case when the surface is a plane, any curve on the surface is plain and the vector of its spatial curvature coincides with the vector of the geodetic curvature. If in this case, we introduce the Cartesian coordinates $\alpha^1 = x$, $\alpha^2 = y$, then $\Gamma^k_{ij} = 0$ and

$$\Gamma^1_{(t)} = \frac{d^2x}{ds^2}, \quad \Gamma^2_{(t)} = \frac{d^2y}{ds^2}.$$

The well-known expression for the curvature of a plane curve follows from this

$$|\vec{\Gamma}| = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}.$$

In the preceding section we used the term "geodetic lines of a surface" for lines where the quality $\varkappa_t = 0$ was satisfied at each point. In acccrdance with (8.7) the latter equality may be replaced by the following: $\vec{\Gamma}_{(t)} = 0$. We note that the following relationship follows from $(7.17)_1$ for the geodetic line

$$\frac{d\vec{t}}{ds_t} = \sigma_t \vec{n}, \tag{8.9}$$

whose analog is the Rodriguez theorem (see (7.58)).

Let us clarify the conditions under which the coordinate line is a geodetic line. First of all along this line

$$t^1=\frac{d\alpha^1}{ds_t};\ t^2=0.$$

Furthermore, according to (7.26), (5.19)

$$\varkappa_{(1)}=\sqrt{a}\,t^1\Gamma^2_{(1)},$$

but

$$\Gamma^2_{(1)}=t^\beta\left(\frac{\partial t^2}{\partial\alpha^\beta}+\Gamma^2_{\beta\alpha}t^\alpha\right)=\Gamma^2_{11}\,(t^1)^2.$$

Therefore the condition for the geodetic property of a considered coordinate line is written in the form /58

$$\varkappa_{(1)}=\sqrt{a}\,\Gamma^2_{11}(t^1)^3=0 \text{ or } \Gamma^2_{11}=0. \tag{8.10}$$

In the same way a condition for the geodetic nature of a coordinate line α^2 is the equality

$$\Gamma^1_{22}=0. \tag{8.11}$$

In the future we shall be limited almost exclusively to the consideration of orthogonal systems of coordinates. For these, according to (6.4), conditions (8.10) and (8.11) are replaced, respectively, by the following

$$\frac{\partial A}{\partial\beta}=0; \tag{8.12}$$

$$\frac{\partial B}{\partial\alpha}=0. \tag{8.13}$$

We note that if conditions (8.12) and (8.13) are satisfied simultaneously, we have according to (7.79) equality K = 0. It follows from this that the orthogonal geodetic mesh may be established only on a surface with zero Gaussian curvature (a surface which unfolds into a plane).

We shall show how we can reduce the quadratic form of the surface

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \tag{8.14}$$

into a canonical form

$$ds^2 = dr^2 + \Psi^2(r, \psi)\, d\psi^2. \tag{8.15}$$

Let us determine the transformation of the coordinates

$$r = r(\alpha, \beta), \quad \psi = \psi(\alpha, \beta), \tag{8.16}$$

which realizes a conversion from equations (8.14) to (8.15). (Below it will be shown that this transformation corresponds to a conversion from the initial orthogonal system to a polar system (semigeodetic system)). For this purpose we replace r and ψ in (8.15) by their expressions from (8.16). By equating the resulting expression to (8.14), we obtain an equality which determines the unknown functions

$$A^2 d\alpha^2 + B^2 d\beta^2 = \left(\frac{\partial r}{\partial \alpha} d\alpha + \frac{\partial r}{\partial \beta} d\beta\right)^2 + \Psi^2 \left(\frac{\partial \psi}{\partial \alpha} d\alpha + \frac{\partial \psi}{\partial \beta} d\beta\right)^2$$

and which is transformed to

$$\left[A^2 - \left(\frac{\partial r}{\partial \alpha}\right)^2\right] d\alpha^2 + \left[B^2 - \left(\frac{\partial r}{\partial \beta}\right)^2\right] d\beta^2 - 2 \frac{\partial r}{\partial \alpha} \frac{\partial r}{\partial \beta} d\alpha d\beta = \Psi^2 \left(\frac{\partial \psi}{\partial \alpha} d\alpha + \frac{\partial \psi}{\partial \beta} d\beta\right)^2. \tag{8.17}$$

The right side is the total square of the linear double term with respect to $d\alpha$ and $d\beta$; consequently the left side must also be the same. On the other hand, the necessary and sufficient condition for this is the satisfaction of equality /59

$$\left[A^2 - \left(\frac{\partial r}{\partial \alpha}\right)^2\right]\left[B^2 - \left(\frac{\partial r}{\partial \beta}\right)^2\right] - \left(\frac{\partial r}{\partial \alpha}\right)^2 \left(\frac{\partial r}{\partial \beta}\right)^2 = 0$$

or

$$A^2\left(\frac{\partial r}{\partial \beta}\right)^2+B^2\left(\frac{\partial r}{\partial \alpha}\right)^2=A^2B^2. \tag{8.18}$$

It can be satisfied, if we let

$$\frac{\partial r}{\partial \alpha}=A\cos\gamma,\ \frac{\partial r}{\partial \beta}=B\sin\gamma. \tag{8.19}$$

Eliminating r from this system, we obtain an equation which determines γ (α, β)

$$\frac{\partial}{\partial \beta}(A\cos\gamma)-\frac{\partial}{\partial \alpha}(B\sin\gamma)=0. \tag{8.20}$$

Furthermore, by replacing $\frac{\partial r}{\partial \alpha}$ and $\frac{\partial r}{\partial \beta}$ in (8.17) with expressions (8.19), we obtain

$$A^2\sin^2\gamma d\alpha^2+B^2\cos^2\gamma d\beta^2-2AB\sin\gamma\cos\gamma d\alpha d\beta=$$
$$=\Psi^2\left(\frac{\partial \psi}{\partial \alpha}d\alpha+\frac{\partial \psi}{\partial \beta}d\beta\right)^2$$

or

$$B\cos\gamma d\beta-A\sin\gamma d\alpha=\Psi\left(\frac{\partial \psi}{\partial \alpha}d\alpha+\frac{\partial \psi}{\partial \beta}d\beta\right). \tag{8.21}$$

From this it follows that

$$\frac{\partial \psi}{\partial \alpha}=-\frac{A}{\Psi}\sin\gamma;\ \frac{\partial \psi}{\partial \beta}=\frac{B}{\Psi}\cos\gamma. \tag{8.22}$$

By eliminating ψ we obtain an equation which determines Ψ (α, β)

$$\frac{\partial}{\partial \alpha}\left(\frac{B}{\Psi}\cos\gamma\right)+\frac{\partial}{\partial \beta}\left(\frac{A}{\Psi}\sin\gamma\right)=0. \tag{8.23}$$

The transformation which has been found during the solution of the above system reduces the first quadratic form to the canonical form (8.15). Since in this case condition (8.12) is satisfied, we can maintain that the lines r

(ψ = const) are geodetic lines, while the lines ψ, which are orthogonal with respect to the geodetic line, are geodetic circles. Thus the constructed coordinate system is indeed polar.

Since in the new coordinate system $dr = ds_\gamma$, equation (8.19) may be written in the form

$$\frac{ds_r}{ds_\alpha} = \cos\gamma; \quad \frac{ds_r}{ds_\beta} = \sin\gamma. \tag{8.24}$$

Then it is easy to see from figure 10 that γ is the angle between the initial first coordinate line and the unknown geodetic line. Since $\Psi d\psi = ds_\psi$, equations (8.22) are written in the following manner /60

$$\frac{ds_\psi}{ds_\alpha} = -\sin\gamma; \quad \frac{ds_\psi}{ds_\beta} = \cos\gamma. \tag{8.25}$$

From figure 10 and from equations (8.24) and (8.25) it now follows that the lines r are orthogonal with respect to the lines ψ. In addition, the validity of the converse relationships is clear

$$\begin{aligned} \frac{\partial\alpha}{\partial r} &= \frac{1}{A}\cos\gamma; \quad \frac{\partial\alpha}{\partial\psi} = -\frac{\Psi}{A}\sin\gamma; \\ \frac{\partial\beta}{\partial r} &= \frac{1}{B}\sin\gamma; \quad \frac{\partial\beta}{\partial\psi} = \frac{\Psi}{B}\cos\gamma. \end{aligned} \tag{8.26}$$

The polar system of coordinates on the surface of the general form is introduced in the following manner: a family of geodetic lines is drawn from the pole 0 (fig. 11). As a coordinate it is convenient to take the angle which is measured at the pole in the positive direction from the first coordinate line of the initial system. With this selection the second coordinate is

$$\lim_{r\to 0}\gamma = \psi. \tag{8.27}$$

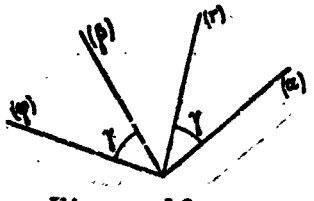

Figure 10

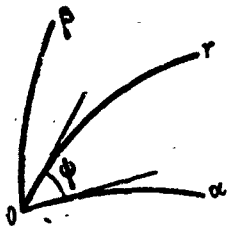

Figure 11

Since as the pole is approached the same increment in the angle $d\psi$ is associated with an ever-decreasing increment in the arc $ds_\psi = \Psi\, d\psi$, any solution of equations (8.23) should be subjected to the following limitation

$$\Psi_0 = \lim_{r \to 0} \Psi(r, \psi) = 0. \tag{8.28}$$

Let us clarify in more detail the structure of the function $\Psi(r, \psi)$ in the neighborhood of the pole. The Codazzi-Gauss relationships (7.78)-(7.79) have the following form in terms of the polar coordinates

$$\begin{aligned}
&\Psi \frac{\partial}{\partial r}\left(\frac{1}{R_\psi}\right) + \frac{\partial}{\partial \psi}\left(\frac{1}{R_{r\psi}}\right) = \left(\frac{1}{R_r} - \frac{1}{R_\psi}\right)\frac{\partial \Psi}{\partial r};\\
&\Psi \frac{\partial}{\partial r}\left(\frac{1}{R_{r\psi}}\right) + \frac{\partial}{\partial \psi}\left(\frac{1}{R_r}\right) = -\frac{2}{R_{r\psi}}\frac{\partial \Psi}{\partial r};\\
&\frac{\partial^2 \Psi}{\partial r^2} = -\Psi K.
\end{aligned} \tag{8.29}$$

If we let r approach zero in the first two equations and remember (8.28), we obtain /61

$$\begin{aligned}
&\frac{\partial}{\partial \psi}\left(\frac{1}{R^0_{r\psi}}\right) = \left(\frac{1}{R^0_r} - \frac{1}{R^0_\psi}\right)\left(\frac{\partial \Psi}{\partial r}\right)_0;\\
&\frac{\partial}{\partial \psi}\left(\frac{1}{R^0_r}\right) = -\frac{2}{R^0_{r\psi}}\left(\frac{\partial \Psi}{\partial r}\right)_0,
\end{aligned} \tag{8.30}$$

where

$$\{R^0_r, R^0_\psi, \ldots\} = \lim_{r \to 0} \{R_r, R_\psi, \ldots\}.$$

If we let r tend to zero in relationships (7.69) and take into account equality (8.27), we obtain

$$\begin{aligned}
&\frac{1}{R^0_r} = \frac{\cos^2 \psi}{R^0_\alpha} - \frac{2 \sin \psi \cos \psi}{R^0_{\alpha\beta}} + \frac{\sin^2 \psi}{R^0_\beta};\\
&\frac{1}{R^0_{r\psi}} = \sin \psi \cos \psi \left(\frac{1}{R^0_\alpha} - \frac{1}{R^0_\beta}\right) + \frac{\cos^2 \psi - \sin^2 \psi}{R^0_{\alpha\beta}};\\
&\frac{1}{R^0_\psi} = \frac{\sin^2 \psi}{R^0_\alpha} + \frac{2 \sin \psi \cos \psi}{R^0_{\alpha\beta}} + \frac{\cos^2 \psi}{R^0_\beta}.
\end{aligned} \tag{8.31}$$

Differentiating we establish the equalities

$$\frac{\partial}{\partial\psi}\left(\frac{1}{R_r^0}\right)=-\frac{2}{R_{r\psi}^0};\ \frac{\partial}{\partial\psi}\left(\frac{1}{R_{r\psi}^0}\right)=\left(\frac{1}{R_r^0}-\frac{1}{R_\psi^0}\right).$$

Substituting these into (8.30), we find

$$\left(\frac{1}{R_r^0}-\frac{1}{R_\psi^0}\right)\left[\left(\frac{\partial\Psi}{\partial r}\right)_0-1\right]=0;$$
$$\frac{2}{R_{r\psi}^0}\left[\left(\frac{\partial\Psi}{\partial r}\right)_0-1\right]=0.$$

From this (if we do not consider points of the spherical type which will be considered in a special manner below) it follows that

$$\left(\frac{\partial\Psi}{\partial r}\right)_0=1. \tag{8.32}$$

Furthermore, it follows from $(8.29)_3$ and (8.28) that

$$\left(\frac{\partial^2\Psi}{\partial r^2}\right)_0=0. \tag{8.33}$$

On the other hand by differentiating $(8.29)_3$ with respect to r and letting /62
r tend to zero we find, by using the relationships just obtained, that

$$\left(\frac{\partial^3\Psi}{\partial r^3}\right)_0=-K_0. \tag{8.34}$$

Finally,

$$\left(\frac{\partial\Psi}{\partial\psi}\right)_0=\frac{\partial\Psi_0}{\partial\psi}=0;\ \left(\frac{\partial^2\Psi}{\partial\psi\partial r}\right)_0=\frac{\partial}{\partial\psi}\left(\frac{\partial\Psi}{\partial r}\right)_0=0. \tag{8.35}$$

In the same way it is shown that

$$\left(\frac{\partial^2\Psi}{\partial\psi^2}\right)_0=\left(\frac{\partial^3\Psi}{\partial\psi^3}\right)_0=\left(\frac{\partial^3\Psi}{\partial\psi^2\partial r}\right)_0=\left(\frac{\partial^3\Psi}{\partial\psi\partial r^2}\right)_0=0.$$

By using the resulting relationships we can represent $\Psi(r, \psi)$ in the form

$$\Psi(r, \psi) = r - \frac{1}{6} K_0 r^3 + r^4 g(r, \psi), \tag{8.36}$$

where, as we recall, K_0 is the value of the Gaussian curvature at the pole of the coordinates.

For surfaces of the general form the actual integration of differential equations (8.19), (8.20), (8.22) and (8.23) is a rather complicated mathematical problem. Here we shall limit ourselves to rather brief considerations of two cases which are of most interest in application.

I. Let us consider a cylindrical shell for which (section 1, Chapter VI, figure 23)

$$A = B = r_0; \quad \alpha = \xi = \frac{x}{r_0}; \quad \beta = \eta = \frac{y}{r_0}. \tag{8.37}$$

By using relationships (8.26), equation (8.20) is transformed in the following manner

$$0 = \frac{1}{r_0}\left(\sin\gamma \frac{\partial\gamma}{\partial\beta} + \cos\gamma \frac{\partial\gamma}{\partial\alpha}\right) = \frac{\partial\gamma}{\partial\beta}\frac{d\beta}{dr} + \frac{\partial\gamma}{\partial\alpha}\frac{d\alpha}{dr} = \frac{\partial\gamma}{\partial r},$$

i.e.,

$$\gamma = \gamma(\psi).$$

The limiting equality (8.27) determines the form of this function

$$\gamma(\psi) \equiv \psi. \tag{8.38}$$

It follows from this that the geodetic lines on a cylindrical surface intersect all generatrices of the cylinder at the same angle and therefore represent helical lines.

Equation (8.23) is transformed in the same way ($\gamma = \psi$)

$$0=\left\{-\frac{\sin\psi}{\Psi}\frac{\partial\psi}{\partial\alpha}+\frac{\cos\psi}{\Psi}\frac{\partial\psi}{\partial\beta}+\cos\psi\frac{\partial}{\partial\alpha}\left(\frac{1}{\Psi}\right)+\sin\psi\frac{\partial}{\partial\beta}\left(\frac{1}{\Psi}\right)\right\}\frac{1}{r_0}=$$
$$=\frac{1}{\Psi^2}\left\{\frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial\psi}+\frac{\partial\psi}{\partial\beta}\frac{\partial\beta}{\partial\psi}\right\}+\frac{\partial}{\partial r}\left(\frac{1}{\Psi}\right)=\frac{1}{\Psi^2}-\frac{1}{\Psi^2}\frac{\partial\Psi}{\partial r}$$

or

$$\frac{\partial\Psi}{\partial r}=1.$$

From this it follows that

$$\Psi = r + \Phi\ (\psi).$$

In view of the limiting equality (8.28), $\Phi\ (\psi) \equiv 0$ and

$$\Psi\ (r,\ \psi) = r. \tag{8.39}$$

Now equalities (8.26) can be written in the form

$$\frac{\partial\xi}{\partial r}=\frac{1}{r_0}\cos\psi;\quad \frac{\partial\xi}{\partial\psi}=-\frac{r}{r_0}\sin\psi;$$
$$\frac{\partial\eta}{\partial r}=\frac{1}{r_0}\sin\psi;\quad \frac{\partial\eta}{\partial\psi}=\frac{r}{r_0}\cos\psi. \tag{8.40}$$

From this we obtain

$$r_0(\xi-\xi_0)=r\cos\psi;\quad r_0(\eta-\eta_0)=r\sin\psi, \tag{8.41}$$

i.e., on the development of the cylinder the introduced system of coordinates transforms into a conventional polar system of coordinates with the pole at the point $(\xi_0,\ \eta_0)$. /63

II. Let us consider briefly the question concerning the polar system of coordinates on a surface of revolution. For this system (section 1, Chapter V, figure 20)

$$\alpha=\theta,\ \beta=\varphi,\ A=R_1(\theta),\ B=R_2(\theta)\sin\theta. \tag{8.42}$$

The significant situation here is that the Lamé parameters and the principal radii of curvature do not depend on the second coordinate $\varphi = \beta$. Let us transform equation (8.20). Multiplying it by $(-\frac{\cos\gamma}{A})$ and making use of equalities (8.26), we obtain

$$0 = \frac{\cos\gamma}{A}\left\{\frac{\partial B\sin\gamma}{\partial\alpha} + A\sin\gamma\,\frac{\partial\gamma}{\partial\beta}\right\} = \frac{\cos\gamma}{A}\left\{\frac{\partial B\sin\gamma}{\partial\alpha}\right\} +$$
$$+\frac{\sin\gamma}{B}B\cos\gamma\,\frac{\partial\gamma}{\partial\beta} = \frac{\cos\gamma}{A}\frac{\partial B\sin\gamma}{\partial\alpha} + \frac{\sin\gamma}{B}\frac{\partial B\sin\gamma}{\partial\beta} = \frac{\partial B\sin\gamma}{\partial r}$$

or

$$B\sin\gamma = C(\psi). \tag{8.43}$$

Here $B = R_2\sin\theta$ is the distance to the surface point from the axis of rotation. Letting $r \to 0$, we determine by means of (8.27) the form of the arbitrary function $C(\psi)$

$$C(\psi) = B_0\sin\psi, \tag{8.44}$$

where B_0 is the distance to the point (which is taken as the pole) from the axis of rotation. The equalities (8.43)-(8.44) represent the Clairaut theorem: on a surface of revolution the geodetic line which intersects the meridians deviates more and more from the meridian as the radius of the parallel decreases. In this case the angle of deviation γ is determined by the Clairaut relationship (8.43)-(8.44). Assuming that $\psi = 0$ in (8.44), we obtain from (8.43) $\gamma = 0$. In other words, the meridians on the surface of rotation are the geodetic lines.

Furthermore, from (8.26) and (8.43) we obtain

$$\frac{d\beta}{d\alpha} = \frac{A}{B}\operatorname{tg}\gamma = \frac{A\cdot C(\psi)}{B\sqrt{B^2 - C^2(\psi)}}.$$

Integrating this equation and replacing the Lamé parameters by their values from (8.42), we obtain the equation for a geodetic line with its pole at the point (θ_0, φ_0) /64

$$\varphi = C(\psi)\int_{\theta_0}^{\theta} \frac{R_1 d\theta}{R_2 \sin\theta \sqrt{R_2^2 \sin^2\theta - C^2(\psi)}} + \varphi_0. \qquad (8.45)$$

Let us now return to the general theory. In considering the geodetic lines the question arises as to whether we can always join two points on the surface with a geodetic line in a unique fashion. The answer to this question in the general case is negative. In particular it is well known that two points on a cylindrical surface may be joined by an infinite number of geodetic (helical) lines which envelop the cylinder. To eliminate such cases a concept of the geodetic neighborhood of a point is introduced, which is interpreted to mean the part of this surface adjoining the given point (pole). This surface is confined inside the curve r = r* (geodetic circle) whose radius is selected in such a way that one and only one geodetic line passes through any point inside of it and connects it to the pole. This sole geodetic line is frequently called the geodetic normal. In the case of a sphere, for example, the geodetic neighborhood of the point M envelops the entire sphere with the exception of the point opposite to M.

Let us assume that two points M and N, one of which lies in the geodetic neighborhood of the other, are connected by some line. The length of this line in polar coordinates having a pole at M is computed by means of equation

$$\int_0^{r_N} \sqrt{dr^2 + \Psi^2 d\psi^2} = \int_0^{r_N} dr \sqrt{1 + \Psi^2 \left(\frac{\partial\psi}{\partial r}\right)^2}. \qquad (8.46)$$

If the line is a geodetic line, then along it $\frac{\partial\psi}{\partial r} = 0$ and its length is equal to r_N. In the opposite case, according to (8.46), its length exceeds r_N. From this follows the remarkable property of geodetic lines: if the point N lies in the geodetic neighborhood of another point M, the normal geodetic line which connects them is shorter than any other line which connects these two points.

Thus the normal geodetic lines, which are the lines of the shortest distances, play the same role on the surface as straight lines do in a plane. In this case the length of the normal geodetic line connecting two points on the surface is adopted as the (geodetic) distance between these points.

In considering some of the problems associated with the edge of the shell it is useful to introduce the so-called system of parallel coordinates. It is constructed in the following manner. We let LM (fig. 12) be the base curve which we shall call the axis of the abscissas. On this axis we select some point 0 (origin of the coordinate system). Through each point of the base curve we pass a geodetic line (geodetic normal) which is orthogonal with respect to it. For the coordinates which fix the position of a point in the

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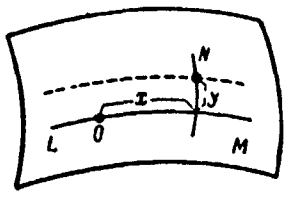

Figure 12

surface we adopt the distance along the axis of the abscissas from the origin of the coordinates to the geodetic normal ($x = s_t$) which passes through the considered point and the distance from this geodetic normal from the point to the base curve $y = s_v$.

Since the lines ($x = \text{const}$) are geodetic lines, the first quadratic form is written in the following manner

$$ds^2 = a_{11}dx^2 + 2a_{12}dxdy + dy^2. \tag{8.47}$$

In addition, according to (8.11) and (5.38) ($a_{22} = 1$)

$$0 = \Gamma^1_{22} = \frac{1}{2}\left\{\frac{\partial a_{2\alpha}}{\partial \alpha^2} + \frac{\partial a_{\alpha 2}}{\partial \alpha^2} - \frac{\partial a_{22}}{\partial \alpha^\alpha}\right\} a^{\alpha 1} = a^{11}\frac{\partial a_{12}}{\partial y}.$$

Thus the quantity a_{12} does not depend on y. However, when $y = 0$, it is equal to zero, because all of the geodetic lines are orthogonal to the axis of the abscissas. From this it follows that

$$a_{12} = 0, \tag{8.48}$$

i.e., the system of coordinates which has been constructed is orthogonal. All coordinate lines $y = \text{const}$ intersect the geodetic normals at a right angle with respect to the base curve and thus are parallel to each other. This actually justifies the term "parallel coordinates." We note that all points on a parallel curve are, by definition, equidistant from the base curve.

The condition of orthogonality (8.48) makes it possible to write the first quadratic form as

$$ds^2 = A^2(x, y)dx^2 + dy^2. \tag{8.49}$$

In this case, since the coordinate x on the base curve coincides with the length of the arc s_t, the Lamé parameter satisfies the condition

$$X|_{y=0}=1. \tag{8.50}$$

Let us consider the structure of the Lamé parameter X (x, y). To do this, we first compute the geodetic curvature of the parallel lines. Because for /66 these lines

$$t^1=\frac{dx}{Xdx}=\frac{1}{X};\quad t^2=0, \tag{8.51}$$

we find from (7.26) and (5.19)

$$\varkappa_t=\sqrt{a}\,\frac{1}{X}\Gamma^2_{(1)}=\sqrt{a}\,\frac{1}{X}t^\beta\left(\frac{\partial t^2}{\partial\alpha^\beta}+\Gamma^2_{\beta\gamma}t^\gamma\right)=$$
$$=\sqrt{a}\,\frac{1}{X}(t^1)^2\Gamma^2_{11}=\frac{\sqrt{a}}{X^3}\Gamma^2_{11}.$$

Since the parallel system of coordinates is orthogonal, we have from (6.2) and (6.4) (A = X, B = 1)

$$\sqrt{a}=X;\ \Gamma^2_{11}=-X\frac{\partial X}{\partial y}$$

and

$$\varkappa_t=-\frac{1}{X}\frac{\partial X}{\partial y}. \tag{8.52}$$

Furthermore, from (7.79) we find

$$\frac{\partial^2 X}{\partial y^2}=-XK. \tag{8.53}$$

If we designate the values of these quantities on the reference curve by $\varkappa_{t0}$ and K_0, i.e., when y = 0, we obtain

$$X=1-\varkappa_{t0}y-\frac{1}{2}K_0y^2+\ldots \tag{8.54}$$

If we select as the reference curve a geodetic line for which $\varkappa_{t0} = 0$, then in the assumed system of coordinates

$$Y\equiv 1,\ X(x,\ y)=1-\frac{1}{2}K(x,\ 0)y^2+\ldots \tag{8.55}$$

Locally (in the neighborhood of the reference line) the constructed system of coordinates may be considered as a Cartesian system and is the analog of the latter on surfaces of the general form. For shells with zero Gaussian curvature ($K = 0$) it is a Cartesian system in the usual interpretation of this term.

We shall adopt the following more convenient designations in the future

$$X=B;\ y=-s_\nu;\ x=s_t. \tag{8.56}$$

Then the relationships characterizing the parallel system of coordinates may be written in the following manner

$$A=1;\ \varkappa_t=\frac{1}{B}\frac{\partial B}{\partial s_\nu};\ B(s_\nu,\ s_t)=1+\varkappa_{t0}s_\nu-\frac{1}{2}K_0s_\nu^2+\ldots \tag{8.57}$$

It is proposed that the reader himself utilize equations (7.26)-(7.27) and (7.64) to obtain the following expression for the geodetic curvature of a curve, which is assigned by the relationship $\gamma = \gamma(s_t\ (\alpha,\ \beta))$ in the arbitrary orthogonal system of coordinates $(\alpha,\ \beta)$

$$\varkappa_t=\frac{d\gamma}{ds_t}+\frac{\cos\gamma}{AB}\frac{\partial B}{\partial\alpha}+\frac{\sin\gamma}{AB}\frac{\partial A}{\partial\beta}. \tag{8.58}$$

It is easy to see that equation $(8.57)_2$ is one of its partial cases (when $\gamma = 0$, $A = 1$).

9. Isothermal Coordinates

In some problems, specifically when determining the values of stress concentrations near a hole, it is convenient to make use of isothermal coordinates. In terms of these the first quadratic form is given by /67

$$ds^2 = \Lambda^2(\xi, \eta)(d\xi^2 + d\eta^2). \tag{9.1}$$

Let us find the transformation of the coordinates

$$\xi = \xi(\alpha, \beta); \quad \eta = \eta(\alpha, \beta), \tag{9.2}$$

which leads the first quadratic form of the surface (referred to the orthogonal coordinates α and β)

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \tag{9.3}$$

to the form (9.1).

Substituting expressions (9.2) into (9.1) and equating the resulting quadratic form to (9.3), we obtain after transformations expressions which are completely analogous to those presented in the preceding section

$$\frac{d\xi}{d\alpha} = \frac{A}{\Lambda}\cos\gamma; \quad \frac{d\xi}{d\beta} = \frac{B}{\Lambda}\sin\gamma; \tag{9.4}$$

$$\frac{\partial\eta}{\partial\alpha} = -\frac{A}{\Lambda}\sin\gamma; \quad \frac{\partial\eta}{\partial\beta} = \frac{B}{\Lambda}\cos\gamma. \tag{9.5}$$

These relationships may be looked upon as the differential equations which determine ξ and η, if we assume that the unknown functions Λ and γ satisfy the conditions of compatibility which follow from (9.4) and (9.5)

$$\begin{aligned} &\frac{\partial}{\partial\beta}\left(\frac{A}{\Lambda}\cos\gamma\right) - \frac{\partial}{\partial\alpha}\left(\frac{B}{\Lambda}\sin\gamma\right) = 0; \\ &\frac{\partial}{\partial\alpha}\left(\frac{B}{\Lambda}\cos\gamma\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{\Lambda}\sin\gamma\right) = 0. \end{aligned} \tag{9.6}$$

Introducing

$$ds_\alpha = A d\alpha; \; ds_\beta = B d\beta; \; ds_\xi = \Lambda d\xi; \; ds_\eta = \Lambda d\eta,$$

we write relationships (9.4)-(9.5) in a somewhat different fashion

$$\frac{ds_\xi}{ds_\alpha}=\cos\gamma;\quad \frac{ds_\xi}{ds_\beta}=\sin\gamma;$$
$$\frac{ds_\eta}{ds_\alpha}=-\sin\gamma;\quad \frac{ds_\eta}{ds_\beta}=\cos\gamma. \tag{9.7}$$

From this and from figure 10 (if we replace r by ξ and ψ by η) we can see that as in the preceding section γ is the angle between the initial first coordinate line and the unknown isothermal line. We may also regard equality (9.7) as the condition for the orthogonality of the unknown isothermal lines ξ and η. The converse relationships follow from the same drawing

$$\frac{\partial\alpha}{\partial\xi}=\frac{\Lambda}{A}\cos\gamma;\quad \frac{\partial\beta}{\partial\xi}=\frac{\Lambda}{B}\sin\gamma;$$
$$\frac{\partial\alpha}{\partial\eta}=-\frac{\Lambda}{A}\sin\gamma;\quad \frac{\partial\beta}{\partial\eta}=\frac{\Lambda}{B}\cos\gamma. \tag{9.8}$$

Let us return to equation (9.4)-(9.5). Eliminating γ and Λ from these /68 we arrive at the system

$$\frac{\partial\xi}{\partial\alpha}=\frac{A}{B}\frac{\partial\eta}{\partial\beta},\quad \frac{\partial\xi}{\partial\beta}=-\frac{B}{A}\frac{\partial\eta}{\partial\alpha}. \tag{9.9}$$

If we eliminate ξ in these equations, we obtain the equation

$$AB\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial\eta}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial\eta}{\partial\beta}\right\}=0, \tag{9.10}$$

which shows that the unknown function $\eta(\alpha, \beta)$ is harmonic.

To determine the isothermal system of coordinates we can pursue the following path. Let us adopt some harmonic function $\eta(\alpha, \beta)$. We determine the corresponding ξ from system (9.9).

We know that the function $\xi(\alpha, \beta)$ obtained in this way is also harmonic. It is easy to see this by eliminating η from (9.9).

The functions $\xi(\alpha, \beta)$ and $\eta(\alpha, \beta)$ constructed in this manner solve the formulated problem. Furthermore, from (9.4) and (9.5) we obtain the following relationships

$$\operatorname{tg}\gamma=\frac{A}{B}\frac{\frac{\partial\xi}{\partial\beta}}{\frac{\partial\xi}{\partial\alpha}}=-\frac{B}{A}\cdot\frac{\frac{\partial\eta}{\partial\alpha}}{\frac{\partial\eta}{\partial\beta}}; \tag{9.11}$$

$$\frac{1}{\Lambda^2}=\frac{1}{A^2}\left(\frac{\partial\xi}{\partial\alpha}\right)^2+\frac{1}{B^2}\left(\frac{\partial\xi}{\partial\beta}\right)^2=\frac{1}{A^2}\left(\frac{\partial\eta}{\partial\alpha}\right)^2+\frac{1}{B^2}\left(\frac{\partial\eta}{\partial\beta}\right)^2. \tag{9.12}$$

Sometimes it is more convenient when determining Λ and γ to utilize relationships which follow from (9.8)

$$\begin{gathered}\Lambda^2=A^2\left(\frac{\partial\alpha}{\partial\xi}\right)^2+B^2\left(\frac{\partial\beta}{\partial\xi}\right)^2=A^2\left(\frac{\partial\alpha}{\partial\eta}\right)^2+B^2\left(\frac{\partial\beta}{\partial\eta}\right)^2;\\ \operatorname{tg}\gamma=\frac{B}{A}\frac{\frac{\partial\beta}{\partial\xi}}{\frac{\partial\alpha}{\partial\xi}}=-\frac{A}{B}\frac{\frac{\partial\alpha}{\partial\eta}}{\frac{\partial\beta}{\partial\eta}}.\end{gathered} \tag{9.13}$$

The term "isothermal coordinates" is associated with the fact that in a stationary temperature field the temperature of the middle surface $T(\alpha, \beta)$ is a harmonic function. Therefore the line $\xi(\alpha, \beta) = \text{const}$, where $\xi(\alpha, \beta)$ is an arbitrary harmonic function, may be considered as a line of uniform temperature--isothermal line--for the temperature field $T = \xi(\alpha, \beta)$. Furthermore, it follows from (9.1) that the isothermal mesh divides the surface into infinitely small squares. The term "isothermal coordinates," which is sometimes used, is associated with this property. /69

Let us assume that the surface is referred to the isothermal coordinates ξ, η and that we wish to transform to some other system of isothermal coordinates ρ, ψ, in which the first quadratic form is written in the following way

$$ds^2=H^2(\rho, \psi)(d\rho^2+d\psi^2) \tag{9.14}$$

We find the relationships corresponding to this transformation by replacing in these relationships the quantities

$$\alpha, \beta; \xi, \eta; A, B; \Lambda \tag{9.15}_1$$

respectively, with

$$\xi, \eta; \rho, \psi; \Lambda, \Lambda; H. \tag{9.15}_2$$

Thus equality (9.9) is transformed into the Cauchy-Riemann relationships

$$\frac{\partial\rho}{\partial\xi}=\frac{\partial\psi}{\partial\eta};\ \frac{\partial\rho}{\partial\eta}=-\frac{\partial\psi}{\partial\xi}. \tag{9.16}$$

which show that harmonic functions $\rho(\xi, \eta)$ and $\psi(\xi, \eta)$ are conjugate and may be considered as the real and imaginary parts of some function of a complex variable

$$\sigma = \rho + i\psi = \sigma(z), \tag{9.17}$$

where

$$z = \xi + i\eta. \tag{9.17'}$$

Conversely

$$z = \xi + i\eta = z(\sigma). \tag{9.17"}$$

The association established between the transformation of isothermal coordinates and the theory of the functions of a complex variable discloses wide possibilities for the application of conformal transformation (ref. 225), which is quite well developed for the two-dimensional problem. For example, let us assume that we wish to determine the value of stress concentration along a hole whose contour does not coincide with the coordinate line. Since we command the coefficients a_1, a_{-1}, ..., a_{-n} in expression

$$z = \xi + i\eta = x(\zeta) = a_1\zeta + \frac{a_{-1}}{\zeta} + \ldots + \frac{a_{-n}}{\zeta^n}, \tag{9.18}$$

where

$$\zeta = e^{\sigma} = e^{\rho + i\psi}, \tag{9.19}$$

we can transform approximately the region with the hole into the "inside of the circle" in such a way that the contour of the hole will be represented by a "circle" $\rho = \rho_0$ (ref. 225).

Separating the real and imaginary parts in relationships (9.18)-(9.19), we obtain

$$\begin{aligned} \xi &= \left(a_1 e^{\rho} + \frac{a_{-1}}{e^{\rho}}\right)\cos\psi + \frac{a_{-2}}{e^{2\rho}}\cos 2\psi + \ldots + \frac{a_{-n}}{e^{n\rho}}\cos n\psi; \\ \eta &= \left(a_1 e^{\rho} - \frac{a_{-1}}{e^{\rho}}\right)\sin\psi - \frac{a_{-2}}{e^{2\rho}}\sin 2\psi - \ldots - \frac{a_{-n}}{e^{n\rho}}\sin n\psi. \end{aligned} \tag{9.20}$$

Assuming that in the above $\rho = \rho_0$ we obtain a parametric equation for the /70
contour of the hole in the "plane" ρ, ψ

$$\xi = \xi(\psi), \quad \eta = \eta(\psi). \tag{9.21}$$

As far as the Lamé parameter H is concerned, according to (9.13) and (9.15) it may be computed by means of equation

$$H^2 = A^2\left[\left(\frac{\partial \xi}{\partial \rho}\right)^2 + \left(\frac{\partial \eta}{\partial \rho}\right)^2\right]. \tag{9.22}$$

Since (see (9.18) and (9.19))

$$\frac{\partial \xi}{\partial \rho} + i\frac{\partial \eta}{\partial \rho} = x'(\zeta)\frac{\partial \zeta}{\partial \rho} = x'(\zeta)\, e^{\rho + i\psi}, \tag{9.23}$$

we have

$$\left(\frac{\partial \xi}{\partial \rho}\right)^2 + \left(\frac{\partial \eta}{\partial \rho}\right)^2 = e^{2\rho}\,|x'(\zeta)|^2$$

and

$$H^2 = A^2(\xi(\alpha, \beta), \eta(\alpha, \beta))\, e^{2\rho}\,[(\operatorname{Re} x'(\zeta))^2 + (\operatorname{Im} x'(\zeta))^2], \tag{9.24}$$

where

$$\begin{aligned} \operatorname{Re} x'(\zeta) &= a_1 - \frac{a_{-1}}{e^{2\rho}}\cos 2\psi - \ldots - \frac{na_{-n}}{e^{(n+1)\rho}}\cos(n+1)\psi; \\ \operatorname{Im} x'(\zeta) &= \frac{a_{-1}}{e^{2\rho}}\sin 2\psi + \ldots + \frac{na_{-n}}{e^{(n+1)\rho}}\sin(n+1)\psi. \end{aligned} \tag{9.25}$$

The second of equalities (9.13), taking into account the substitution (9.15), is written in the form

$$\operatorname{tg}\gamma=\frac{\frac{\partial\eta}{\partial\rho}}{\frac{\partial\xi}{\partial\rho}}. \tag{9.26}$$

Separating the real and imaginary parts in expression (9.23) and substituting them into (9.26) we obtain

$$\operatorname{tg}\gamma=\frac{\operatorname{tg}\psi\operatorname{Re}\varkappa'(\zeta)+\operatorname{Im}\varkappa'(\zeta)}{\operatorname{Re}\varkappa'(\zeta)-\operatorname{tg}\psi\operatorname{Im}\varkappa'(\zeta)}. \tag{9.27}$$

Using this equation to determine γ, it becomes possible for us to compute (see (7.69)) the values of the radii of curvature of the surface in terms of the new coordinates.

In addition to the surface curvature effect, the value of stress concentrations is affected by the geodetic curvature of the hole contour. Let us compute it. First of all, since we are considering the coordinate line $\psi\ (\rho=\rho_0)$

$$ds_t=Hd\psi;\quad \alpha^1=\alpha=\rho;\quad \alpha^2=\beta=\psi;\quad A=B=H \tag{9.28}$$

and

$$\frac{d\alpha^1}{ds_t}=0;\quad \frac{d\alpha^2}{ds_t}=\frac{1}{H}. \tag{9.29}$$

According to (8.6)

$$\begin{aligned}\Gamma^1_{(t)}&=\frac{d^2\alpha^1}{(ds_t)^2}+\Gamma^1_{\gamma\beta}\frac{d\alpha^\gamma}{ds_t}\frac{d\alpha^\beta}{ds_t};\\ \Gamma^2_{(t)}&=\frac{d^2\alpha^2}{(ds_t)^2}+\Gamma^2_{\gamma\beta}\frac{d\alpha^\gamma}{ds_t}\frac{d\alpha^\beta}{ds_t}.\end{aligned} \tag{9.30}$$

Taking into account equality (9.29) and replacing the Christoffels by their /71 expressions from (6.4), we find

$$\Gamma^1_{(t)}=-\frac{1}{H^3}\frac{\partial H}{\partial\rho};\quad \Gamma^2_{(t)}=0 \tag{9.31}$$

and

$$\vec{\Gamma}_{(l)} = -\frac{1}{H^3}\frac{\partial H}{\partial \rho}\vec{r}_\rho = -\frac{1}{H^2}\frac{\partial H}{\partial \rho}\vec{e}_\rho. \tag{9.32}$$

Thus the geodetic curvature of the boundary contour is equal to

$$|\vec{\Gamma}_{(l)}| = \left|\frac{1}{H^2}\frac{\partial H}{\partial \rho}\right|_{\rho=\rho_0} \tag{9.33}$$

and its vector is directed along the normal to the contour.

10. Spiral Surfaces. Helicoid

In this section we shall consider the relatively little known shells with a spiral middle surface. This spiral surface is one which is formed by rotating the arc of some curve around a straight line (axis of rotation), with the simultaneous uniform movement of this arc along the axis. The radius vector of a point on the spiral surface may be represented in the form (figs. 13 and 14)

$$\vec{r} = \rho(u)\vec{e}_\rho + [z(u) + l\varphi]\vec{e}_z. \tag{10.1}$$

Here u is the parameter of the generating curve $\vec{R}(u) = \rho(u)\vec{e}_\rho + z(u)\vec{e}_z$;

l is a constant associated with the pitch of the helical surface L by the relationship

$$l2\pi = L. \tag{10.2}$$

From figure 14 we can establish the equalities

$$\vec{e}_\rho = \vec{e}_x\cos\varphi + \vec{e}_y\sin\varphi;\quad \vec{e}_\varphi = -\vec{e}_x\sin\varphi + \vec{e}_y\cos\varphi, \tag{10.3}$$

which are used to find

$$\frac{\partial\vec{e}_\rho}{\partial\varphi} = \vec{e}_\varphi;\quad \frac{\partial\vec{e}_\varphi}{\partial\varphi} = -\vec{e}_\rho. \tag{10.4}$$

Differentiating $\vec{r}$ and taking into account the last relationships we obtain /72 the equalities

$$\vec{r}_1=\rho'\vec{e}_\rho+z'\vec{e}_z;\quad \vec{r}_2=\rho\vec{e}_\varphi+l\vec{e}_z; \tag{10.5}$$

$$\vec{r}_{11}=\rho''\vec{e}_\rho+z''\vec{e}_z;\quad \vec{r}_{12}=\rho'\vec{e}_\varphi;\quad \vec{r}_{22}=-\rho\vec{e}_\rho \tag{10.6}$$

($(')$ designates that the differentiation is taken with respect to the parameter u). By using these equalities and (5.6), (5.13) and (5.27) we find the expressions which follow from (5.18) and (5.19)

$$\vec{n}=\frac{[\vec{r}_1,\vec{r}_2]}{\sqrt{a}}; \tag{10.7}$$

$$a_{11}=\rho'^2+z'^2;\quad a_{12}=lz';\quad a_{22}=\rho^2+l^2; \tag{10.8}$$

$$a=\rho'^2(\rho^2+l^2)+z'^2\rho^2>0;$$

$$b_{11}=\frac{1}{\sqrt{a}}\{-z'\rho\rho''+z''\rho\rho'\}; \tag{10.9}$$

$$b_{12}=\frac{1}{\sqrt{a}}\{-l\rho'^2\};\quad b_{22}=\frac{1}{\sqrt{a}}\{z'\rho^2\}; \tag{10.10}$$

$$K=\frac{z'\rho^2(-z'\rho\rho''+z''\rho\rho')-l^2(\rho')^4}{[\rho'^2(\rho^2+l^2)+z'^2\rho^2]^2}. \tag{10.11}$$

In these relationships it is frequently convenient (as in the case when we consider a shell of revolution) to use the angle θ as the parameter u (fig. 13). Then

$$\rho'=R\cos\theta;\quad z'=-R\sin\theta, \qquad \text{(Chapter V, 1.6)}$$

where the positive quantity R = R (θ) is the radius of curvature of the generating (plane) curve. The preceding relationships assume a more simple /73 form

$$a_{11}=R^2;\quad a_{12}=-Rl\sin\theta;\quad a_{22}=\rho^2+l^2; \tag{10.12}$$

$$a=R^2\{\rho^2+l^2\cos^2\theta\}>0; \tag{10.13}$$

$$b_{11}=-\frac{R^2\rho}{\sqrt{a}};\quad b_{12}=-\frac{lR^2\cos^2\theta}{\sqrt{a}};\quad b_{22}=-\frac{R\rho^2\sin\theta}{\sqrt{a}}; \tag{10.14}$$

$$K=\frac{\sin\theta\rho^3-l^2R\cos^4\theta}{R(\rho^2+l^2\cos^2\theta)^2}. \tag{10.15}$$

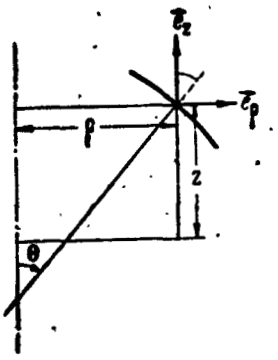

Figure 13

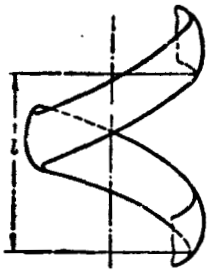

Figure 14

The considered surface is a generalization of the surface of rotation considered in detail in Chapter V. The respective expressions for the surface of revolution are obtained by letting $l = 0$ in these relationships.

Since we have the expressions for a_{ij}, it is not difficult to use (5.14) and (5.38) and obtain the following expressions for the Christoffels

$$\begin{aligned}
&\Gamma_{11}^1 = \frac{R'}{R} - \frac{l^2 \sin\theta\cos\theta}{\rho^2 + l^2\cos^2\theta}; \quad \Gamma_{11}^2 = -\frac{lR\cos\theta}{\rho^2 + l^2\cos^2\theta};\\
&\Gamma_{12}^1 = \frac{l\rho\sin\theta\cos\theta}{\rho^2 + l^2\cos^2\theta}; \quad \Gamma_{12}^2 = \frac{R\rho\cos\theta}{\rho^2 + l^2\cos^2\theta};\\
&\Gamma_{22}^1 = -\frac{\rho\cos\theta(\rho^2 + l^2)}{R(\rho^2 + l^2\cos^2\theta)}; \quad \Gamma_{22}^2 = -\frac{l\rho\sin\theta\cos\theta}{\rho^2 + l^2\cos^2\theta}.
\end{aligned} \tag{10.16}$$

The helicoid, which is the simplest of the helix surfaces, is used most frequently. This is a name given to a helix surface for which the generatrix is a straight line intersecting the axis of rotation at a right angle. For this surface

$$\vec{r} = r\vec{e}_\rho + l\varphi\vec{e}_z. \tag{10.17}$$

We obtain the geometric characteristics for a helicoid by assuming that in (10.8)-(10.11)

$$\rho(u) \equiv r; \quad z(u) \equiv 0; \quad (') = \frac{\partial}{\partial r}; \tag{10.18}$$

$$a_{11} = 1; \quad a_{12} = 0; \quad a_{22} = r^2 + l^2; \tag{10.19}$$

$$a = r^2 + l^2; \tag{10.20}$$

$$b_{11}=b_{22}=0;\quad b_{12}=-\frac{l}{\sqrt{r^2+l^2}}; \tag{10.21}$$

$$K=-\frac{l^2}{(r^2+l^2)^2}. \tag{10.22}$$

We can give these expressions a more convenient form by introducing a new independent variable

$$r = l\,\mathrm{tg}\alpha; \tag{10.23}$$

$$a_{11}=1;\quad a_{12}=0;\quad a_{22}=\frac{l^2}{\cos^2\alpha}=a; \tag{10.24}$$

$$b_{11}=b_{22}=0;\quad b_{12}=-\cos\alpha; \tag{10.25}$$

$$K=-\frac{\cos^4\alpha}{l^2}. \tag{10.26}$$

From these relationships we can see that the coordinate lines on a helicoid are orthogonal to each other and are asymptotic. The helicoid itself is a surface with a negative Gaussian curvature. A series of works are devoted to the computation of a helicoid shell (refs. 226, 229, 230, 231 and 240). /74

CHAPTER VIII. RELATIONSHIPS OF THE THEORY OF SHELLS USING COORDINATES OF THE GENERAL FORM

In the present chapter we briefly present the theory of shells in the invariant form, i.e., in a form which does not depend on a specific system /75
of coordinates associated with the middle surface. In addition to the natural purpose of generalizing the relationships obtained in Part I for the lines of curvature in the case of arbitrary coordinates, it has also been our purpose to reduce to a minimum the number of references to the first part of the book. Thus section 8 may be looked upon as a summary of the basic equations for the orthogonal system of coordinates.

The relationships utilizing physical components in the oblique system of coordinates are not presented. They can be obtained by the reader with the aid of expressions presented at the end of section 6, Chapter VII. Some of the basic relationships for this case are presented in reference 11.

1. Relationships of the Theory of Elasticity Expressed in Coordinates of the General Form

The relationships of the theory of elasticity in an invariant form (which does not depend on the specific system of coordinates) can be obtained most easily by proceeding from the well-known equalities of the theory of elasticity expressed in terms of the Cartesian orthogonal coordinates. Thus the deformation components in the Cartesian system of coordinates have the form

$$e_{ij}=\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right) \tag{1.1}$$

(in this section i, j, α, β = 1, 2, 3).

However, in the invariant form they are written in the following manner

$$e_{ij}=\frac{1}{2}(v_{j|i}+v_{i|j}). \tag{1.2}$$

Indeed, the structure of the quantities (1.2) is such that they are components of a tensor. In the Cartesian system where (section 4, Chapter VII) the covariant derivatives coincide with the conventional ones, they are equal to /76

the components of deformation (1.1). In this manner we constructed the tensor of deformations (see the beginning of section 2, Chapter VII).

We can transform to this tensor directly without turning to the Cartesian coordinates. Indeed, let the radius vector of a point in a deformed body $\vec{R}^*$ be equal to

$$\vec{R}^* = \vec{R} + \vec{V}, \tag{1.3}$$

where $\vec{V}$ designates the displacement vector of points in an elastic body when it is deformed

$$\vec{V} = v^\alpha \vec{R}_\alpha. \tag{1.4}$$

Differentiating expression (1.3), we obtain (see Chapter VII, 4.7)

$$\vec{R}_i^* = \vec{R}_i + \frac{\partial \vec{V}}{\partial \theta^i} = \vec{R}_i + v_{/i}^\alpha \vec{R}_\alpha. \tag{1.5}$$

Let us now compute the components of a metric tensor for the deformed space referred to the same coordinates which existed before the deformation

$$\begin{aligned} g_{ij}^* = \vec{R}_i^* \vec{R}_j^* = \left(\vec{R}_i + v_{/i}^\alpha \vec{R}_\alpha\right)\left(\vec{R}_j + v_{/j}^\beta \vec{R}_\beta\right) = \\ = \vec{R}_i \vec{R}_j + v_{/i}^\alpha \left(\vec{R}_j \vec{R}_\alpha\right) + v_{/j}^\beta \left(\vec{R}_i \vec{R}_\beta\right) + \underline{v_{/i}^\alpha v_{/j}^\beta \left(\vec{R}_\alpha \vec{R}_\beta\right)}. \end{aligned} \tag{1.6}$$

If we remain within the framework of the linear theory, we should drop the underlined terms which are nonlinear with respect to the displacements. Utilizing equalities (1.2) we then find

$$g_{ij}^* = g_{ij} + g_{j\alpha} v_{/i}^\alpha + g_{i\beta} v_{/j}^\beta = g_{ij} + v_{j/i} + v_{i/j} = g_{ij} + 2e_{ij}. \tag{1.7}$$

From this it follows that

$$e_{ij} = \frac{g_{ij}^* - g_{ij}}{2}. \tag{1.8}$$

Thus the deformation components are equal to half of the increments (during deformation) of components for the basic metric tensor. This relationship makes

it possible to establish the geometric meaning for the deformation components. Thus, according to (Chapter VII, 3.25)

$$\frac{ds_i^* - ds_i}{ds_i} = \frac{\sqrt{g_{ii}^*} - \sqrt{g_{ii}}}{\sqrt{g_{ii}}} = \sqrt{1 + 2\frac{e_{ii}}{g_{ii}}} - 1 \approx \frac{e_{ii}}{\sqrt{g_{ii}}\sqrt{g_{ii}}},$$

i.e., the physical components of the deformation tensor (with the same indices) may be identified with the relative elongations of the fibers which are parallel to the respective coordinate lines.

Furthermore, in accordance with (Chapter VII, 3.24) the cosine of the angle between the two directions $(d\theta^1, d\theta^2, d\theta^3)$ and $(\delta\theta^1\ \delta\theta^2, \delta\theta^3)$ is determined by /77 equation

$$\cos\chi = \frac{g_{\alpha\beta}d\theta^\alpha\delta\theta^\beta}{\sqrt{g_{\alpha\beta}d\theta^\alpha d\theta^\beta}\sqrt{g_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta}}. \tag{1.9}$$

From this we find

$$\sin\chi = \sqrt{1-\cos^2\chi} = \frac{\sqrt{g_{\alpha\beta}d\theta^\alpha d\theta^\beta \cdot g_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta - \left(g_{\alpha\beta}d\theta^\alpha\delta\theta^\beta\right)^2}}{\sqrt{g_{\alpha\beta}d\theta^\alpha d\theta^\beta}\cdot\sqrt{g_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta}}. \tag{1.10}$$

During the deformation process the angle between the considered directions assumes some value χ^*, decreasing by an amount

$$\omega = \chi - \chi^*. \tag{1.11}$$

In order to compute ω we make use of the known reduction equation

$$\sin\omega = \sin(\chi - \chi^*) = \cos\chi^*\sin\chi - \sin\chi^*\cos\chi. \tag{1.12}$$

The quantities $\cos\chi^*$ and $\sin\chi^*$ contained in it are computed by means of equations (1.9) and (1.10), if in the latter we replace g_{ij} by g^*_{ij}. Replacing g^*_{ij} by their expressions (1.7) and dropping terms which are nonlinear with respect to the deformation components we obtain, after simple transformations

$$\cos\chi^* = \cos\chi\left\{1 - \frac{e_{\alpha\beta}d\theta^\alpha d\theta^3}{g_{\alpha\beta}d\theta^\alpha d\theta^3} - \frac{e_{\alpha\beta}\delta\theta^\alpha\delta\theta^3}{g_{\alpha\beta}\delta\theta^\alpha\delta\theta^3} + 2\frac{e_{\alpha\beta}d\theta^\alpha\delta\theta^3}{g_{\alpha\beta}d\theta^\alpha\delta\theta^3}\right\};$$

$$\sin\chi^* = \sin\chi\left\{1 - \frac{e_{\alpha\beta}d\theta^\alpha d\theta^\beta}{g_{\alpha\beta}d\theta^\alpha d\theta^3} - \frac{e_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta}{g_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta} + \right.$$

$$\left. + \frac{g_{\alpha\beta}d\theta^\alpha d\theta^\beta\, e_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta + g_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta\, e_{\alpha\beta}d\theta^\alpha d\theta^3 - 2g_{\alpha 3}d\theta^\alpha\delta\theta^\beta\, e_{\alpha\beta}d\theta^\alpha\delta\theta^\beta}{g_{\alpha\beta}d\theta^\alpha d\theta^\beta g_{\alpha 3}\delta\theta^\alpha\delta\theta^3 - (g_{\alpha 3}d\theta^\alpha\delta\theta^3)^2}\right\}.$$

By substituting the resulting expression into (1.12) and taking into account the fact that $\sin\omega \approx \omega$ because ω is small, we obtain the following equality

$$\omega = \frac{2e_{\alpha\beta}d\theta^\alpha\delta\theta^3 - g_{\alpha 3}d\theta^\alpha\delta\theta^3\left(\frac{e_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta}{g_{\alpha\beta}\delta\theta^\alpha\delta\theta^3} + \frac{e_{\alpha\beta}d\theta^\alpha d\theta^3}{g_{\alpha\beta}d\theta^\alpha d\theta^3}\right)}{\sqrt{g_{\alpha\beta}d\theta^\alpha d\theta^\beta g_{\alpha\beta}\delta\theta^\alpha\delta\theta^3 - (g_{\alpha 3}d\theta^\alpha\delta\theta^3)^2}}. \qquad (1.13)$$

For orthogonal directions ($g_{\alpha\beta}d\theta^\alpha\delta\theta^\beta = 0$) it is substantially simplified

$$\omega = 2\frac{e_{\alpha\beta}d\theta^\alpha\delta\theta^\beta}{\sqrt{g_{\alpha\beta}d\theta^\alpha d\theta^\beta}\sqrt{g_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta}}. \qquad (1.14)$$

In particular, if the considered orthogonal directions coincide with the coordinate directions /78

$$d\theta^i \neq 0;\ d\theta^j = d\theta^k = 0;\quad \delta\theta^j \neq 0;\quad \delta\theta^i = \delta\theta^k = 0\quad (i \neq j \neq k),$$

then

$$\omega^{(A)} = \frac{2e_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}}. \qquad (1.15)$$

Thus, in the case of orthogonal coordinates the doubled physical components (with different indices) of the deformation tensor may be identified with the value of the decrease (during the deformation process) in the angles between the corresponding coordinate lines.

Let us now turn to the determination of the rotation angles. For this purpose we consider the vector product $[\vec{R}_i, \vec{R}^*_i]$. According to the determination of the vector product of two vectors it represents a vector which is orthogonal with respect to $\vec{R}_i$ and $\vec{R}^*_i$, equal in value

$$|\vec{R}_i| \cdot |\vec{R}_i^*| \sin \omega_{(i)},$$

where $\omega_{(i)}$ is the angle between $\vec{R}_i$ and $\vec{R}^*_i$, i.e., the angle of rotation of the i-th coordinate axis.

We designate by $\vec{\omega}_{(i)}$ the vector equal in magnitude to $\omega_{(i)}$ and codirected with the vector product $[\vec{R}_i, \vec{R}^*_i]$. Since the angle is small $\sin \omega_{(i)} = \omega_{(i)}$; it then follows from the preceding expression that

$$\vec{\omega}_{(i)} = \frac{[\vec{R}_i, \vec{R}_i^*]}{|\vec{R}_i||\vec{R}_i^*|}.$$

By using equalities (Chapter VII, 3.32) (1.5), we compute the introduced quantity

$$[\vec{R}_i, \vec{R}_i^*] = [\vec{R}_i, \vec{R}_i + v_{ii}^{\alpha}\vec{R}_\alpha] = v_{ii}^{\alpha}[\vec{R}_i, \vec{R}_\alpha] = \varepsilon_{i\alpha\beta}v_{ii}^{\alpha}\vec{R}^{\beta};$$
$$|\vec{R}_i| = \sqrt{g_{ii}};\ |\vec{R}_i^*| = \sqrt{g_{ii}}\left(1 + \frac{e_{ii}}{g_{ii}}\right) \approx \sqrt{g_{ii}}.$$

From this follows that

$$\vec{\omega}_{(i)} = \frac{\varepsilon_{i\alpha\beta}v_{ii}^{\alpha}\vec{R}^{\beta}}{g_{ii}}. \qquad (1.16)$$

It is easy to see that the extensive variable

$$\vec{\omega}_{ij} = \varepsilon_{i\alpha\beta}v_{ij}^{\alpha}\vec{R}^{\beta} \qquad (1.17)$$

is a doubly covariant tensor with vector components which may be appropriately called the tensor of the rotation vectors. By representing equality (1.16) in the form /79

$$\vec{\omega}_{(ij)} = \frac{\vec{\omega}_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}}, \qquad (1.18)$$

we see that the physical components (with the same indices) of the introduced tensor are equal to the rotation vectors of fibers which coincide with the corresponding coordinate lines.

Raising and lowering the indices we arrive at the mixed tensor

$$\vec{\omega}^{i}_{.j} = \varepsilon^{i\alpha\beta} v_{\alpha|j} \vec{R}_{\beta}. \tag{1.19}$$

From this equality, in particular, we can see that the extensive variable

$$\omega^{ik}_{..j} = \varepsilon^{i\alpha k} v_{\alpha|j} \tag{1.20}$$

represents a tensor which we shall call the tensor of the rotation angles. From these statements it follows that its physical components $\omega_{(iki)}$ are identified with the rotation of the fiber which coincides with the i-th coordinate axis, around the k-th coordinate axis.

Generally speaking, the value of the rotation angle varies as we go from one fiber to another. In order to give certain integral characteristics for the body, we introduce the concept of the mean rotation of a coordinate layer.

Thus the mean rotation of the coordinate layer θ^3 = const will be interpreted to mean the half-sum of the rotation of two fibers which are parallel, respectively, to the θ^1 and θ^2 axis around θ^3, i.e.,

$$\omega^3 = \frac{1}{2}\left(\omega^{13}_{..1} + \omega^{23}_{..2}\right) = \frac{1}{2}\left(\varepsilon^{1\alpha 3} v_{\alpha|1} + \varepsilon^{2\alpha 3} v_{\alpha|2}\right) = \frac{1}{2}\varepsilon^{123}\left(v_{2|1} - v_{1|2}\right).$$

We also determine the mean rotations of the remaining coordinate layers and become convinced that the introduced quantitites are the components of the contravariant vector of rotation for the neighborhood of the body's point

$$\omega^k = \frac{1}{2}\omega^{ak}_{..a} = \frac{1}{2}\varepsilon^{\beta ak} v_{\alpha|\beta}. \tag{1.21}$$

The examination of the equilibrium conditions for the tetrahedron element (fig. 2) isolated from the deformed body leads to the tensor of stresses. We designate by $\vec{K}_{\nu}$ the vector of forces acting on a unit area of the cut boundary. We designate by $\vec{K}_{(i)}$ the analagous quantities for the side boundaries. The

conditions that the principal vector of forces acting on the isolated tetra- /80
hedron is equal to zero lead to the relationship

$$\vec{K}_{(\nu)}dS_\nu = \sum_{i=1}^{3} K_{(i)}dS_i. \tag{1.22}$$

From here, by means of (Chapter VII, 3.37), we find

$$\vec{K}_{(\nu)} = \sum_{i=1}^{3} \vec{K}_{(i)} \frac{dS_i}{dS_\nu} = \sum_{i=1}^{3} \nu_i \sqrt{g^{ii}}\, \vec{K}_{(i)}. \tag{1.23}$$

Since during contraction with the covariant vector of the normal the quantity $\sqrt{g^{ii}}\,\vec{K}_{(i)}$ produces an invariant vector,[1] it may be looked upon as a tensor of the first rank with vector components and may be represented in the form

$$\sqrt{g^{ii}}\vec{K}_{(i)} = \sigma^{i\beta}\vec{R}_\beta. \tag{1.24}$$

It follows from this that the quantity σ^{ij} determines the tensor of rank two--the so-called stress tensor. In this case we find from (1.23) an expression for the stress tensor $(\vec{K}_{(\nu)})$

$$\begin{aligned} &\vec{K}_{(\nu)} = \nu_\alpha \sigma^{\alpha\beta}\vec{R}_\beta; \\ &K^i_{(\nu)} = \nu_\alpha \sigma^{\alpha i}; \quad K^{(\nu)}_i = \nu^\alpha \sigma_{\alpha i}, \end{aligned} \tag{1.25}$$

which acts on the surface determined by the normal

$$\vec{\nu} = \nu_\alpha \vec{R}^\alpha = \nu^\alpha \vec{R}_\alpha. \tag{1.26}$$

Assuming that the following expressions from the theory of elasticity in the Cartesian system of coordinates are well known

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[1] When $\vec{\nu}$ is fixed.

equilibrium equations:

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}+F_x=0,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

the law of parity for the tangential stresses:

$$\sigma_{xy}=\sigma_{yx},\quad \sigma_{xz}=\sigma_{zx},\quad \sigma_{yz}=\sigma_{zy},$$

Hooke's law for a homogeneous isotropic elastic body:

$$\sigma_{xx}=\lambda\theta+2\mu e_{xx},\quad \sigma_{xy}=2\mu e_{xy}$$
$$\cdots\cdots\cdots\quad\cdots\cdots$$
$$\cdots\cdots\cdots\quad\cdots\cdots$$
$$(\theta=e_{xx}+e_{yy}+e_{zz}),$$

expressions for the deformation energy and its variations:

$$V=\frac{1}{2}\iiint(\sigma_{xx}e_{xx}+\ldots+\sigma_{zz}e_{zz})\,dxdydz,$$
$$\delta V=\iiint(\sigma_{xx}\delta e_{xx}+\ldots+\sigma_{zz}\delta e_{zz})\,dxdydz,$$

we impart to them an invariant form by means of (Chapter VII, 3.35)

$$\sigma^{i\alpha}_{|\alpha}+F^i=0; \tag{1.27}$$
$$\sigma_{ij}=\sigma_{ji}; \tag{1.28}$$
$$\sigma_{ij}=\lambda\theta g_{ij}+2\mu e_{ij}\quad(\theta=g^{\alpha\beta}e_{\alpha\beta}); \tag{1.29}$$
$$V=\frac{1}{2}\iiint\sigma^{\alpha\beta}e_{\alpha\beta}\sqrt{g}\,d\theta^1d\theta^2d\theta^3; \tag{1.30}$$
$$\delta V=\iiint\sigma^{\alpha\beta}\delta e_{\alpha\beta}\sqrt{g}\,d\theta^1d\theta^2d\theta^3. \tag{1.31}$$

2. Law Governing Variations of Displacements Across the Thickness of the Shell. Deformations of the Shell

We began the development of the theory of shells by establishing the law for the variation of displacements along the thickness. First of all, we find the following relationships for the angles of rotation of the neighborhood /81

associated with the point on the middle surface by utilizing (1.21) and (Chapter VII, 5.18 and 5.19)

$$\Omega^1 = \omega^1|_{\zeta=0} = \frac{1}{2\sqrt{a}}[(v_{3,2})_0 - (v_{2,3})_0];$$
$$\Omega^2 = \omega^2|_{\zeta=0} = \frac{1}{2\sqrt{a}}[(v_{1,3})_0 - (v_{3,1})_0] \tag{2.1}$$

(the zero indicates that the corresponding quantity is computed on the middle surface). Furthermore, from equalities (1.2) we obtain

$$e_{13}|_{\zeta=0} = \frac{1}{2}[(v_{3/1})_0 + (v_{1,3})_0];$$
$$e_{23}|_{\zeta=0} = \frac{1}{2}[(v_{3/2})_0 + (v_{2/3})_0];$$
$$e_{33}|_{\zeta=0} = (v_{3/3})_0.$$

Comparing these expressions with the preceding ones, we see that the angles of rotation may substantially exceed the components of deformation written out above (this takes place for thin shells) only in the case when (see section 1, Chapter II)

$$(v_{3/1})_0 + (v_{1,3})_0 \approx 0; \quad (v_{3/2})_0 + (v_{2/3})_0 \approx 0, \quad (v_{3,3})_0 \approx 0. \tag{2.2}$$

From this and from equalities (Chapter VII, 5.48) we obtain the following approximate relationships

$$(v_{1/3})_0 = -(v_{3/1})_0 = -\left[\frac{\partial (v_3)_0}{\partial \alpha^1} + b_1^\alpha (v_\alpha)_0\right];$$
$$(v_{2/3})_0 = -(v_{3/2})_0 = -\left[\frac{\partial (v_3)_0}{\partial \alpha^2} + b_2^\alpha (v_\alpha)_0\right]. \tag{2.3}$$

We introduce the quantities

$$u_1 = (v_1)_0; \quad u_2 = (v_2)_0; \quad w = (v_3)_0, \tag{2.4}$$

which are components of the vector for the displacements of points on the middle surface /82

$$\vec{U}=u_\alpha\vec{r}^{\,\alpha}+w\vec{n}=u^\alpha\vec{r}_\alpha+w\vec{n}. \tag{2.5}$$

By using these relationships for the angles of rotations (2.1) we find the following expressions

$$\Omega^1=-\frac{1}{\sqrt{a}}\vartheta_2;\quad \Omega^2=\frac{1}{\sqrt{a}}\vartheta_1, \tag{2.6}$$

where

$$\vartheta_i=-\left(\frac{\partial w}{\partial\alpha^i}+b_i^\alpha u_\alpha\right). \tag{2.7}$$

They can be written (see Chapter VII, 5.19) in a slightly different form

$$\Omega^i=c^{\alpha i}\vartheta_\alpha, \tag{2.8}$$

which shows their vector nature.

The angle of rotation around the normal to the middle surface, according to (1.21), is equal to (see Chapter VII, 5.48)

$$\Omega_n=\omega^3|_{\zeta=0}=\frac{1}{2}c^{\beta\alpha}\left(\nabla_\beta u_\alpha-\underline{b_{\alpha\beta}w}\right)=\frac{1}{2}c^{\beta\alpha}\nabla_\beta u_\alpha. \tag{2.9}$$

The underlined terms drop out, because they represent the contraction of the symmetric and skew-symmetric tensors (section 2, Chapter VII). However, in the expanded form Ω_n is written in the following manner (Chapter VII, 5.42)

$$\Omega_n=\frac{1}{2\sqrt{a}}\left(\nabla_1u_2-\nabla_2u_1\right)=\frac{1}{2\sqrt{a}}\left(\frac{\partial u_2}{\partial\alpha^1}-\frac{\partial u_1}{\partial\alpha^2}\right). \tag{2.10}$$

The vector of the angles of rotation for the neighborhood of point on the middle surface may then be represented in the form

$$\vec{\Omega}=c^{\alpha\beta}\vartheta_\alpha\vec{r}_\beta+\Omega_n\vec{n}. \tag{2.11}$$

Let us expand the components of the displacement vector in a series containing powers of ζ and limit ourselves (considering the thinness of the shell's walls) to two terms of the expansion

$$v_1=(v_1)_0+\zeta(v_{1/3})_0;$$
$$v_2=(v_2)_0+\zeta(v_{2/3})_0;$$
$$v_3=(v_3)_0+\zeta(v_{3/3})_0.$$

Utilizing equalities (2.2) in the above designations we obtain a law for the variation of displacements along the thickness of the shell

$$\begin{aligned} v_1&=u_1+\zeta\vartheta_1;\\ v_2&=u_2+\zeta\vartheta_2;\\ v_3&=w. \end{aligned} \tag{2.12}$$

It may also be given a vector form

$$\vec{V}=\vec{U}+\zeta\vartheta_\alpha\vec{r}^{\,\alpha}. \tag{2.13}$$

Here $\vec{V} = v_\alpha\vec{r}^{\,\alpha} + w\vec{n}$ is the displacement vector for the points of the shell referred to the coordinate system of the middle surface. /83

By means of relationships (Chapter VII, 5.21 and 5.23) we establish the equality

$$\left[\vec{\Omega},\ \vec{n}\right]=\left[c^{\alpha\beta}\vartheta_\alpha\vec{r}_\beta+\Omega_n\vec{n},\ \vec{n}\right]=\vartheta_\alpha\left(c^{\alpha\beta}c_{\gamma\beta}\right)\vec{r}^{\,\gamma}=\vartheta_\alpha\delta^\alpha_\gamma\vec{r}^{\,\gamma}=\vartheta_\alpha\vec{r}^{\,\alpha},$$

which makes it possible to give (2.13) a slightly different form

$$\vec{V}=\vec{U}+\left[\vec{\Omega},\ \zeta\vec{n}\right]. \tag{2.14}$$

Let us now consider the deformation of the shell. According to equalities (2.13) and (Chapter VII, 5.1)

$$\vec{R}^*=\vec{R}+\vec{V}=\vec{r}+\zeta\vec{n}+\vec{U}+\zeta\vartheta_\alpha\vec{r}^{\,\alpha}. \tag{2.15}$$

From this

$$g^*_{ij}=\vec{R}^*_i\cdot\vec{R}^*_j=\left\{\vec{r}_i+\zeta\vec{n}_i+\frac{\partial\vec{U}}{\partial\alpha^i}+\zeta\frac{\partial}{\partial\alpha^i}\left(\vartheta_\alpha\vec{r}^\alpha\right)\right\}\times$$
$$\times\left\{\vec{r}_j+\zeta\vec{n}_j+\frac{\partial\vec{U}}{\partial\alpha^j}+\zeta\frac{\partial}{\partial\alpha^j}\left(\vartheta_\beta\vec{r}^\beta\right)\right\}. \tag{2.16}$$

In the preceding section we have shown that the deformation of the shell is determined completely as soon as we know the variations (changes) in the components of the basic metric tensor. On the other hand, we can see from relationships (Chapter VII, 5.32 and 5.26) that the variation of the metric tensor is determined in terms of the variations of the first and second surface tensors $\delta a_{ij} = a^*_{ij} - a_{ij}$, $\delta b_{ij} = b^*_{ij} - b_{ij}$. Introducing the designations.

$$\varepsilon_{ij}=\frac{\delta a_{ij}}{2}=\frac{a^*_{ij}-a_{ij}}{2};\quad \beta_{ij}=\delta b_{ij}=b^*_{ij}-b_{ij}, \tag{2.17}$$

we see from (Chapter VII, 5.32)

$$\varepsilon_{ij}=\frac{1}{2}\left(g^*_{ij}-g_{ij}\right)\Big|_{\zeta=0};\quad \beta_{ij}=-\frac{1}{2}\left\{\frac{\partial\left(g^*_{ij}-g_{ij}\right)}{\partial\zeta}\right\}\Big|_{\zeta=0}. \tag{2.18}$$

Let us return to equality (2.16). Expanding it by means of (Chapter VII, 5.51 and 5.28) and (2.5) and retaining only those terms which are linear with respect to displacements and their derivatives, we obtain

$$g^*_{ij}-g_{ij}=2\left\{\frac{1}{2}\left(\nabla_i u_j+\nabla_j u_i\right)-b_{ij}w\right\}+$$
$$+\zeta\left\{-b^\alpha_i\left(\nabla_j u_\alpha-b_{j\alpha}w\right)-b^\alpha_j\left(\nabla_i u_\alpha-b_{i\alpha}w\right)+\right.$$
$$\left.+\nabla_j\vartheta_i+\nabla_i\vartheta_j\right\}+\zeta^2\left\{-b^\alpha_i\nabla_j\vartheta_\alpha-b^\alpha_j\nabla_i\vartheta_\alpha\right\}. \tag{2.19}$$

From here, according to (2.18), we have

$$\varepsilon_{ij}=\frac{1}{2}\left(\nabla_i u_j+\nabla_j u_i\right)-b_{ij}w;$$
$$\beta_{ij}=-\frac{1}{2}\left\{\nabla_j\vartheta_i+\nabla_i\vartheta_j-b^\alpha_i\nabla_j u_\alpha-\right.$$
$$\left.-b^\alpha_j\nabla_i u_\alpha+\left(b^\alpha_i b_{j\alpha}+b^\alpha_j b_{i\alpha}\right)\dot{w}\right\}. \tag{2.20}$$

By using equality (2.9) we can obtain a slightly different expression /84 for the quantity ε_{ij}

$$\varepsilon_{ij} = \nabla_i u_j - b_{ij} w - c_{ij} \Omega_n. \tag{2.21}$$

In order to gain experience in the use of tensor notation, the reader should be convinced (Chapter VII, 5.56) of the validity of the identity

$$-\left(b_i^\alpha \nabla_j \vartheta_\alpha + b_j^\alpha \nabla_i \vartheta_\alpha\right) = b_i^\beta \beta_{j\beta} + b_j^\alpha \beta_{i\alpha} - 2b_i^\alpha b_j^\beta \varepsilon_{\alpha\beta}.$$

By using these expressions we obtain the equality

$$\begin{aligned} e_{ij} = \frac{1}{2}\left(g_{ij}^* - g_{ij}\right) = \varepsilon_{ij} - \zeta\beta_{ij} + \\ + \zeta^2\left(\frac{b_i^\beta \beta_{j\beta} + b_j^\alpha \beta_{i\alpha}}{2} - b_i^\alpha b_j^\beta \varepsilon_{\alpha\beta}\right), \end{aligned} \tag{2.22}$$

which shows that the six quantities ε_{11}, $\varepsilon_{12} = \varepsilon_{21}$, ε_{22} and β_{11}, $\beta_{12} = \beta_{21}$, β_{22} completely determine the deformation of the shell. In this case the first three characterize the deformation of the middle surface (its extension and shear) and are therefore called the components of tangential deformation. The last three determine the variation in the coefficients of the second quadratic surface form $(2.17)_2$ and characterize the flexure of the middle surface. They are therefore called components of flexural deformation.

Let us obtain the continuity equations for the middle surface. The most natural approach to this is to vary the Codazzi-Gauss relationships. However, we shall select a different path yielding several useful relationships. For this purpose we developed the obvious identities

$$\frac{\partial^2 \vec{\Omega}}{\partial\alpha^1 \partial\alpha^2} = \frac{\partial^2 \vec{\Omega}}{\partial\alpha^2 \partial\alpha^1}; \quad \frac{\partial^2 \vec{U}}{\partial\alpha^1 \partial\alpha^2} = \frac{\partial^2 \vec{U}}{\partial\alpha^2 \partial\alpha^1}. \tag{2.23}$$

First of all, by using equalities (Chapter VII, 5.51 and 5.46) we find from (2.11)

$$\frac{\partial \vec{\Omega}}{\partial s^j} = \left(c^{\alpha\beta} \nabla_j \vartheta_\alpha - b_j^\beta \Omega_n\right) \vec{r}_\beta + \left(\frac{\partial \Omega_n}{\partial \alpha^j} + b_{j\beta} c^{\alpha\beta} \vartheta_\alpha\right) \vec{n}.$$

Introducing the designations

$$\nu_{ij} = \nabla_i \vartheta_j - c_{j_1} b_i^{j_1} \Omega_n; \tag{2.24}$$

$$\zeta_j = \frac{\partial \Omega_n}{\partial \alpha^j} + c^{\alpha\beta} b_{j\beta} \vartheta_\alpha, \tag{2.25}$$

we write the preceding equality in the form

$$\frac{\partial \vec{\Omega}}{\partial \alpha^j} = c^{\alpha\beta} \nu_{j\alpha} \vec{r}_\beta + \zeta_j \vec{n}. \tag{2.26}$$

Furthermore, (see (2.5))

$$\begin{aligned} \frac{\partial \vec{U}}{\partial \alpha^j} &= (\nabla_j u_\alpha - b_{j\alpha} w) \vec{r}^\alpha + \left(\frac{\partial w}{\partial \alpha^j} + b_j^\alpha u_\alpha \right) \vec{n} = \\ &= (\nabla_j u_\alpha - b_{j\alpha} \tau w) \vec{r}^\alpha - \vartheta_j \vec{n}. \end{aligned} \tag{2.27}$$

Since

$$\begin{aligned} \nabla_j u_\alpha - b_{j\alpha} w &= \left\{ \frac{1}{2} (\nabla_j u_\alpha + \nabla_\alpha u_j) - b_{j\alpha} w \right\} + \\ &+ \frac{1}{2} (\nabla_j u_\alpha - \nabla_\alpha u_j) = \varepsilon_{j\alpha} + \frac{1}{2} (\nabla_j u_\alpha - \nabla_\alpha u_j), \\ \frac{\partial \vec{U}}{\partial \alpha^j} &= \varepsilon_{j\alpha} \vec{r}^\alpha + \left\{ \frac{1}{2} (\nabla_j u_\alpha - \nabla_\alpha u_j) \vec{r}^\alpha - \vartheta_j \vec{n} \right\}. \end{aligned} \tag{2.28}$$

We use equalities (Chapter VII, 5.20, 5.21 and 5.23) and (2.11) to compute the vector product /85

$$\begin{aligned} \left[\vec{\Omega}, \vec{r}_j \right] &= \left[c^{\alpha\beta} \vartheta_\alpha \vec{r}_\beta + \Omega_n \vec{n}, \vec{r}_j \right] = c^{\alpha\beta} \vartheta_\alpha c_{\beta j} \vec{n} + \\ &+ \Omega_n c_{j\alpha} \vec{r}^\alpha = - (c^{\alpha\beta} c_{j\beta}) \vartheta_\alpha \vec{n} + \frac{1}{2} c_{j\alpha} c^{\beta\gamma} \nabla_\beta u_\gamma \vec{r}^\alpha = \\ &= - \delta_j^\alpha \vartheta_\alpha \vec{n} + \frac{1}{2} (\delta_j^\beta \delta_\alpha^\gamma - \delta_\alpha^\beta \delta_j^\gamma) \nabla_\beta u_\gamma \vec{r}^\alpha = - \vartheta_j \vec{n} + \frac{1}{2} (\nabla_j u_\alpha - \nabla_\alpha u_j) \vec{r}^\alpha. \end{aligned}$$

The preceding expression may now be represented in the form

$$\frac{\partial \vec{U}}{\partial \alpha^j} = \varepsilon_{j\alpha} \vec{r}^\alpha + \left[\vec{\Omega}, \vec{r}_j \right]. \tag{2.29}$$

Since we have the expressions for the derivatives of the displacement vectors and the angles of rotation, it is easy to obtain the continuity equations. Thus from (2.26) we find (Chapter VII, 5.51)

$$\frac{\partial}{\partial\alpha^2}\left(\frac{\partial\vec{\Omega}}{\partial\alpha^1}\right)=\left(\nabla_2 c^{\beta\alpha}\mu_{1\beta}-b_2^{\alpha}\zeta_1\right)\vec{r}_{\alpha}+\left(\frac{\partial\zeta_1}{\partial\alpha^2}+b_{2\alpha}c^{\beta\alpha}\mu_{1\beta}\right)\vec{n};$$
$$\frac{\partial}{\partial\alpha^1}\left(\frac{\partial\vec{\Omega}}{\partial\alpha^2}\right)=\left(\nabla_1 c^{\beta\alpha}\mu_{2\beta}-b_1^{\alpha}\zeta_2\right)\vec{r}_{\alpha}+\left(\frac{\partial\zeta_2}{\partial\alpha^1}+b_{1\alpha}c^{\beta\alpha}\mu_{2\beta}\right)\vec{n}.$$

Equating both expressions we obtain the vector continuity equation

$$\begin{aligned}&\left\{\left(\nabla_2 c^{\beta\alpha}\mu_{1\beta}-\nabla_1 c^{\beta\alpha}\mu_{2\beta}\right)+\left(b_1^{\alpha}\zeta_2-b_2^{\alpha}\zeta_1\right)\right\}\vec{r}_{\alpha}+\\&+\left\{\left(\frac{\partial\zeta_1}{\partial\alpha^2}-\frac{\partial\zeta_2}{\partial\alpha^1}\right)+\left(b_{2\alpha}c^{\beta\alpha}\mu_{1\beta}-b_{1\alpha}c^{\beta\alpha}\mu_{2\beta}\right)\right\}\vec{n}=0.\end{aligned}\tag{2.30}$$

It must be transformed slightly. According to (Chapter VII, 5.19)

$$\nabla_2 c^{\beta\alpha}\mu_{1\beta}-\nabla_1 c^{\beta\alpha}\mu_{2\beta}=\sqrt{a}c^{\gamma\delta}\nabla_{\delta}c^{\beta\alpha}\mu_{\gamma\beta}=\sqrt{a}\nabla_{\delta}\left(c^{\gamma\delta}c^{\beta\alpha}\mu_{\gamma\beta}\right).$$

Carrying out similar transformations for the remaining pairs of values, we reduce equation (2.30) to the form

$$\begin{aligned}&\sqrt{a}\left\{\nabla_{\delta}\left(c^{\gamma\delta}c^{\beta\alpha}\mu_{\gamma\beta}\right)+b_{\gamma}^{\alpha}\left(c^{\gamma\delta}\zeta_{\delta}\right)\right\}\vec{r}_{\alpha}+\\&+\sqrt{a}\left\{-\nabla_{\gamma}\left(c^{\gamma\delta}\zeta_{\delta}\right)+b_{\delta\alpha}\left(c^{\gamma\delta}c^{\beta\alpha}\mu_{\gamma\beta}\right)\right\}\vec{n}=0.\end{aligned}$$

From this we obtain three scalar equations

$$\nabla_{\delta}\left(c^{i\beta}c^{\gamma\delta}\mu_{\gamma\beta}\right)+b_{\gamma}^{i}\left(c^{\delta\gamma}\zeta_{\delta}\right)=0\quad(i=1,2);\tag{2.31}$$
$$\nabla_{\gamma}\left(c^{\delta\gamma}\zeta_{\delta}\right)-b_{\delta\alpha}\left(c^{\alpha\beta}c^{\gamma\delta}\mu_{\gamma\beta}\right)=0.\tag{2.32}$$

Furthermore, from (2.29) we find

$$\frac{\partial}{\partial\alpha^2}\left(\frac{\partial\vec{U}}{\partial\alpha^1}\right)=\nabla_2 e_{1\alpha}\vec{r}^{\alpha}+b_2^{\alpha}e_{1\alpha}\vec{n}+\left[\frac{\partial\vec{\Omega}}{\partial\alpha^2},\ \vec{r}_1\right]+\left[\vec{\Omega},\ \nabla_2\vec{r}_1\right];$$
$$\frac{\partial}{\partial\alpha^1}\left(\frac{\partial\vec{U}}{\partial\alpha^2}\right)=\nabla_1 e_{2\alpha}\vec{r}^{\alpha}+b_1^{\alpha}e_{2\alpha}\vec{n}+\left[\frac{\partial\vec{\Omega}}{\partial\alpha^1},\ \vec{r}_2\right]+\left[\vec{\Omega},\ \nabla_1\vec{r}_2\right].$$

By equating these quantities and using transformations analogous to those presented above, we obtain (see also (Chapter VII, 5.52)$_1$) three more scalar equations /86

$$\begin{aligned} c^{\gamma\delta}\left(\mu_{\delta\gamma}-b^{\alpha}_{\delta}\varepsilon_{\gamma\alpha}\right)&=0;\\ c^{\gamma\delta}\nabla_{\delta}\varepsilon_{\gamma l}+\zeta_{l}&=0. \end{aligned} \tag{2.33}$$

It is convenient to represent the last of these in a different form. For this purpose we multiply its left and right sides by c^{ji} and take the sum over the index i (replacing the latter, as established in Chapter VII, by the Greek symbol β). As a result we obtain two equations

$$\nabla_{\delta}(c^{\gamma\delta}c^{\beta\beta}\varepsilon_{\gamma\beta})+c^{\beta\beta}\zeta_{\beta}=0. \tag{2.34}$$

Let us examine these relationships in more detail. It is recommended that the reader use equalities (2.20) and (2.24) to become convinced that the following relationship is valid

$$\mu_{ij}=-\beta_{ij}+b^{\gamma}_{j}\varepsilon_{i\gamma}. \tag{2.35}$$

By using this relationship, equation (2.33)$_1$ is reduced to the form

$$-\beta_{\delta\gamma}c^{\gamma\delta}=0.$$

Since its left side represents the contraction of the skew-symmetric tensor with a symmetric tensor, the considered equation is identically satisfied. We drop it and collect the remaining five equations, making the following changes in the indices as required

$$\begin{aligned} \nabla_{\alpha}\left(c^{\alpha\gamma}c^{\beta\beta}\mu_{\gamma\beta}\right)+b^{\beta}_{\alpha}\left(c^{\alpha\beta}\zeta_{\beta}\right)&=0;\\ -\nabla_{\alpha}\left(c^{\alpha\beta}\zeta_{\beta}\right)+b^{\gamma}_{\alpha\delta}\left(c^{\alpha\gamma}c^{\delta\delta}\mu_{\gamma\beta}\right)&=0;\\ -\nabla_{\delta}\left(c^{\delta\gamma}c^{\beta\beta}\varepsilon_{\gamma\beta}\right)+c^{\beta\beta}\zeta_{\beta}&=0. \end{aligned} \tag{2.36}$$

If we use the last equations to eliminate the quantities $c^{j\beta}\zeta_{\beta}$ from the first three equations, we shall obtain three equations which must satisfy the seven quantities

$$\varepsilon_{11},\ \varepsilon_{12}=\varepsilon_{21},\ \varepsilon_{22};\quad \mu_{11},\ \mu_{12},\ \mu_{21},\ \mu_{22}. \tag{2.37}$$

Using equalities (2.35) to eliminate μ_{ij}, we obtain three equations for the six components of deformation introduced earlier

$$\varepsilon_{11},\ \varepsilon_{12}=\varepsilon_{21},\ \varepsilon_{22};\quad \beta_{11},\ \beta_{12}=\beta_{21},\ \beta_{22}. \tag{2.38}$$

The latter, as we established above, have a clear geometric meaning and are more desirable when we consider theoretical problems. Their shortcoming is that the quantities β_{11} and β_{22} are expressed in a rather complex fashion in terms of the displacements. Therefore, in the theory of shells the usual practice is to use mixed quantities as the components of flexural deformation /87

$$\mu_{11},\ \mu_{22};\quad (-\beta_{12})=(-\beta_{21}), \tag{2.39}$$

and to replace μ_{12} and μ_{21} in the relationships obtained above by expressions

$$\mu_{12}=(-\beta_{12})+b_1^1\varepsilon_{21};\quad \mu_{21}=(-\beta_{12})+b_2^1\varepsilon_{11}. \tag{2.40}$$

3. Deformation of the Edge

When investigating the deformation of the edge it is convenient to deal with the coordinate system introduced in section 7, Chapter VII. In this system the vector of displacements is expanded into the form

$$\vec{U}=u_\nu\vec{\nu}+u_t\vec{t}+w\vec{n}, \tag{3.1}$$

where according to (Chapter VII, 7.11, 7.12 and 7.15)

$$u_\nu=\nu^\alpha u_\alpha;\quad u_t=t^\alpha u_\alpha; \tag{3.2}$$

$$u_l=\nu_l u_\nu+t_l u_t. \tag{3.3}$$

The vector of rotation of the neighborhood associated with a point on the middle surface is represented in a similar manner

$$\vec{\Omega}=\Omega_\nu\vec{\nu}+\Omega_t\vec{t}+\Omega_n\vec{n}. \tag{3.4}$$

Here, according to (Chapter VII, 7.5 and 7.6) and (2.11)

$$\Omega_\nu = \nu_\beta c^{\alpha\beta}\vartheta_\alpha = -t^\alpha\vartheta_\alpha = -\vartheta_t;$$
$$\Omega_t = t_\beta c^\alpha\ \vartheta_\alpha = \nu^\alpha\vartheta_\alpha = \vartheta_\nu. \tag{3.5}$$

As we have stated in the preceding section, the quantity Ω_n characterizes the mean rotation of the neighborhood around the normal to the surface. On the other hand, the rotation of the normal element fixed with respect to the line α^i is characterized by the quantity $\omega^i_{\cdot i}$ where ((1.20),(Chapter VII, 5.48_1))

$$\omega^i_{\cdot j} = (\omega^{i3}_{\cdot\cdot j})_{\zeta=0} = c^{i\alpha}(\vartheta_{\alpha j})_0 = c^{i\alpha}(\nabla_j u_\alpha - b_{\alpha j}w). \tag{3.6}$$

(We note in passing that

$$\omega^1_{\cdot 2} = c^{12}(\nabla_2 u_2 - b_{22}w) = c^{12}\varepsilon_{22};$$
$$\omega^2_{\cdot 1} = c^{21}(\nabla_1 u_1 - b_{11}w) = c^{21}\varepsilon_{11}, \tag{3.7}$$

i.e., in terms of the physical components ($c_{(12)} = +1$, $c_{(21)} = -1$) the components with different indices coincide, within the accuracy of the sign, with the relative elongation in the direction determined by the second index.)

Thus the vector for the rotation of the normal element fixed with respect to the line Γ should be taken in the form /88

$$\vec{\Omega}_t = \Omega_\nu\vec{\nu} + \Omega_t\vec{t} + \omega_{tt}\vec{n}, \tag{3.8}$$

where

$$\omega_{tt} = t_\alpha t^\beta\omega^\alpha_{\cdot\beta}. \tag{3.9}$$

Let us establish a relationship between $\vec{\Omega}$ and $\vec{\Omega}_t$. First of all,

$$\vec{\Omega}_t = \vec{\Omega} + (\omega_{tt} - \Omega_n)\vec{n}. \tag{3.10}$$

A simple calculation taking into account the qualities (2.10) and (Chapter VII, 7.5) gives us

$$\omega_{tt} - \Omega_n = -t^\beta\nu^\gamma\varepsilon_{\beta\gamma} = -\varepsilon_{t\nu}. \tag{3.11}$$

Therefore,

$$\vec{\Omega}_t = \vec{\Omega} - \varepsilon_{t\nu}\vec{n}. \tag{3.12}$$

Furthermore, according to (2.27)

$$\frac{d\vec{U}}{ds_t} = \frac{\partial \vec{U}}{\partial \alpha^\beta}\frac{d\alpha^\beta}{ds_t} = t^\beta\left\{(\nabla_\beta u_\alpha - b_{\beta\alpha}w)\vec{r}^\alpha - \vartheta_\beta\vec{n}\right\}.$$

First of all, in view of equality (3.5)

$$-t^\beta\vartheta_\beta = -\vartheta_t = \Omega_\nu.$$

In addition, by means of (Chapter VII, 7.14, 7.5 and 7.6) and (3.6) the first term is written in the form

$$\begin{aligned} &t^\beta(\Delta_\beta u_\alpha - b_{\alpha\beta}w)(\nu^\alpha\vec{\nu} + t^\alpha\vec{t}) = \\ &= t^\beta t_\gamma c^{\alpha\gamma}(\nabla_\beta u_\alpha - b_{\beta\alpha}w)\vec{\nu} + t^\beta\nu_\gamma c^{\gamma\alpha}(\nabla_\beta u_\alpha - b_{\beta\alpha}w)\vec{t} = \\ &= -t^\beta t_\gamma\omega^\gamma_\beta\vec{\nu} + t^\beta\nu_\gamma\omega_\beta^\gamma\vec{t} = -\omega_{tt}\vec{\nu} + \omega_{\nu t}\vec{t}. \end{aligned}$$

In view of the remark made above (3.7), we should let $\omega_{\nu t} = \varepsilon_{tt}$. By collecting everything we obtain

$$\frac{d\vec{U}}{ds_t} = \varepsilon_{tt}\vec{t} - \omega_{tt}\vec{\nu} + \Omega_\nu\vec{n}. \tag{3.13}$$

By using equalities (Chapter VII, 7.2) and (3.8) this equality may be written in the form

$$\frac{d\vec{U}}{ds_t} = \varepsilon_{tt}\vec{t} + [\vec{\Omega}_t, \vec{t}]. \tag{3.14}$$

Relationship (3.13) makes it possible for us to represent the introduced quantities in terms of the components u_ν, u_t, w. For this purpose we differentiate expression (3.1) along the arc, taking into account the rule for the differentiation of unit orthogonal vectors (Chapter VII, 7.17). As a result we find /89

$$\begin{aligned}
\varepsilon_{tt} &= \frac{du_t}{ds_t} + \varkappa_t u_\nu - \sigma_t w;\\
-\omega_{tt} &= \frac{du_\nu}{ds_t} - \varkappa_t u_t + \tau_t w;\\
\Omega_\nu &= \frac{dw}{ds_t} + \sigma_t u_t - \tau_t u_\nu = -\vartheta_t.
\end{aligned} \tag{3.15}$$

Furthermore, according to (2.7), (3.5) and (Chapter VII, 7.15)

$$\begin{aligned}
\Omega_t = \vartheta_\nu = \nu^\alpha \vartheta_\alpha = -\nu^\alpha \left(\frac{\partial w}{\partial \alpha^\alpha} + b_\alpha^\beta u_\beta\right) =\\
= -\nu^\alpha \frac{\partial w}{\partial \alpha^\alpha} - \nu^\alpha \nu_\beta b_\alpha^\beta u_\nu - \nu^\alpha t_\beta b_\alpha^\beta u_t
\end{aligned}$$

or (Chapter VII, 7.29 and 7.30)

$$\Omega_t = -\frac{dw}{ds_\nu} - \sigma_\nu u_\nu + \tau_\nu u_t = \vartheta_\nu. \tag{3.16}$$

Let us now turn to the curvature vector and determine it as a derivative of the vector for the angles of rotation along the arc

$$\vec{\varkappa}_t = \frac{d\vec{\Omega}_t}{ds_t} = -\varkappa_{tt}\vec{\nu} + \varkappa_{t\nu}\vec{t} - \varkappa_{tn}\vec{n}. \tag{3.17}$$

Here, according to (Chapter VII, 7.17),

$$\begin{aligned}
-\varkappa_{tt} &= \frac{d\Omega_\nu}{ds_t} - \varkappa_t \Omega_t + \tau_t \omega_{tt};\\
\varkappa_{t\nu} &= \frac{d\Omega_t}{ds_t} + \varkappa_t \Omega_\nu - \sigma_t \omega_{tt};\\
-\varkappa_{tn} &= \frac{d\omega_{tt}}{ds_t} - \tau_t \Omega_\nu + \sigma_t \Omega_t.
\end{aligned} \tag{3.18}$$

Then, by utilizing equalities (3.12) and (Chapter VII, 7.17), we find

$$\vec{\varkappa}_t = t^\beta \frac{\partial \vec{\Omega}}{\partial \alpha^\beta} - \frac{d\varepsilon_{t\nu}}{ds_t}\vec{n} - \varepsilon_{t\nu}\left(\tau_t \vec{\nu} - \sigma_t \vec{t}\right).$$

However, (2.26) and (Chapter VII, 7.14, 7.5 and 7 6)

$$t^\beta \frac{\partial \vec{\Omega}}{\partial \alpha^\beta} = t^\beta \left(c^{\alpha\tau} \mu_{\beta\alpha} \vec{r}_\tau + \zeta_\beta \vec{n} \right) =$$
$$= t^\beta c^{\alpha\tau} \nu_\tau \mu_{\beta\alpha} \vec{\nu} + t^\beta c^{\alpha\tau} t_\tau \mu_{\beta\alpha} \vec{t} + t^\beta \zeta_\beta \vec{n} =$$
$$= - t^\alpha t^\beta \mu_{\beta\alpha} \vec{\nu} + t^\beta \nu^\alpha \mu_{\beta\alpha} \vec{t} + t^\beta \zeta_\beta \vec{n} =$$
$$= - \mu_{tt} \vec{\nu} + \mu_{t\nu} \vec{t} + \zeta_t \vec{n}.$$

Thus the components of the flexural deformation of a normal element determined by the curve Γ are expressed in terms of the conventional components of the shell deformation by means of the following relationships

$$\begin{aligned} \varkappa_{tt} &= \mu_{tt} + \tau_t \varepsilon_{t\nu}; \\ \varkappa_{t\nu} &= \mu_{t\nu} + \sigma_t \varepsilon_{t\nu}; \\ \varkappa_{tn} &= - \zeta_t + \frac{d\varepsilon_{t\nu}}{ds_t}, \end{aligned} \tag{3.19}$$

where, as we recall

$$\begin{aligned} \mu_{tt} &= t^\alpha t^\beta \mu_{\alpha\beta}; \quad \zeta_t = t^\beta \zeta_\beta; \\ \mu_{t\nu} &= t^\alpha \nu^\beta \mu_{\alpha\beta}; \quad \varepsilon_{t\nu} = t^\beta \nu^\tau \varepsilon_{\beta\tau}. \end{aligned} \tag{3.20}$$

Sometimes it is more convenient to refer the deformation of the edge /90 to the directions of the principal normal and of the binormal for the boundary contour (see section 7, Chapter VII); in this case

$$\vec{U} = u_m \vec{m} + u_b \vec{b} + u_t \vec{t}; \tag{3.21}$$
$$\vec{\Omega}_t = \vartheta_m \vec{m} + \vartheta_b \vec{b} + \vartheta_\nu \vec{t}; \tag{3.22}$$
$$\vec{\varkappa}_t = \frac{d\vec{\Omega}_t}{ds_t} = \varkappa_{tm} \vec{m} + \varkappa_{tb} \vec{b} + \varkappa_{t\nu} \vec{t}. \tag{3.23}$$

Replacing the orthogonal unit vectors $\vec{\nu}$, $\vec{n}$ in (3.1), (3.8), (3.5) and (3.17) by their expressions (Chapter VII, 7.40) and comparing the relationships obtained in this manner with those written out above, we obtain the following equalities

$$u_m = u_\nu \sin\varphi + w \cos\varphi; \qquad u_b = u_\nu \cos\varphi - w \sin\varphi; \tag{3.24}$$
$$\vartheta_m = -\vartheta_t \sin\varphi + \omega_{tt} \cos\varphi; \qquad \vartheta_b = -\vartheta_t \cos\varphi - \omega_{tt} \sin\varphi; \tag{3.25}$$
$$\varkappa_{tm} = -\varkappa_{tt} \sin\varphi - \varkappa_{tn} \cos\varphi; \qquad \varkappa_{tb} = -\varkappa_{tt} \cos\varphi + \varkappa_{tn} \sin\varphi. \tag{3.26}$$

Furthermore, by utilizing relationships (3.14) and the rule for the differentiation of unit orthogonal vectors (Chapter VII, 7.35), we arrive at the following relationships between the introduced quantities

$$\vartheta_m = -\frac{du_b}{ds_t} - \hat{\tau} u_m; \qquad (3.27)$$

$$\vartheta_b = \frac{du_m}{ds_t} - \hat{\tau} u_b + \hat{\sigma} u_t;$$

$$\varepsilon_{tt} = \frac{du_t}{ds_t} - \hat{\sigma} u_m; \qquad (3.28)$$

$$\varkappa_{t\nu} = \frac{d\vartheta_\nu}{ds_t} - \hat{\sigma}\vartheta_m;$$

$$\varkappa_{tm} = \frac{d\vartheta_m}{ds_t} - \hat{\tau}\vartheta_b + \hat{\sigma}\vartheta_\nu; \qquad (3.29)$$

$$\varkappa_{tb} = \frac{d\vartheta_b}{ds_t} + \hat{\tau}\vartheta_m.$$

4. Forces and Moments. Equilibrium Equations

Let us consider a section of the shell whose normal is directed along the line α^i. A force $\vec{K}_{(i)}$ (section 1) acts on the unit area of this section. The element of the lateral surface is subjected to a force $\int_{-h/2}^{h/2} \vec{K}_{(i)} dS_i$. We designate by $\vec{T}_{(i)}$ the force acting on a unit length of the middle line of the considered element

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$$\vec{T}_{(i)} = \frac{1}{ds_k} \int_{-h/2}^{h/2} \vec{K}_{(i)} dS_i \quad (i \neq k). \qquad (4.1)$$

Let us consider a triangular element on the middle surface (fig. 15). Repeating the reasoning presented at the end of section 1, we obtain

$$\vec{T}_{(\nu)} = \sum_{i=1}^{2} \nu_i \left(\sqrt{a^{ii}}\, \vec{T}_{(i)} \right). \qquad (4.2)$$

Here $\vec{\nu} = \nu_\alpha \vec{r}^{\,\alpha} = \nu^\alpha \vec{r}_\alpha$ is the unit normal vector to the hypotenuse (fig. 15).

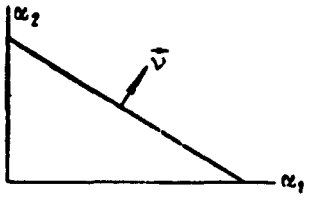

Figure 15

The contraction of the quantity $(\sqrt{a^{II}}\,\vec{T}_{(i)})$ with the covariant normal vector produces an invariant (when $\vec{\nu}$ is fixed). For this reason this quantity represents a tensor of the first rank with vector components. It can therefore be represented in the form

$$\sqrt{a^{ii}}\,\vec{T}_{(i)} = T^{i\alpha}\vec{r}_\alpha + T^{in}\vec{n}. \tag{4.3}$$

Let us substitute equalities (Chapter VII, 3.34 and 5.15) and (1.24) into (4.1), and then substitute the resulting expression into (4.3). As a result we obtain relationships (in (Chapter VII, 3.34) $d\theta^3 = d\zeta$)

$$T^{ij} = \int_{-h/2}^{h/2} (\sigma^{ij} - \zeta b_1^j \sigma^{i1})\sqrt{\frac{g}{a}}\,d\zeta;\quad T^{in} = \int_{-h/2}^{h/2} \sigma^{in}\sqrt{\frac{g}{a}}\,d\zeta. \tag{4.4}$$

The form of the latter and also equality (Chapter VII, 5.33) confirm that the introduced tangential forces (T^{ij}) and cutting forces ($T^{(in)}$) have a tensor nature. At the same time (4.3) shows that the force vectors $\vec{T}_{(i)}$ are the physical components for the contravariant tensor assigned on the middle surface, and that the vector components of this tensor are

$$\vec{T}^i = T^{i\alpha}\vec{r}_\alpha + T^{in}\vec{r}. \tag{4.5}$$

Finally, from equalities (4.2) and (4.3) we obtain expressions for the components of the vector of forces acting on the skew area

$$T^j_{(\nu)} = \nu_\beta T^{\beta j};\quad T^n_{(\nu)} = \nu_\beta T^{\beta n}. \tag{4.6}$$

The moment produced relative to the middle line by the stresses is equal to (computed for its unit length) /92

$$\vec{M}_{(i)}=\frac{1}{ds_k}\int_{-h/2}^{h/2}[\zeta\vec{n},\ \vec{K}_{(i)}]\,dS_i=\left[\vec{n},\ \frac{1}{ds_k}\int_{h/2}^{h/2}\vec{K}_{(i)}ds_i\zeta d\zeta\right].$$

As above, it is shown that

$$\sqrt{a^{ii}}\,\vec{M}_{(i)}=M^{i\alpha}[\vec{n},\ \vec{r}_\alpha]=c_{\alpha\beta}M^{i\alpha}\vec{r}^{\beta}, \tag{4.7}$$

where

$$M^{ij}=\int_{-h/2}^{h/2}(\delta^{ij}-\zeta b_\gamma^j\delta^{i\gamma})\sqrt{\frac{g}{a}}\,\zeta d\zeta \tag{4.8}$$

are the components of the tensor for the bending moments (M^{ii}) and twisting moments (M^{ik}). In this case the vectors of the moments are written in the form

$$\vec{M}^i=c_{\alpha\beta}M^{i\alpha}\vec{r}^{\beta}. \tag{4.9}$$

From equality (4.7) we can see that the introduced vectors $\vec{M}_{(i)}$ are the physical components of the tensor with vector components $\vec{M}^i$. In exactly the same manner as in (4.6) equalities are established which associate the components of the vector of moments acting in the skew section with components of the tensor for the moments

$$M_j^{(\nu)}=\nu_\beta c_{\alpha j}M^{\beta\alpha}. \tag{4.10}$$

Let us obtain the equilibrium conditions for the element of area on the middle surface bounded by the lines α^1, $\alpha^1+d\alpha^1$; α^2, $\alpha^2+d\alpha^2$. The side α^1 whose length is $\sqrt{a_{22}}\,d\alpha^2$ is subjected to the force (according to (Chapter VII, 5.14) (fig. 16)

$$-\vec{T}_{(1)}\sqrt{a_{22}}\,d\alpha^2 = -\frac{\vec{T}^1}{\sqrt{a^{11}}}\sqrt{a_{22}}\,d\alpha^2 = -\sqrt{a}\,\vec{T}^1 d\alpha^2.$$

The sign (-) shows that the normal to the considered area is directed toward the side where the coordinate α^1 decreases. The following force acts on the side $\alpha^1 + d\alpha^1$

$$\left(\sqrt{a}\,\vec{T}^1 + \frac{\partial\sqrt{a}\,\vec{T}^1}{\partial\alpha^1}\,d\alpha^1\right)d\alpha^2.$$

Adding these we obtain the total force

$$\frac{\partial\sqrt{a}\,\vec{T}^1}{\partial\alpha^1}\,d\alpha^1 d\alpha^2.$$

In exactly the same manner it is shown that the total force acting on the two other sides is equal to /93

$$\frac{\partial\sqrt{a}\,\vec{T}^2}{\partial\alpha^2}\,d\alpha^1 d\alpha^2.$$

In addition, an external force acts on the considered area

$$\vec{q}\,\sqrt{a}\,d\alpha^1 d\alpha^2,$$

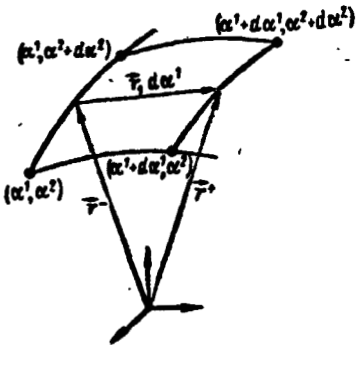

Figure 16

where $\vec{q}$ is the external load per unit area reduced to the middle surface. Putting together the established expressions and setting their sum equal to zero we obtain the condition that the principal vector of all forces applied to the considered element of the middle surface is equal to zero

$$\frac{\partial \sqrt{a}\,\vec{T}^{\beta}}{\partial \alpha^{\beta}} + \sqrt{a}\,\vec{q} = 0. \tag{4.11}$$

Let us compute the principal moment. As in the case above, we find that the moments applied to the edge of the element produce the principal moment

$$\left(\frac{\partial \sqrt{a}\,\vec{M}^{1}}{\partial \alpha^{1}} + \frac{\partial \sqrt{a}\,\vec{M}^{2}}{\partial \alpha^{2}}\right)\sqrt{a}\,d\alpha^{1}d\alpha^{2}.$$

It is also necessary to take into account the moment produced by the forces.

Thus, if we consider the sides α^1 and $\alpha^1 + d\alpha^1$, then the forces applied to them produce the moment (fig. 16)

$$\left[\vec{r}^{+}, \left(\sqrt{a}\,\vec{T}^{1} + \frac{\partial \sqrt{a}\,\vec{T}^{1}}{\partial \alpha^{1}} d\alpha^{1}\right) d\alpha^{2}\right] + \\ + \left[\vec{r}^{-}, \left(-\sqrt{a}\,\vec{T}^{1}\right) d\alpha^{2}\right] \approx \left[\vec{r}^{+} - \vec{r}^{-}, \vec{T}^{1}\right] \sqrt{a}\,d\alpha^{2},$$

since

$$\vec{r}^{+} - \vec{r}^{-} = \vec{r}_{1} d\alpha^{1},$$

it can be written in the form

$$\left[\vec{r}_{1}, \vec{T}^{1}\right] \sqrt{a}\,d\alpha^{1}d\alpha^{2}.$$

Analogous conditions applied to the two other sides give us a moment

$$\left[\vec{r}_{2}, \vec{T}^{2}\right] \sqrt{a}\,d\alpha^{1}d\alpha^{2}.$$

Generally speaking, it is also necessary to take into account the moment produced by the external load

$$\vec{m}\sqrt{a}\,d\alpha^1 d\alpha^2.$$

However, as shown by the analysis conducted in section 2, Chapter III, the role of surface moments is small for most of the problems of practical interest. /94
Therefore, in the future we shall not take them into account, although it would not be difficult to do so. Then, by collecting the obtained expressions and setting their sum equal to zero we obtain the condition that the principal moment of all forces and moments applied to the considered element of the middle surface is equal to zero

$$\frac{\partial \sqrt{a}\,\vec{M}^\beta}{\partial \alpha^\beta} + \left[\vec{r}_\beta, \vec{T}^\beta\right]\sqrt{a} = 0. \tag{4.12}$$

It is convenient to transform equations (4.11) and (4.12) slightly. Thus, by taking into account identity (Chapter VII, 5.40), we obtain

$$\begin{aligned}\frac{\partial \sqrt{a}\,\vec{T}^\beta}{\partial \alpha^\beta} &= \sqrt{a}\,\frac{\partial \vec{T}^\beta}{\partial \alpha^\beta} + \frac{\partial \sqrt{a}}{\partial \alpha^\beta}\vec{T}^\beta = \\ &= \sqrt{a}\left\{\frac{\partial \vec{T}^\beta}{\partial \alpha^\beta} + \Gamma^\alpha_{\alpha\beta}\vec{T}^\beta\right\} = \sqrt{a}\,\nabla_\beta \vec{T}^\beta.\end{aligned} \tag{4.13}$$

Substituting this expression into (4.11), we reduce the latter to the form

$$\nabla_\beta \vec{T}^\beta + \vec{q} = 0. \tag{4.14}$$

Furthermore, replacing $\vec{T}^\beta$ with its expression given by (4.5) and taking into account equations (Chapter VII, 5.52) for the covariant differentiation, we obtain

$$\begin{aligned}\nabla_\beta \vec{T}^\beta = \nabla_\beta\left(T^{\beta\alpha}\vec{r}_\alpha + T^{\beta n}\vec{n}\right) = \left(\nabla_\beta T^{\beta\alpha} - b^\alpha_\beta T^{\beta n}\right)\vec{r}_\alpha + \\ + \left(\nabla_\beta T^{\beta n} + b_{\beta\alpha}T^{\beta\alpha}\right)\vec{n}.\end{aligned}$$

This equality makes it possible for us to replace the vector equation (4.14) with a system consisting of three scalars

$$\begin{aligned}&\nabla_\beta T^{\beta l} - b^l_\beta T^{\beta n} + q^l = 0; \\ &\nabla_\beta T^{\beta n} + b_{\beta\alpha}T^{\beta\alpha} + q_n = 0.\end{aligned} \tag{4.15}$$

Carrying out a similar transformations for equation (4.12), we obtain by means of equality (Chapter VII, 5.23)

$$\nabla_\beta \vec{M}^\beta + \left[\vec{r}_\beta, \vec{T}^\beta\right] = 0 \tag{4.16}$$

and

$$\nabla_\beta M^{\beta j} - T^{jn} = 0; \tag{4.17}$$

$$c_{\alpha\beta}\left(T^{\alpha\beta} + b_1^\beta M^{1\alpha}\right) = 0. \tag{4.18}$$

We shall show that the last equality, in essence, is identically satisfied. /95 Indeed, let us formulate the expression

$$S^{ij} = T^{ij} + b_1^j M^{1i} = \int_{-h/2}^{h/2} \left\{\sigma^{ij} - \zeta b_\beta^i \sigma^{\beta j} + \zeta b_1^j \left(\sigma^{1i} - \zeta b_\beta^i \sigma^{1\beta}\right)\right\} \sqrt{\frac{g}{a}}\, d\zeta =$$

$$= \int_{-h/2}^{h/2} \left\{\sigma^{ij} + \underline{\zeta b_\beta^j \left(\sigma^{\beta i} - \sigma^{i\beta}\right)} - \zeta^2 b_1^j b_\beta^i \sigma^{1\beta}\right\} \sqrt{\frac{g}{a}}\, d\zeta.$$

Since the stress tensor is symmetric, the underlined terms drop out and the simplified expression

$$S^{ij} = T^{ij} + b_1^j M^{1i} = \int_{-h/2}^{h/2} \left\{\sigma^{ij} - \zeta^2 b_1^j b_\beta^i \sigma^{1\beta}\right\} \sqrt{\frac{g}{a}}\, d\zeta \tag{4.19}_1$$

shows that expressions S^{ij} are the components of a symmetric tensor. Then equality (4.18) is satisfied identically, since its left part represents the contraction of a skew-symmetric tensor and a symmetric tensor. Thus relationship (4.18), which is known in the literature as the sixth equilibrium equation, is an integral expression for the symmetric property of the stress tensor. We drop it and collect the remaining equilibrium equations, making appropriate changes in the indices

$$\begin{aligned} &\nabla_\alpha T^{\alpha j} - b_\alpha^j T^{\alpha n} + q^j = 0; \\ &\nabla_\alpha T^{\alpha n} + b_{\alpha\beta} T^{\alpha\beta} + q_n = 0; \\ &\nabla_\beta M^{\beta j} - T^{jn} = 0. \end{aligned} \tag{4.20}$$

Using the last two relationships to exclude the quantities T^{jn} from the first three, we arrive at a system of three equilibrium equations (j = 1, 2)

$$\begin{aligned} &\nabla_\alpha T^{\alpha j} - b^j_\alpha \nabla_\beta M^{\beta\alpha} + q^j = 0, \\ &\nabla_\alpha \nabla_\beta M^{\beta\gamma} + b_{\alpha\beta} T^{\alpha\beta} + q_n = 0 \end{aligned} \tag{4.21}$$

containing the eight quantities

$$T^{11},\ T^{12},\ T^{21},\ T^{22};\ M^{11},\ M^{12},\ M^{21},\ M^{22}. \tag{4.22}$$

It can be shown that in this system the number of unknowns may be decreased to six, if in addition to the symmetric forces $(4.19)_1$ we introduce the symmetric moments

$$H^{ij} = \frac{1}{2}(M^{ij} + M^{ji}) = \int_{-h/2}^{h/2} \left\{ \sigma^{ij} - \zeta \frac{b^i_\gamma \sigma^{j\gamma} + b^j_\gamma \sigma^{i\gamma}}{2} \right\} \sqrt{\frac{g}{a}}\, \zeta\, d\zeta. \tag{4.19}_2$$

The quantities

$$S^{11},\ S^{12} = S^{21},\ S^{22};\ H^{11},\ H^{12} = H^{21},\ H^{22} \tag{4.23}$$

are very convenient for the purpose of theoretical problems, but are somewhat cumbersome. Therefore the following mixed quantities are usually used in their place

$$T^{11},\ S^{12} = S^{21},\ T^{22},\ M^{11},\ H^{12} = H^{21},\ M^{22}. \tag{4.24}$$

Below (section 6) we shall show that within the error associated with the theory of thin shells we may assume that $M^{ij} = M^{ji}$. In the future we shall therefore utilize quantities (4.22), assuming that in these

$$T^{12} = S^{12} - b^2_1 M^{11};\ T^{21} = S^{12} - b^1_1 M^{22},\ M^{12} = M^{21}. \tag{4.25}$$

5. The Energy of Deformation of a Shell. Static Boundary Conditions

In view of the Kirchhoff hypotheses which we have adopted, the variation in the energy of deformation of a shell which is considered as a three-dimensional body (see (1.31)) is written in the form

$$\delta V=\iint_S\left\{\int_{-h/2}^{h/2}\sigma^{\alpha 3}\delta e_{\alpha 3}\sqrt{\frac{g}{a}}\,d\zeta\right\}\sqrt{a}\,d\alpha^1 d\alpha^2, \tag{5.1}$$

where in this case $\alpha, \beta = 1, 2$.

Replacing $e_{\alpha\beta}$ in the expression under the integral sign by its value given by expression (2.22) and utilizing the identities

$$\sigma^{\alpha 3}\frac{b_\alpha^\gamma\beta_{\gamma\tau}+b_\beta^\gamma\beta_{\alpha\tau}}{2}=\frac{b_\gamma^\alpha\sigma^{\gamma 3}+b_\gamma^\beta\sigma^{\alpha\gamma}}{2}\beta_{\alpha\beta},$$

$$\sigma^{\alpha 3}b_\alpha^\gamma b_\beta^\delta\varepsilon_{\gamma\delta}=\sigma^{\gamma\delta}b_\gamma^\alpha b_\delta^\beta\varepsilon_{\alpha\beta},$$

we obtain

$$\delta V=\iint_S\left\{\int_{-h/2}^{h/2}\left[\left(\sigma^{\alpha 3}-\zeta^2 b_\gamma^\alpha b_\delta^\beta\sigma^{\gamma\delta}\right)\delta e_{\alpha 3}+\right.\right.$$
$$\left.\left.+\zeta\left(\sigma^{\alpha 3}-\zeta\frac{b_\gamma^\alpha\sigma^{\gamma 3}+b_\gamma^\beta\sigma^{\alpha\gamma}}{2}\right)\delta(-\beta_{\alpha 3})\right]\sqrt{\frac{g}{a}}\,d\zeta\right\}\sqrt{a}\,d\alpha^1 d\alpha^2,$$

or by means of equalities (4.19)

$$\delta V=\iint_S\left\{S^{\alpha 3}\delta e_{\alpha 3}+H^{\alpha 3}\delta(-\beta_{\alpha 3})\right\}\sqrt{a}\,d\alpha^1 d\alpha^2. \tag{5.2}$$

The resulting equality gives us a basis for calling the components ϵ_{ij} $(-\beta_{ij})$ the energy components with respect to symmetric forces and moments (i.e., corresponding to them in the expression for the energy of deformation).

By using equalities (4.19) and (2.35) it is easy to transform (5.2) into the form

$$\delta V=\int_S\int\left\{T^{\alpha\beta}\delta\varepsilon_{\alpha\beta}+M^{\alpha\beta}\delta\mu_{\alpha\beta}\right\}\sqrt{a}\,d\alpha^1d\alpha^2, \tag{5.2'}$$

which shows that within the accuracy of relationships for the theory of thin shells the quantities ε_{ij} and μ_{ij} may be looked upon as the energy components of deformation for the forces and moments T^{ij}, M^{ij}.

Let us transform (5.2') by integrating it in parts and utilizing the Green equations, which transform the surface integrals over the middle surface S into line integrals over contour Γ bounding S. When integration is carried out in the positive direction (fig. 3), the following equations are valid (ref. 40, vol. 2) /97

$$\begin{aligned}\int_S\int\frac{\partial\Phi}{\partial\alpha^1}\,d\alpha^1d\alpha^2&=\oint_\Gamma\Phi d\alpha^2;\\ \int_S\int\frac{\partial\Phi}{\partial\alpha^2}\,d\alpha^1d\alpha^2&=-\oint_\Gamma\Phi d\alpha^1.\end{aligned} \tag{5.3}$$

Since, according to (Chapter VII, 7.6 and 7.9)

$$\begin{aligned}\frac{d\alpha^i}{ds_t}&=t^i=c^{\alpha i}v_\alpha;\\ d\alpha^1&=t^1ds_t=c^{21}v_2ds_t=-\frac{v_2}{\sqrt{a}}\,ds_t;\\ d\alpha^2&=t^2ds_t=c^{12}v_1ds_t=\frac{v_1}{\sqrt{a}}\,ds_t,\end{aligned} \tag{5.4}$$

they can be transformed to the form

$$\int_S\int\frac{\partial\Phi}{\partial\alpha^i}\,d\alpha^1d\alpha^2=\oint_\Gamma v_i\Phi\frac{ds_t}{\sqrt{a}}. \tag{5.5}$$

This equation, as we can see from its derivation, does not assume that the function Φ is necessarily invariant. It can be any extensive variable. In particular let us assume that Φ^i is a component of a contravariant vector.

Let us apply equation (5.5) to the extensive variable $(\sqrt{a}\,\Phi^i)$. Taking the sum over i we arrive at the Green equation

$$\iint_S \frac{\partial \sqrt{a}\,\Phi^\alpha}{\partial \alpha^\alpha}\,d\alpha^1 d\alpha^2 = \oint_\Gamma \nu_\alpha \Phi^\alpha ds_t. \tag{5.6}$$

However, according to (4.13),

$$\frac{\partial \sqrt{a}\,\Phi^\alpha}{\partial \alpha} = \sqrt{a}\,\nabla_\alpha \Phi^\alpha = \nabla_\alpha (\sqrt{a}\,\Phi^\alpha). \tag{5.7}$$

(The latter transformation is valid, because $\sqrt{a} = c_{12}$, while the quantities c_{ij} according to (Chapter VII, 5.46) behave like constants during covariant differentiation. By using the last equality we obtain one other version of the Green equation

$$\iint_S \nabla_\alpha (\sqrt{a}\,\Phi^\alpha)\,d\alpha^1 d\alpha^2 = \oint_\Gamma \nu_\alpha \Phi^\alpha ds_t. \tag{5.8}$$

Utilizing the expressions which we have obtained and equalities (2.21) and (2.24) we carry out the required transformation. Thus integrating by parts /98 and applying (5.8), we obtain

$$\iint_S T^{\alpha\beta}\delta\varepsilon_{\alpha\beta}\sqrt{a}\,d\alpha^1 d\alpha^2 = \iint_S \sqrt{a}\,T^{\alpha\beta}\delta\{\nabla_\alpha u_\beta - b_{\alpha\beta}w - c_{\alpha\beta}\Omega_n\}\,d\alpha^1 d\alpha^2 = \iint_S \{-\nabla_\alpha(\sqrt{a}\,T^{\alpha\beta})\delta u_\beta - \sqrt{a}\,b_{\alpha\beta}T^{\alpha\beta}\delta w - \underline{\sqrt{a}\,c_{\alpha\beta}T^{\alpha\beta}\delta\Omega_n}\}\,d\alpha^1 d\alpha^2 + \oint_\Gamma \nu_\alpha T^{\alpha\beta}\delta u_\beta ds_t;$$

$$\iint_S M^{\beta\alpha}\delta\varkappa_{\beta\alpha}\sqrt{a}\,d\alpha^1 d\alpha^2 = \iint_S \sqrt{a}\,M^{\beta\alpha}\delta(\nabla_\beta \vartheta_\alpha - c_{\alpha\gamma}b^\gamma_\beta \delta\Omega_n)\,d\alpha^1 d\alpha^2 = \iint_S \{-\underline{\underline{\nabla_\beta(\sqrt{a}\,M^{\beta\alpha})\delta\vartheta_\alpha}} - \underline{\sqrt{a}\,c_{\alpha\gamma}b^\gamma_\beta M^{\beta\alpha}\delta\Omega_n}\}\,d\alpha^1 d\alpha^2 + \oint_\Gamma \nu_\beta M^{\beta\alpha}\delta\vartheta_\alpha ds_t.$$

In view of equality (4.18) the sum of the terms underlined in both integrals by one line drop out. By further transforming the terms underlined with a double line we find

$$\iint_S \{-\sqrt{a}\,\nabla_\beta M^{3\alpha}\delta\left(-\frac{\partial w}{\partial\alpha^\alpha}-b_\alpha^\gamma u_\gamma\right)d\alpha^1 d\alpha^2 =$$

$$-\iint_S\left\{-\frac{\partial}{\partial\alpha^\alpha}\left(\sqrt{a}\,\nabla_\beta M^{\beta\alpha}\right)\delta w+\sqrt{a}\,b_\alpha^\gamma\nabla_\beta M^{\beta\alpha}\delta u_\gamma\right\}d\alpha^1 d\alpha^2+$$

$$+\oint_\Gamma \nu_\alpha\nabla_\beta M^{\beta\alpha}\delta w\,ds_t.$$

Collecting the transformed expressions we obtain

$$\delta V=\delta V_1+\delta V_2, \tag{5.9}$$

where

$$\delta V_1=-\iint_S\left\{\left(\nabla_\alpha T^{\alpha\beta}-b_\alpha^\beta\nabla_\delta M^{\delta\alpha}\right)\delta u_\beta+\right.$$
$$\left.+\left(\nabla_\alpha\nabla_\delta M^{\delta\alpha}+b_{\alpha\delta}T^{\alpha\delta}\right)\delta w\right\}\sqrt{a}\,d\alpha^1 d\alpha^2 \tag{5.10}$$

and (see $(4.20)_3$, (3.3))

$$\delta V_2=\oint_\Gamma\left\{\nu_\alpha T^{\alpha\beta}\delta u_\beta+\nu_\alpha\nabla_\beta M^{\beta\alpha}\delta w+\nu_\beta M^{\beta\alpha}\delta\vartheta_\alpha\right\}ds_t=$$
$$=\oint_\Gamma\left\{\nu_\alpha\nu_\beta T^{\alpha\beta}\delta u_\nu+\nu_\alpha t_\beta T^{\alpha\beta}\delta u_t+\nu_\alpha T^{\alpha n}\delta w+\right. \tag{5.11}$$
$$\left.+\nu_\beta\nu_\alpha M^{\beta\alpha}\delta\vartheta_\nu+\nu_\beta t_\alpha M^{\beta\alpha}\delta\vartheta_t\right\}ds_t=$$
$$=\oint_\Gamma\left\{T_{\nu\nu}\delta u_\nu+T_{\nu t}\delta u_t+T_{\nu n}\delta w+M_{\nu\nu}\delta\vartheta_\nu+\underline{M_{\nu t}\delta\vartheta_t}\right\}ds_t.$$

Transforming the underlined term by means of $(3.15)_3$ and assuming for simplicity that the twisting moment $M_{\nu t}$ has no discontinuity at the contour, we obtain /99

$$\oint_\Gamma M_{\nu t}\delta\vartheta_t ds_t=\oint M_{\nu t}\left(-\frac{d\delta w}{ds_t}+\tau_t\delta u_\nu-\sigma_t\delta u_t\right)ds_t=$$
$$=\oint_\Gamma\left\{M_{\nu t}\left(\tau_t\delta u_\nu-\sigma_t\delta u_t\right)+\frac{dM_{\nu t}}{ds_t}\delta w\right\}ds_t.$$

By substituting the transformed expression into (5.11) we write δV_2 in the form

$$\delta V_2 = \oint_\Gamma \Big\{ (T_{\nu\nu} + \tau_t M_{\nu t})\,\delta u_\nu + (T_{\nu t} - \sigma_t M_{\nu t})\,\delta u_t + \Big(T_{\nu n} + \frac{dM_{\nu t}}{ds_t} \Big) \delta w + M_{\nu\nu}\delta\vartheta_\nu \Big\}\, ds_t. \tag{5.12}$$

From this we can see that within the framework of the theory utilizing the Kirchhoff hypothesis the generalized edge displacements u_ν, u_t, w, ϑ_ν are associated with the generalized forces

$$\begin{aligned} &Q_{\nu\nu} \equiv T_{\nu\nu} + \tau_t M_{\nu t};\\ &Q_{\nu t} \equiv T_{\nu t} - \sigma_t M_{\nu t};\\ &Q_{\nu n} \equiv T_{\nu n} + \frac{dM_{\nu t}}{ds_t};\\ &M_{\nu\nu},\ \vec{Q}_\nu = Q_{\nu\nu}\vec{\nu} + Q_{\nu t}\vec{t} + Q_{\nu n}\vec{n}. \end{aligned} \tag{5.13}$$

The Lagrange variational equation (Chapter IX, 4.3) is written in the invariant form

$$\begin{aligned} &\delta V - \int\int_S \{ q^\alpha \delta u_\alpha + q_n \delta w \} \sqrt{a}\, d\alpha^1 d\alpha^2 -\\ &- \oint_\Gamma \{ Q^0_{\nu\nu}\delta u_\nu + Q^0_{\nu t}\delta u_t + Q^0_{\nu n}\delta w + M^0_{\nu\nu}\delta\vartheta_\nu \}\, ds_t = 0. \end{aligned} \tag{5.14}$$

Substituting here the transformed expressions for δV_1 and δV_2 and assuming that the variations in the displacements and in the angle of rotation are arbitrary, we become convinced that the equilibrium equations (4.21) and the static boundary conditions follow from the variational equation

$$\begin{aligned} &Q_{\nu\nu} \equiv T_{\nu\nu} + \tau_t M_{\nu t} = Q^0_{\nu\nu};\\ &Q_{\nu t} \equiv T_{\nu t} - \sigma_t M_{\nu t} = Q^0_{\nu t};\\ &Q_{\nu n} \equiv \nu_\alpha \nabla_\beta M^{\beta\alpha} + \frac{dM_{\nu t}}{ds_t} = Q^0_{\nu n};\\ &M_{\nu\nu} = M^0_{\nu\nu}; \end{aligned} \tag{5.15}$$

$$\vec{Q}_\nu \equiv Q_{\nu\nu}\vec{\nu} + Q_{\nu t}\vec{t} + Q_{\nu n}\vec{n} = \vec{Q}^0_\nu. \tag{5.16}$$

Sometimes it is convenient to utilize the directions associated with the boundary contour (section 7, Chapter VII). In this system of coordinates /100

$$\vec{Q}_\nu = Q_{\nu m}\vec{m} + Q_{\nu b}\vec{b} + Q_{\nu t}\vec{t}, \tag{5.17}$$

and

$$Q_{\nu m} = Q_{\nu_1}\sin\varphi + Q_{\nu_2}\cos\varphi,\ Q_{\nu b} = Q_{\nu_1}\cos\varphi - Q_{\nu_2}\sin\varphi. \tag{5.18}$$

6. Hooke's Law

The second (static) Kirchhoff hypothesis consists of the proposition that $\sigma_{33} \ll \sigma_{ij}$. By adopting it, we obtain the following approximate equality from (1.29)

$$\sigma_{33} = 0 = \lambda(g^{\alpha\beta}e_{\alpha\beta} + e_{33}) + 2\mu e_{33} \quad (\alpha, \beta = 1, 2).$$

From this we find

$$e_{33} = -\frac{\lambda}{\lambda + 2\mu}(g^{\alpha\beta}e_{\alpha\beta});\ \theta = \frac{2\mu}{\lambda + 2\mu}(g^{\alpha\beta}e_{\alpha\beta}).$$

If we substitute the expression obtained for θ into (1.29) and use the well-known relationships

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},\quad \mu = \frac{E}{2(1+\nu)} \tag{6.1}$$

to replace the Lamé constants with the more manageable E and ν, we obtain the relationships of Hooke's law for the principal stresses

$$\begin{aligned} \sigma_{ij} &= \frac{E}{1-\nu^2}\{\nu e g_{ij} + (1-\nu)e_{ij}\};\\ \sigma^{ij} &= \frac{E}{1-\nu^2}\{\nu e g^{ij} + (1-\nu)e^{ij}\} \quad (e = g^{\alpha\beta}e_{\alpha\beta}). \end{aligned} \tag{6.2}$$

To establish relationships between forces-moments and the deformation components we use the equalities

$$S^{ij}=\int_{-h/2}^{h/2}\left(\sigma^{ij}-\underline{\zeta^2 b_\gamma^i b_\delta^j \sigma^{\gamma\delta}}\right)\sqrt{\frac{g}{a}}\,d\zeta;$$
$$H^{ij}=\int_{-h/2}^{h/2}\left(\sigma^{ij}-\zeta\,\underline{\frac{b_\gamma^i\sigma^{j\gamma}+b_\gamma^j\sigma^{i\gamma}}{2}}\right)\sqrt{\frac{g}{a}}\,\zeta d\zeta. \qquad (4.19)$$

If we substitute

$$\sqrt{\frac{g}{a}}=1-2H\zeta+K\zeta^2 \qquad \text{(Chapter VII, 5.33)}$$

into the expressions under the integral signs and replace the stresses by expressions (6.2), in which we compute the quantities g^{ij} by means of equations (Chapter VII, 5.8, 5.32 and 5.33) and the quantities e^{ij} from relationships (2.22) and

$$e^{ij}=g^{i\alpha}g^{j\beta}e_{\alpha\beta},$$

the expressions under the integral sign will appear in the form of some known functions of ζ. Such integrals, even in cases when they are easily evaluated, lead to rather awkward expressions containing a large number of small terms.

Bearing this in mind, we shall derive the simplest version of Hooke's law. For this purpose we shall neglect quantitites of the order $|\zeta b^i_j| \leq \frac{h}{2} b_{(ij)}$ compared with unity. In this case we should, first of all, let $\sqrt{\frac{g}{a}} = 1$. /101

Furthermore, by substituting this expression into equalities (6.2) we become convinced that after the small terms are dropped the stress compcnents appear in the form

$$\sigma^{ij}=\sigma^{ij}_{(0)}+\zeta\sigma^{ij}_{(1)}, \qquad (6.3)$$

where

$$\sigma^{ij}_{(0)}=\frac{E}{1-\nu^2}\left[\nu a^{ij}a^{\alpha\beta}e_{\alpha\beta}+(1-\nu)\,e^{ij}\right];$$
$$\sigma^{ij}_{(1)}=\frac{E}{1-\nu^2}\left[\nu a^{ij}a^{\alpha\beta}(-\beta_{\alpha\beta})+(1-\nu)(-\beta^{ij})\right]. \qquad (6.4)$$

Substituting the resulting simplified relationships into the expressions under the integral sign and neglecting the small underlined terms, we find, after integration,

$$S^{ij} = h\sigma^{ij}_{(0)}; \quad H^{ij} = \frac{h^3}{12}\sigma^{ij}_{(1)}, \tag{6.5}$$

or in expanded form

$$\begin{aligned} S^{ij} &= \frac{Eh}{1-\nu^2}[\nu a^{ij}a^{\alpha\beta}\varepsilon_{\alpha\beta} + (1-\nu)\varepsilon^{ij}]; \\ H^{ij} &= \frac{Eh^3}{12(1-\nu^2)}[\nu a^{ij}a^{\alpha\beta}(-\beta_{\alpha\beta}) + (1-\nu)(-\beta^{ij})]. \end{aligned} \tag{6.6}$$

Eliminating $\sigma^{ij}_{(0)}$ and $\sigma^{ij}_{(1)}$ from relationships (6.5) and (6.3) we obtain

$$\sigma^{ij} = \frac{S^{ij}}{h} + \zeta\frac{12}{h^3}H^{ij}, \tag{6.7}$$

or

$$\begin{aligned} \sigma^{ij} = \frac{E}{1-\nu^2}[&\nu a^{ij}a^{\alpha\beta}\varepsilon_{\alpha\beta} + (1-\nu)\varepsilon^{ij} + \\ &+ \zeta[\nu a^{ij}a^{\alpha\beta}(-\beta_{\alpha\beta}) + (1-\nu)(-\beta^{ij})]]. \end{aligned} \tag{6.8}$$

We note, first of all, that in deriving relationships (6.6) we have neglected the underlined term which distinguishes H^{ij} from M^{ij} and M^{ji} in $(4.19)_2$. Thus, within the limits of the accuracy inherent in the theory of thin shells, we may assume that (ref. 33, p. 55)

$$M^{ij} = M^{ji} = H^{ij}. \tag{6.9}$$

Let us now turn to the quantities (2.35) and (4.22). First of all, it follows from $(4.19)_1$ and (6.9) that

$$S^{ij} = T^{ij} + b^j_1 M^{1i} = T^{ij} + b^j_1 H^{1i}.$$

Substituting these quantities into (6.7), we obtain

$$\sigma^{ij} = \frac{T^{ij}}{h} + \frac{12}{h^3}\left(\zeta H^{ij} + \underline{h\left(hb_\gamma^i\right)H^{\gamma j}}\right).$$

From this we see that the difference in determining the stresses introduced by replacing S^{ij} with T^{ij} is negligibly small. Therefore we may adopt the following law for the forces T^{ij} /102

$$T^{ij} = \frac{Eh}{1-\nu^2}\left[\nu a^{ij}a^{\alpha\beta}\varepsilon_{\alpha\beta} + (1-\nu)\varepsilon^{ij}\right]. \qquad (6.10)_1$$

Furthermore, by replacing $(-\beta_{ij})$ in $(6.6)_2$ with their expressions from (2.35) we obtain, in place of (6.8), the following equality

$$\sigma^{ij} = \frac{E}{1-\nu^2}\{\nu a^{ij}a^{\alpha\beta}\varepsilon_{\alpha\beta} + (1-\nu)\varepsilon^{ij} + \\ + \zeta\left[\nu a^{ij}a^{\alpha\beta}\left(\mu_{\alpha\beta} - \underline{b_\alpha^\gamma\varepsilon_{\beta\gamma}}\right) + (1-\nu)\left(\mu^{ij} - \underline{b_\gamma^i\varepsilon^{j\gamma}}\right)\right]\}.$$

By dropping the small underlined terms in this equality we may assume that

$$M^{ij} = \frac{Eh^3}{12(1-\nu^2)}\left[\nu a^{ij}a^{\alpha\beta}\mu_{\alpha\beta} + (1-\nu)\mu^{ij}\right]. \qquad (6.10)_2$$

If we make use of the mixed quantities (2.39) and (4.24), we should use Hooke's law in the form (6.6), retain the quantities $S^{12} = S^{21}$, $(-\beta_{12}) = (-\beta_{21})$ and replace

$$S^{11},\ S^{22},\ H^{11},\ H^{12} = H^{21},\ H^{22},\ (-\beta_{11}),\ (-\beta_{22}) \qquad (6.11)$$

respectively, with

$$T^{11},\ T^{22},\ M^{11},\ M^{12} = M^{21},\ M^{22},\ \mu_{11},\ \mu_{22}. \qquad (6.12)$$

Let us solve relationships $(6.6)_1$ with respect to the components of deformation. For this purpose we multiply the left and right side by a_{ij} and

make a summation over both signs. As a result we obtain (remembering that $a^{\gamma\delta} a_{\gamma\delta} = \delta^{\gamma}_{\gamma} = 2$)

$$a^{\alpha\beta}\varepsilon_{\alpha\beta} = \frac{1-\nu}{Eh} S^{\alpha\beta} a_{\alpha\beta}. \tag{6.13}$$

Substituting the resulting expression into $(6.6)_1$ and solving it with respect to ε^{ij} we obtain, after dropping the indices,

$$\varepsilon_{ij} = \frac{1}{Eh}[(1+\nu) S_{ij} - \nu a_{ij} a^{\alpha\beta} S_{\alpha\beta}]. \tag{6.14}_1$$

By using analogous transformations we find from $(6.6)_2$

$$(-\beta_{ij}) = \frac{1}{Eh^3}[(1+\nu) H_{ij} - \nu a_{ij} a^{\alpha\beta} H_{\alpha\beta}]. \tag{6.14}_2$$

In the same manner we find the relationships which are converse to (6.10)

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{Eh}[(1+\nu) T_{ij} - \nu a_{ij} a^{\alpha\beta} T_{\alpha\beta}]; \\ \beta_{ij} &= \frac{1}{Eh^3}[(1+\nu) M_{ij} - \nu a_{ij} a^{\alpha\beta} M_{\alpha\beta}]. \end{aligned} \tag{6.15}$$

On the other hand, to utilize the mixed components of deformation and forces it is necessary to use relationships (6.14) making the replacement (6.11) and (6.12) in them.

7. Static Geometry Analogy. Complex Equations

Comparing the homogeneous equations (4.20) with (2.36) we can see that they are of one type and change from one to another when we replace /103

$$\begin{aligned} T^{ij} &\text{ by } c^{i\gamma} c^{j\beta} \mu_{\gamma\beta}; \\ M^{ij} &\text{ by } -c^{i\gamma} c^{j\beta} \varepsilon_{\gamma\beta}; \\ T^{in} &\text{ by } -c^{i\beta} \zeta_{\beta}. \end{aligned}$$

This noted property is usually called the static-geometric analogy and makes it possible for us to introduce the stress functions. Let us consider the expressions

$$\begin{aligned} T^{ij} &= T^{ij*} + Ehc\, c^{i\gamma} c^{j\beta} \overline{\mu}_{\gamma\beta}; \\ M^{ij} &= M^{ij*} - Ehc\, c^{i\gamma} c^{j\beta} \overline{\varepsilon}_{\gamma\beta}; \\ T^{in} &= T^{in*} - Ehc\, c^{i\beta} \overline{\zeta}_{\beta} \quad \left(c = \frac{h}{\sqrt{12(1-\nu^2)}}\right). \end{aligned} \tag{7.1}$$

Here the quantities T^{ij*}, M^{ij*}, T^{in*} constitute some solution of the nonhomogeneous system of equilibrium equations, while $\overline{\mu}_{ij}$, $\overline{\varepsilon}_{ij}$, $\overline{\zeta}_i$ are the expressions obtained above, which, however, are not constructed in terms of displacement vector components, but rather in terms of arbitrary functions $\overline{u}_1$, $\overline{u}_2$, $\overline{w}$. Substituting these expressions into equation (4.20), we become convinced that they satisfy them, in view of equations (2.36), for any values $\overline{u}_1$, $\overline{u}_2$, $\overline{w}$. This gives us the basis for considering them as stress functions.

Let us introduce the complex forces and moments interpreting them to mean the following complex combinations

$$\begin{aligned} \widetilde{T}^{ij} &= T^{ij} - \sqrt{-1}\, Ehc\, c^{i\gamma} c^{j\beta} \mu_{\gamma\beta}, \\ \widetilde{M}^{ij} &= M^{ij} + \sqrt{-1}\, Ehc\, c^{i\gamma} c^{j\beta} \varepsilon_{\gamma\beta}, \\ \widetilde{T}^{in} &= T^{in} + \sqrt{-1}\, Ehc\, c^{i\beta} \zeta_{\beta}. \end{aligned} \tag{7.2}$$

Here, by replacing the forces and moments with their expressions from (7.1), we obtain

$$\begin{aligned} \widetilde{T}^{ij} &= T^{ij*} - \sqrt{-1}\, Ehc\, c^{i\gamma} c^{j\beta} \widetilde{\mu}_{\gamma\beta}; \\ \widetilde{M}^{ij} &= M^{ij*} + \sqrt{-1}\, Ehc\, c^{i\gamma} c^{j\beta} \widetilde{\varepsilon}_{\gamma\beta}; \\ \widetilde{T}^{in} &= T^{in*} + \sqrt{-1}\, Ehc\, c^{i\beta} \widetilde{\zeta}_{\beta}. \end{aligned} \tag{7.3}$$

where $\widetilde{\mu}_{ij}$, $\widetilde{\varepsilon}_{ij}$, $\widetilde{\zeta}_i$ are the respective expressions constructed from the following complex combinations of displacements and stress functions

$$\widetilde{u}_i = u_i + \sqrt{-1}\, \overline{u}_i; \quad \widetilde{w} = w + \sqrt{-1}\, \overline{w}, \tag{7.4}$$

which are called complex displacements.

The resulting relationships establish a coincidence (analogy) betwe n /104
geometric static and complex quantities. It is perti ent to call this correspondence the complex analogy.

To find equations which are satisfied by the introduced complex quantities, we multiply equations (2.36) by $(-\sqrt{-1}\ Ehc)$ and add them to the corresponding equations of system (4.20). The system obtained in this manner is written in the following form, using the designations (7.2)

$$\begin{aligned} &\nabla_\alpha \widetilde{T}^{\alpha j} - b_\alpha^\beta \widetilde{T}^{\alpha n} + q^j = 0; \\ &\nabla_\alpha \widetilde{T}^{\alpha n} + b_{\alpha\delta} \widetilde{T}^{\alpha\delta} + q_n = 0; \\ &\nabla_\delta \widetilde{M}^{ij} - \widetilde{T}^{jn} = 0. \end{aligned} \tag{7.5}$$

We obtain the nondifferential relationship

$$c_{\alpha\beta}(\widetilde{T}^{\alpha\beta} + b_\gamma^\beta \widetilde{M}^{\gamma\alpha}) = 0. \tag{7.6}$$

from equations $(2.33)_1$ and (4.18).

The introduced complex moments $\widetilde{M}^{ij}$ are actually not independent quantities and may be expressed in terms of the complex forces $\widetilde{T}^{ij}$ by means of Hooke's law relationships. Indeed, according to $(6.10)_2$ and $(6.15)_1$

$$\begin{aligned} \widetilde{M}^{ij} = Ehc^2 [\nu a^{ij} a^{\alpha\beta} \mu_{\alpha\beta} + (1-\nu)\mu^{ij}] - \\ + \sqrt{-1}\, c\, c^{i\gamma} c^{j\beta} [(1+\nu) \operatorname{Re} \widetilde{T}_{\beta\gamma} - \nu a_{\beta\gamma} a^{\delta\nu} \operatorname{Re} \widetilde{T}_{\delta\nu}] \\ \left(c = \frac{h}{\sqrt{12(1-\nu^2)}}\right). \end{aligned} \tag{7.7}$$

From (7.3) we find

$$\operatorname{Im} \widetilde{T}^{ik} = -Ehc\, c^{i\gamma} c^{k\beta} \mu_{\beta\gamma}.$$

Multiplying the left and right sides by $c_{ij} c_{ki}$ and taking the summation over i and k we, obtain, taking into account equality (Chapter VII, 5.21),

$$\mu_{ij} = -\frac{1}{Ehc} c_{\alpha j} c_{\nu i} \operatorname{Im} \widetilde{T}^{\alpha\nu}. \tag{7.8}$$

Substituting these expressions into (7.7), we obtain after simple transformations (which the readers should carry out)

$$\widetilde{M}^{ij}=\sqrt{-1}\,c\left\{c^{i\gamma}c^{j\beta}\widetilde{T}_{\beta\gamma}-\nu a^{i\beta}a^{j\gamma}\overline{\widetilde{T}}_{\beta\gamma}\right\} \tag{7.9}$$

(the bar over the wave designates the conjugation sign for the complex quantity). By using identities (Chapter VII, 5.20) equality (7.9) may be given a slightly different form

$$\widetilde{M}^{ij}=\sqrt{-1}\,c\left\{a^{ij}a^{\gamma\beta}\widetilde{T}_{\gamma\beta}-\left(\widetilde{T}^{ij}+\nu\overline{\widetilde{T}}^{ij}\right)\right\}. \tag{7.10}$$

The resulting relationships make it possible to eliminate the complex moments from system (7.5). In this case, instead of $(7.5)_3$ we obtain the equation /105

$$\sqrt{-1}\,c\left\{\nabla_\delta a^{\delta j}a^{\gamma\beta}\widetilde{T}_{\gamma\beta}-\underline{\nabla_\delta\widetilde{T}^{\delta j}-\nu\nabla_\delta\overline{\widetilde{T}}^{\delta j}}\right\}-\widetilde{T}^{jn}=0.$$

Replacing the underlined terms by their expressions from $(7.5)_1$ we find

$$\sqrt{-1}\,c\left\{\nabla_\delta a^{\delta j}a^{\gamma\beta}\widetilde{T}_{\gamma\beta}-\underline{b^j_\alpha\widetilde{T}^{\alpha n}-\nu b^j_\alpha\overline{\widetilde{T}}^{\alpha n}}+(1+\nu)\,q^j\right\}-\widetilde{T}^{jn}=0.$$

Since for thin shells $|cb^i_j|\ll 1$, we can neglect the underlined terms, which gives us a simpler equation for the cutting complex forces

$$\widetilde{T}^{jn}=\sqrt{-1}\,c\left\{a^{j\beta}\nabla_\delta\left(a^{\gamma\beta}\widetilde{T}_{\gamma\beta}\right)+(1+\nu)\,q^j\right\}. \tag{7.11}$$

Using these to eliminate $\widetilde{T}^{jn}$ from the remaining equations, we obtain the simplified system of equations in terms of the complex forces

$$\begin{aligned}&\nabla_\alpha\widetilde{T}^{\alpha j}-\sqrt{-1}\,cb^j_\alpha a^{\alpha\delta}\nabla_\delta\widetilde{T}=-q^j+\sqrt{-1}\,(1+\nu)\,cb^j_\alpha q^\alpha\approx-q^j;\\&\sqrt{-1}\,ca^{\alpha\delta}\nabla_\alpha\nabla_\delta\widetilde{T}+b_{\alpha\delta}\widetilde{T}^{\alpha\delta}=-q_n-\sqrt{-1}\,(1+\nu)\,c\nabla_\alpha q^\alpha\approx-q_n.\end{aligned} \tag{7.12}$$

Here the expression

$$\widetilde{T}=a^{ij}\widetilde{T}_{ij} \tag{7.13}$$

is used to designate the basic V. V. Novozhilov complex function.

Substituting expressions (7.9) into the nondifferential relationships (7.6), we see that with the same degree of accuracy it can be taken in the form

$$c_{\alpha\beta}\widetilde{T}^{\alpha\beta}=0 \quad \left(\widetilde{T}^{12}-\widetilde{T}^{21}=0\right). \tag{7.14}$$

Thus, in the simplified system we may assume that

$$\widetilde{T}^{12}=\widetilde{T}^{21}=\widetilde{S}^{12}=\widetilde{S}. \tag{7.15}$$

Returning to relationship (7.9) we see that the terms containing the Poisson ratio ν as a factor have not entered the system of complex equations (7.12), which is a complete system of equations for the theory of shells. This gives us a basis to consider these terms insignificant. They can be dropped. However, we shall retain them with slight modifications, thus assuming the /106 following expressions for the complex moments (section 7, Chapter IV; Chapter VII, 5.20_2)

$$\begin{aligned}\widetilde{M}^{ij}&=\sqrt{-1}\,c\left\{c^{i\tau}c^{j\beta}\widetilde{T}_{\beta\tau}-\nu a^{i\beta}a^{j\tau}\widetilde{T}_{\beta\tau}\right\}=\\&=\sqrt{-1}\,c\left\{c^{i\tau}c^{j\beta}\widetilde{T}_{\beta\tau}-\nu\widetilde{T}^{ij}\right\}=\sqrt{-1}\,c\left\{a^{ij}\widetilde{T}-(1+\nu)\widetilde{T}^{ij}\right\}.\end{aligned} \tag{7.16}$$

Now equalities $(7.3)_{1;2}$ may be written in the form

$$\begin{aligned}\sqrt{-1}\,c\left\{c^{i\tau}c^{j\beta}\widetilde{T}_{\beta\tau}-\nu\widetilde{T}^{ij}\right\}&=M^{ij*}+\sqrt{-1}\,Ehc\,c^{i\tau}c^{j\beta}\widetilde{\varepsilon}_{\beta\tau};\\ \widetilde{T}^{ij}&=T^{ij*}-\sqrt{-1}\,Ehc\,c^{i\tau}c^{j\beta}\widetilde{\mu}_{\tau\beta}.\end{aligned}$$

Eliminating the complex forces from these equations, we obtain a system of equations containing the complex displacements

$$\begin{aligned}&c^{i\tau}c^{j\beta}\widetilde{\varepsilon}_{\beta\tau}+\sqrt{-1}\,c\left\{\widetilde{\mu}^{ji}-\nu c^{i\tau}c^{j\beta}\widetilde{\mu}_{\tau\beta}\right\}=\\&=\frac{1}{Eh}\left\{c^{i\tau}c^{j\beta}T^{*}_{\beta\tau}-\nu T^{ij*}+\frac{\sqrt{-1}}{c}M^{ij*}\right\}.\end{aligned} \tag{7.17}$$

We note that in this system we may assume approximately that

$$\widetilde{\mu}_{12}=\widetilde{\mu}_{21}=-\widetilde{\beta}_{12}. \tag{7.18}$$

Therefore, there are only three equations which differ substantially.

Let us return to the boundary quantities (5.13), (3.19) and $(3.15)_1$. They are associated with the following complex quantities

$$\begin{aligned}
&\widetilde{Q}_{vv}=\nu_\alpha\nu_\beta\widetilde{T}^{\alpha\beta}+\tau_t\widetilde{M}_{vt};\\
&\widetilde{Q}_{vt}=\nu_\alpha t_\beta\widetilde{T}^{\alpha\beta}-\tau_t\widetilde{M}_{vt};\\
&\widetilde{Q}_{vn}=\nu_\alpha\widetilde{T}^{\alpha n}+\frac{d\widetilde{M}_{vt}}{ds_t};\\
&\widetilde{M}_{vv};\\
&\vec{\widetilde{Q}}_v=\widetilde{Q}_{vv}\vec{\nu}+\widetilde{Q}_{vt}\vec{t}+\widetilde{Q}_{vn}\vec{n}.
\end{aligned} \tag{7.19}$$

To clarify their static-geometric meaning, we separate the real and imaginary parts. Let us, for example, consider the quantity $\nu_\alpha\nu_\beta\widetilde{T}^{\alpha\beta}$, which is contained in the first equation. According to (7.2) and (Chapter VII, 7.6)

$$\begin{aligned}
\nu_\alpha\nu_\beta\widetilde{T}^{\alpha\beta}&=\nu_\alpha\nu_\beta T^{\alpha\beta}-\sqrt{-1}\,Ehc\left\{\nu_\alpha\nu_\beta c^{\alpha\gamma}c^{\beta\delta}\mu_{\gamma\delta}\right\}=\\
&=\nu_\alpha\nu_\beta T^{\alpha\beta}-\sqrt{-1}\,Ehc\left\{t^\gamma t^\delta\mu_{\gamma\delta}\right\}=T_{vv}-\sqrt{-1}\,Ehc\mu_{tt}.
\end{aligned}$$

By applying similar transformations to the other terms we obtain /107

$$\begin{aligned}
&\widetilde{Q}_{vv}=Q_{vv}-\sqrt{-1}\,Ehc\varkappa_{tt};\\
&\widetilde{Q}_{vt}=Q_{vt}+\sqrt{-1}\,Ehc\varkappa_{tv};\\
&\widetilde{Q}_{vn}=Q_{vn}-\sqrt{-1}\,Ehc\varkappa_{tn};\\
&\widetilde{M}_{vv}=M_{vv}+\sqrt{-1}\,Ehc\varepsilon_{tt};\quad\vec{\widetilde{Q}}_v=\vec{Q}_v+\sqrt{-1}\,Ehc\vec{\varkappa}_t.
\end{aligned} \tag{7.20}$$

From this we find that

$$\begin{aligned}
&Q_{vv}=Q^*_{vv}+Ehc\bar{\varkappa}_{tt};\\
&Q_{vt}=Q^*_{vt}-Ehc\bar{\varkappa}_{tv};
\end{aligned} \tag{7.21}$$

$$Q_{\nu n}=Q^{*}_{\nu n}+Ehc\bar{\varkappa}_{tn};$$
$$M_{\nu\nu}=M^{*}_{\nu\nu}-Ehc\bar{\varepsilon}_{tt};\quad \vec{Q}_{\nu}=\vec{Q}^{*}_{\nu}-Ehc\vec{\bar{\varkappa}}_{t}$$

and

$$\begin{aligned}
&\widetilde{Q}_{\nu\nu}=Q^{*}_{\nu\nu}-\sqrt{-1}\,Ehc\widetilde{\varkappa}_{tt};\\
&\widetilde{Q}_{\nu t}=Q^{*}_{\nu t}+\sqrt{-1}\,Ehc\widetilde{\varkappa}_{tv};\\
&\widetilde{Q}_{\nu n}=Q^{*}_{\nu n}-\sqrt{-1}\,Ehc\widetilde{\varkappa}_{tn};\\
&\widetilde{M}_{\nu\nu}=M^{*}_{\nu\nu}+\sqrt{-1}\,Ehc\widetilde{\varepsilon}_{tt};\quad \vec{\widetilde{Q}}_{\nu}=\vec{Q}^{*}_{\nu}+\sqrt{-1}\,Ehc\vec{\widetilde{\varkappa}}_{t}.
\end{aligned}\tag{7.22}$$

Let us simplify expressions (7.19). First of all according to (7.16)

$$\widetilde{M}_{\nu\nu}=\sqrt{-1}\,c\left\{\nu_{\alpha}\nu_{\beta}a^{\alpha\beta}\widetilde{T}-(1+\nu)\,\nu_{\alpha}\nu_{\beta}\widetilde{T}^{\alpha\beta}\right\};$$
$$\widetilde{M}_{\nu t}=\sqrt{-1}\,c\left\{\nu_{\alpha}t_{\beta}a^{\alpha\beta}\widetilde{T}-(1+\nu)\,\nu_{\alpha}t_{\beta}\widetilde{T}^{\alpha\beta}\right\}.$$

Since

$$\begin{aligned}
&a^{\alpha\beta}\nu_{\alpha}\nu_{\beta}=\vec{\nu}\,\vec{\nu}=1,\quad a^{\alpha\beta}\nu_{\alpha}t_{\beta}=\vec{\nu}\,\vec{t}=0,\\
&\widetilde{M}_{\nu\nu}=\sqrt{-1}\,c\left\{\widetilde{T}-(1+\nu)\,\widetilde{T}_{\nu\nu}\right\},\\
&\widetilde{M}_{\nu t}=\sqrt{-1}\,c\left\{-(1+\nu)\,\widetilde{T}_{\nu t}\right\}.
\end{aligned}\tag{7.23}$$

Substituting these equalities into the first two expressions of (7.19), we obtain

$$\begin{aligned}
&\widetilde{Q}_{\nu\nu}=\widetilde{T}_{\nu\nu}-\sqrt{-1}\,(1+\nu)\,(c\tau_{t})\,\widetilde{T}_{\nu t}\approx\widetilde{T}_{\nu\nu};\\
&\widetilde{Q}_{\nu t}=\widetilde{T}_{\nu t}+\sqrt{-1}\,(1+\nu)\,(c\tau_{t})\,\widetilde{T}_{\nu t}\approx\widetilde{T}_{\nu t},
\end{aligned}\tag{7.24}$$

since

$$c\tau_{t}=\frac{1}{\sqrt{12(1-\nu^{2})}}(h\tau_{t})\ll 1.$$

Let us now consider the third quantity. In view of (7.11)

$$\widetilde{Q}_{\nu n}=\nu_\alpha\sqrt{-1}\,c\left\{a^{\alpha\delta}\nabla_\delta\widetilde{T}+(1+\nu)\,q^\alpha\right\}-$$
$$-\sqrt{-1}\,c(1+\nu)\frac{d\widetilde{T}_{\nu t}}{ds_t}.$$

Now by putting all the results together we obtain the following simplified expressions for the complex boundary conditions /108

$$\widetilde{Q}_{\nu\nu}=\widetilde{T}_{\nu\nu};\ \widetilde{Q}_{\nu t}=\widetilde{T}_{\nu t};$$
$$\widetilde{Q}_{\nu n}+\sqrt{-1}(1+\nu)c\left\{\frac{d\widetilde{T}_{\nu t}}{ds_t}-q_\nu\right\}=\sqrt{-1}\,c\frac{d\widetilde{T}}{ds_\nu};\qquad(7.25)$$
$$\tilde{M}_{\nu\nu}+\sqrt{-1}(1+\nu)c\widetilde{T}_{\nu\nu}=\sqrt{-1}\,c\,\widetilde{T}\quad\left(\frac{d}{ds_\nu}=\nu^\delta\nabla_\delta,\ q_\nu=\nu_\alpha q^\alpha\right).$$

8. Relationships of the Theory of Shells in Terms of the Orthogonal Coordinates

Most frequently we have to deal with shells whose middle surfaces are referred to an orthogonal system of coordinates. We shall write down the most important expressions obtained above, utilizing the physical components of vectors and tensors. First of all we recall that

$$a_{(11)}=a_{(22)}=1;\ a_{(12)}=0;\qquad\text{(Chapter VII, 6.10)}$$
$$c_{(12)}=-c_{(21)}=1;\ c_{(11)}=c_{(22)}=0;\qquad\text{(Chapter VII, 6.11)}$$
$$b_{(11)}=-\frac{1}{R_\alpha};\ b_{(12)}=\frac{1}{R_{\alpha\beta}};\ b_{(22)}=-\frac{1}{R_\beta}.\qquad\text{(Chapter VII, 7.67)}$$

The Codazzi-Gauss relationships are useful in various transformations

$$\frac{\partial}{\partial\beta}\left(\frac{A}{R_\alpha}\right)+\frac{1}{B}\frac{\partial}{\partial\alpha}\left(\frac{B^2}{R_{\alpha\beta}}\right)=\frac{1}{R_\beta}\frac{\partial A}{\partial\beta};$$
$$\frac{\partial}{\partial\alpha}\left(\frac{B}{R_\beta}\right)+\frac{1}{A}\frac{\partial}{\partial\beta}\left(\frac{A^2}{R_{\alpha\beta}}\right)=\frac{1}{R_\alpha}\frac{\partial B}{\partial\alpha};\qquad\text{(Chapter VII, 7.78-7.79)}$$
$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right)=-ABK=$$
$$=-AB\left(\frac{1}{R_\alpha R_\beta}-\frac{1}{R_{\alpha\beta}^2}\right).$$

By designating

$$\vec{r}_{(1)}=\vec{e}_{\alpha},\ \vec{r}_{(2)}=\vec{e}_{\beta}, \tag{8.1}$$

we find $\vec{e}_{\alpha} = \frac{\vec{r}_{\alpha}}{A}$ is a unit vecotr directed along the coordinate line α; $\vec{e}_{\beta} = \frac{\vec{r}_{\beta}}{B}$ is a unit vector directed along β.

If we let /109

$$u_{(1)}=u;\ u_{(2)}=v, \tag{8.2}$$

The vector of displacements takes the form

$$\vec{U}=u\vec{e}_{\alpha}+v\vec{e}_{\beta}+w\vec{n}. \tag{8.3}$$

By using equalities (2.7) and (Chapter VII, 6.6) and by replacing α^1, α^2 with the usual α, β, we find

$$A\vartheta_{(1)}=-\frac{\partial w}{\partial\alpha}-A\left(-\frac{u_{(1)}}{R_{\alpha}}+\frac{u_{(2)}}{R_{\alpha\beta}}\right).$$

By letting

$$\vartheta_{(1)}=\vartheta_{\alpha};\ \vartheta_{(2)}=\vartheta_{\beta}, \tag{8.4}$$

we have

$$\vartheta_{\alpha}=-\frac{\partial w}{A\partial\alpha}+\frac{u}{R_{\alpha}}-\frac{v}{R_{\alpha\beta}};\ \vartheta_{\beta}=-\frac{\partial w}{B\partial\beta}+\frac{v}{R_{\beta}}-\frac{u}{R_{\alpha\beta}}. \tag{8.5}$$

In the same way from (2.10) we find

$$\Omega=\frac{1}{2AB}\left\{\frac{\partial Bv}{\partial\alpha}-\frac{\partial Au}{\partial\beta}\right\}. \tag{8.6}$$

Now the vector of the angles of rotation is written in the form

$$\vec{\Omega} = -\vartheta_\beta \vec{e}_\alpha + \vartheta_\alpha \vec{e}_\beta + \Omega_n \vec{n}. \tag{8.7}$$

Let us now consider the components of deformation. By making use of relationships (2.20), (2.35) and (Chapter VII, 6.12) and introducing the designations

$$\begin{aligned} &\varepsilon_{(11)} = \varepsilon_\alpha, \quad \varepsilon_{(12)} = \frac{\omega}{2}, \quad \varepsilon_{(22)} = \varepsilon_\beta, \\ &\mu_{(11)} = \varkappa_\alpha, \quad (-\beta_{12}) = \tau - \frac{\varepsilon_\alpha + \varepsilon_\beta}{R_{\alpha\beta}}, \quad \mu_{(22)} = \varkappa_\beta, \end{aligned} \tag{8.8}$$

we obtain the following expressions

$$\begin{aligned} \varepsilon_\alpha &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + \frac{w}{R_\alpha}; \\ \omega &= \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - \frac{2w}{R_{\alpha\beta}}; \\ \varepsilon_\beta &= \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + \frac{w}{R_\beta}; \\ \varkappa_\alpha &= \frac{1}{A}\frac{\partial \vartheta_\alpha}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\vartheta_\beta - \frac{\Omega_n}{R_{\alpha\beta}}; \\ \varkappa_\beta &= \frac{1}{B}\frac{\partial \vartheta_\beta}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\vartheta_\alpha + \frac{\Omega_n}{R_{\alpha\beta}}; \\ \tau &= \tau_\beta + \frac{\omega_\alpha}{R_\beta} + \frac{\varepsilon_\beta}{R_{\alpha\beta}} = \tau_\alpha + \frac{\omega_\beta}{R_\alpha} + \frac{\varepsilon_\alpha}{R_{\alpha\beta}}, \end{aligned} \tag{8.9}$$

where

$$\begin{aligned} \omega_\alpha &= \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta} u - \frac{w}{R_{\alpha\beta}}; \quad \tau_\alpha = \frac{1}{A}\frac{\partial \vartheta_\beta}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\vartheta_\alpha; \\ \omega_\beta &= \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha} v - \frac{w}{R_{\alpha\beta}}; \quad \tau_\beta = \frac{1}{B}\frac{\partial \vartheta_\alpha}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\vartheta_\beta \\ &(\omega_\alpha = \omega_{(11)}, \ \omega_\beta = -\omega_{(22)}), \end{aligned} \tag{8.10}$$

In addition,

$$\mu_{(12)} = \tau - \frac{\omega}{2R_\alpha} - \frac{\varepsilon_\alpha}{R_{\alpha\beta}}; \quad \mu_{(21)} = \tau - \frac{\omega}{2R_\beta} - \frac{\varepsilon_\beta}{R_{\alpha\beta}}. \tag{8.11}$$

Let us consider the deformation of a normal element associated with a /110 line on the surface. From relationships of section 3 and by utilizing equalities (Chapter VII, 7.64), we obtain

$$\vec{U} = u_\nu \vec{\nu} + u_t \vec{t} + w\vec{n}, \tag{8.12}$$

where

$$u_\nu = \cos\gamma\, u + \sin\gamma\, v, \quad u_t = -\sin\gamma\, u + \cos\gamma\, v. \tag{8.13}$$

The vector for the rotation of the normal elements is written in the form

$$\vec{\Omega}_t = -\vartheta_t \vec{\nu} + \vartheta_\nu \vec{t} + \omega_{tt}\vec{n}, \tag{8.14}$$

where

$$\begin{aligned} &\vartheta_\nu = \cos\gamma\,\vartheta_\alpha + \sin\gamma\,\vartheta_\beta, \quad \vartheta_t = -\sin\gamma\,\vartheta_\alpha + \cos\gamma\,\vartheta_\beta; \\ &\omega_{tt} = \sin\gamma\cos\gamma\,(\varepsilon_\alpha - \varepsilon_\beta) + \sin^2\gamma\,\omega_\alpha - \cos^2\gamma\,\omega_\beta. \end{aligned} \tag{8.15}$$

The following equation is also valid

$$\frac{d\vec{U}}{ds_t} = \varepsilon_{tt}\vec{t} + [\vec{\Omega}_t, \vec{t}]. \tag{8.16}$$

Here

$$\varepsilon_{tt} = \cos^2\gamma\,\varepsilon_\beta - \sin\gamma\cos\gamma\,\omega + \sin^2\gamma\,\varepsilon_\alpha. \tag{8.17}$$

The vector of curvature is determined from the equality

$$\vec{\varkappa}_t = -\varkappa_{tt}\vec{\nu} + \varkappa_{t\nu}\vec{t} - \varkappa_{tn}\vec{n} = \frac{d\vec{\Omega}_t}{ds_t}, \tag{8.18}$$

where (Chapter VII, 7.69)

$$\begin{aligned} &\varkappa_{tt} = \cos^2\gamma\,\varkappa_\beta - 2\sin\gamma\cos\gamma\,\tau + \sin^2\gamma\,\varkappa_\alpha + \\ &+ \sin\gamma\cos\gamma\left[\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right)\frac{\omega}{2} + \frac{\varepsilon_\alpha + \varepsilon_\beta}{R_{\alpha\beta}}\right] + \frac{\varepsilon_{t\nu}}{R_{wt}}; \\ &\varkappa_{t\nu} = \sin\gamma\cos\gamma\,(\varkappa_\beta - \varkappa_\alpha) + (\cos^2\gamma - \sin^2\gamma)\,\tau + \\ &+ \left(\frac{\sin^2\gamma}{R_\alpha} - \frac{\cos^2\gamma}{R_\beta}\right)\frac{\omega}{2} - \frac{\cos^2\gamma\,\varkappa_\beta - \sin^2\gamma\,\varkappa_\alpha}{R_{\alpha\beta}} - \frac{\varepsilon_{t\nu}}{R_t}; \end{aligned} \tag{8.19}$$

$$\varkappa_{tn}=-\frac{\cos\gamma}{AB}\left\{\frac{\partial B\varepsilon_\beta}{\partial\alpha}-\frac{1}{A}\frac{\partial}{\partial\beta}\left(A^2\frac{\omega}{2}\right)-\frac{\partial B}{\partial\alpha}\varepsilon_\alpha\right\}-$$
$$-\frac{\sin\gamma}{AB}\left\{\frac{\partial A\varepsilon_\alpha}{\partial\beta}-\frac{1}{B}\frac{\partial}{\partial\alpha}\left(B^2\frac{\omega}{2}\right)-\frac{\partial A}{\partial\beta}\varepsilon_\beta\right\}+\frac{d\varepsilon_{tn}}{ds_t}$$
$$\left(\varepsilon_{tn}=\sin\gamma\cos\gamma\,(\varepsilon_\beta-\varepsilon_\alpha)+(\cos^2\gamma-\sin^2\gamma)\frac{\omega}{2};\right.$$
$$\left.\frac{d}{ds_t}=-\frac{\sin\gamma}{A}\frac{\partial}{\partial\alpha}+\frac{\cos\gamma}{B}\frac{\partial}{\partial\beta}\right).$$

By using these components of deformation for the normal element we can find the values of the vector of displacements at an arbitrary point on the middle surface M by means of equation (section 3, Chapter II) /111

$$\vec{U}=\vec{U}^0+[\vec{\Omega}_t^0,\ \vec{r}-\vec{r}^0]+\left[\int_{s_0}^{s_t}\vec{\varkappa}_t ds_t',\ \vec{r}-\vec{r}^0\right]+$$
$$+\int_{s_0}^{s_t}([\vec{r}'-\vec{r}^0,\ \vec{\varkappa}_t]+\varepsilon_{tt}\vec{t}')\,ds_t', \tag{8.20}$$

which may also be given a slightly different form

$$\vec{U}=\vec{U}^0+[\vec{\Omega}_t^0,\ \vec{r}-\vec{r}^0]+$$
$$+\int_{s_0}^{s_t}\left\{\int_{s_0}^{s_t'}[\vec{\varkappa}_t ds_t',\ \vec{t}']+\varepsilon_{tt}\vec{t}'\right\}ds_t'. \tag{8.21}$$

Here the small zeros designate the values of the quantities at some initial point in the curve which is always followed in the integration. In this case, if the dislocation displacements are absent (Chapter X), the values of the displacement vector at an arbitrary point M do not depend on the selection of the curve which joins M and M^0.

As we have already stated in section 4, it is convenient to take the following expressions for the basic forces and moments

$$T_{(11)}=T_\alpha;\ T_{(22)}=T_\beta;\ S_{(12)}=S_{(21)}=S;$$
$$M_{(11)}=M_\alpha;\ M_{(22)}=M_\beta;\ M_{(12)}=M_{(21)}=H. \tag{8.22}$$

In this case

$$T_{(12)}=S-\frac{1}{R_{\alpha\beta}}M_\alpha+\frac{1}{R_\beta}H;\quad T_{(21)}=S-\frac{1}{R_{\alpha\beta}}M_\beta+\frac{1}{R_\alpha}H. \tag{8.23}$$

By utilizing the physical coordinates the system of equilibrium equations (4.21) assumes the form ($q_{(1)}=q_\alpha$, $q_{(2)}=q_\beta$) /112

$$\begin{gathered}
\frac{\partial BT_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2S}{\partial\beta}-\frac{\partial B}{\partial\alpha}T_\beta+\frac{1}{R_\alpha}\left(\frac{\partial BM_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta\right)-\\
-\frac{1}{R_{\alpha\beta}}\left(\frac{\partial AM_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\right)+\frac{\partial}{\partial\beta}\times\\
\times\left(\frac{A}{R_\alpha}H-\frac{A}{R_{\alpha\beta}}M_\beta\right)+\frac{\partial A}{\partial\beta}\left(\frac{1}{R_\beta}H-\frac{1}{R_{\alpha\beta}}M_\alpha\right)+ABq_\alpha=0;\\
\frac{\partial AT_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2S}{\partial\alpha}-\frac{\partial A}{\partial\beta}T_\alpha+\frac{1}{R_\beta}\left(\frac{\partial AM_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\right)-\\
-\frac{1}{R_{\alpha\beta}}\left(\frac{\partial BM_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta\right)+\frac{\partial}{\partial\alpha}\left(\frac{B}{R_\beta}H-\frac{B}{R_{\alpha\beta}}M_\alpha\right)+\\
+\frac{\partial B}{\partial\alpha}\left(\frac{1}{R_\alpha}H-\frac{1}{R_{\alpha\beta}}M_\beta\right)+ABq_\beta=0;\\
\frac{T_\alpha}{R_\alpha}-\frac{2S}{R_{\alpha\beta}}+\frac{T_\beta}{R_\beta}-\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{1}{A}\left(\frac{\partial BM_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta\right)+\right.\\
\left.+\frac{\partial}{\partial\beta}\frac{1}{B}\left[\frac{\partial AM_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\right]\right\}+\\
+\frac{\Gamma}{R_{\alpha\beta}}\left[-\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)H+\frac{M_\alpha+M_\beta}{R_{\alpha\beta}}\right]=q_n.
\end{gathered} \tag{8.24}$$

With regard to the static boundary quantities, according to (5.15) and (Chapter VII, 6.12-6.14) they are expressed in the following manner in terms of the forces and moments

$$\begin{gathered}
Q_{\nu\nu}=\cos^2\gamma T_\alpha+2\sin\gamma\cos\gamma S+\sin^2\gamma T_\beta+\\
+\sin\gamma\cos\gamma\left[\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)H-\frac{M_\alpha+M_\beta}{R_{\alpha\beta}}\right]+\frac{1}{R_{\nu t}}M_{\nu t};\\
Q_{\nu t}=\sin\gamma\cos\gamma\,(T_\beta-T_\alpha)+(\cos^2\gamma-\sin^2\gamma)S+\\
+\left(\frac{\cos^2\gamma}{R_\beta}-\frac{\sin^2\gamma}{R_\alpha}\right)H+\frac{\sin^2\gamma M_\beta-\cos^2\gamma M_\alpha}{R_{\alpha\beta}}+\frac{1}{R_t}M_{\nu t};\\
Q_{\nu n}=\frac{\cos\gamma}{AB}\left(\frac{\partial BM_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta\right)+\\
+\frac{\sin\gamma}{AB}\left(\frac{\partial AM_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\right)+\frac{dM_{\nu t}}{ds_t};\\
M_{\nu\nu}=\cos^2\gamma M_\alpha+2\sin\gamma\cos\gamma H+\sin^2\gamma M_\beta\\
(M_{\nu t}=(\cos^2\gamma-\sin^2\gamma)H+\sin\gamma\cos\gamma\,(M_\beta-M_\alpha)).
\end{gathered} \tag{8.25}$$

Hooke's law assumes the following form

$$T_\alpha = \frac{Eh}{1-\nu^2}(\varepsilon_\alpha + \nu\varepsilon_\beta); \quad M_\alpha = \frac{Eh^3}{12(1-\nu^2)}(\varkappa_\alpha + \nu\varkappa_\beta);$$
$$T_\beta = \frac{Eh}{1-\nu^2}(\varepsilon_\beta + \nu\varepsilon_\alpha); \quad M_\beta = \frac{Eh^3}{12(1-\nu^2)}(\varkappa_\beta + \nu\varkappa_\alpha); \qquad (8.26)$$
$$S = \frac{Eh}{1+\nu}\frac{\omega}{2}; \quad H = \frac{Eh^3}{12(1+\nu)}\tau;$$

$$\varepsilon_\alpha = \frac{1}{Eh}(T_\alpha - \nu T_\beta); \quad \varkappa_\alpha = \frac{12}{Eh^3}(M_\alpha - \nu M_\beta);$$
$$\varepsilon_\beta = \frac{1}{Eh}(T_\beta - \nu T_\alpha); \quad \varkappa_\beta = \frac{12}{Eh^3}(M_\beta - \nu M_\alpha); \qquad (8.27)$$
$$\frac{\omega}{2} = \frac{1+\nu}{Eh}S; \quad \tau = \frac{12(1+\nu)}{Eh^3}H.$$

The stress functions $\bar{u}$, $\bar{v}$, $\bar{w}$ are introduced by means of relationships /113

$$T_\alpha = T_\alpha^* + Ehc\bar{\varkappa}_\beta; \quad M_\alpha = M_\alpha^* - Ehc\bar{\varepsilon}_\beta;$$
$$T_\beta = T_\beta^* + Ehc\bar{\varkappa}_\alpha; \quad M_\beta = M_\beta^* - Ehc\bar{\varepsilon}_\alpha; \qquad (8.28)$$
$$S = S^* - Ehc\bar{\tau}; \quad H = H^* + Ehc\frac{\bar{\omega}}{2}$$
$$(T_{(\alpha\beta)} = T^*_{(\alpha\beta)} - Ehc\bar{\mu}_{(\beta\alpha)}, \quad T_{(\beta\alpha)} = T^*_{(\beta\alpha)} - Ehc\bar{\mu}_{(\alpha\beta)}).$$

In this case

$$Q_{\nu\nu} = Q^*_{\nu\nu} + Ehc\bar{\varkappa}_{tt}; \quad Q_{\nu n} = Q^*_{\nu n} + Ehc\bar{\varkappa}_{tn};$$
$$Q_{nt} = Q^*_{nt} - Ehc\bar{\varkappa}_{t\nu}; \quad M_{\nu\nu} = M^*_{\nu\nu} - Ehc\bar{\varepsilon}_{tt}. \qquad (8.29)$$

The vectors of the stress functions

$$\vec{U} = \bar{u}_\nu\vec{\nu} + \bar{u}_t\vec{t} + \bar{w}\vec{n} \qquad (8.30)$$

and of the "angular stresses"

$$\vec{\Omega}_t = -\bar{\vartheta}_t\vec{\nu} + \bar{\vartheta}_\nu\vec{t} + \bar{\omega}_{tt}\vec{n} \qquad (8.31)$$

are associated with the principal vector (fig. 17)

$$\vec{F}=\int_{s_0}^{s_t}\vec{Q}_\nu ds_t' \tag{8.32}$$

and with the principal moment

$$\vec{B}=\int_{s_0}^{s_t} M_{\nu}\vec{t}'ds_t+\int_{s_0}^{s_t}\left[\vec{r}'-\vec{r},\ \vec{Q}_\nu\right]ds_t' \tag{8.33}$$

of forces and moments, applied to the particular part of the curve (section 2, Chapter IV) by relationships

$$\begin{aligned}\vec{\Omega}_t&=\vec{\Omega}_t^0-\frac{\vec{F}-\vec{F}^*}{Ehc};\\ \vec{U}&=\vec{U}^0+\left[\vec{\Omega}_t^0,\ \vec{r}-\vec{r}^0\right]-\frac{\vec{B}-\vec{B}^*}{Ehc}.\end{aligned} \tag{8.34}$$

Here the zeros also designate quantities which are computed at the initial point of the curve M^0, $\vec{Q}_\nu = Q_{\nu\nu}\vec{\nu} + Q_{\nu t}\vec{t} + Q_{\nu n}\vec{n}$, while $\vec{F}^*$ and $\vec{B}^*$ are the principal vector and the principal moment computed from some solution of the equilibrium equations (8.24). We recall that the principal moment is computed with respect to a running point on curve M. /114

Further, the complex forces ($i = \sqrt{-1}$)

$$\begin{aligned}\tilde{T}_\alpha&=T_\alpha-iEhc\varkappa_\beta=T_\alpha^*-iEhc\tilde{\varkappa}_\beta,\\ \tilde{T}_\beta&=T_\beta-iEhc\varkappa_\alpha=T_\beta^*-iEhc\tilde{\varkappa}_\alpha,\\ \tilde{S}&=S+iEhc\tau=S^*+iEhc\tilde{\tau}\end{aligned} \tag{8.35}$$

are determined from the simplified system of equations containing the complex forces

$$\begin{aligned}&\frac{\partial B\tilde{T}_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2\tilde{S}}{\partial\beta}-\frac{\partial B}{\partial\alpha}\tilde{T}_\beta+ic\left(\frac{B}{R_\alpha}\frac{\partial\tilde{T}}{\partial\alpha}-\frac{A}{R_{\alpha\beta}}\frac{\partial\tilde{T}}{\partial\beta}\right)+ABq_\alpha=0;\\ &\frac{\partial A\tilde{T}_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2\tilde{S}}{\partial\alpha}-\frac{\partial A}{\partial\beta}\tilde{T}_\alpha+ic\left(\frac{A}{R_\beta}\frac{\partial\tilde{T}}{\partial\beta}-\frac{B}{R_{\alpha\beta}}\frac{\partial\tilde{T}}{\partial\alpha}\right)+ABq_\beta=0;\end{aligned} \tag{8.36}$$

$$\frac{\tilde{T}_\alpha}{R_\alpha}-\frac{2\tilde{S}}{R_{\alpha\beta}}+\frac{\tilde{T}_\beta}{R_\beta}-ic\Delta\tilde{T}=q_n;$$

$$\Delta=\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial}{\partial\beta}\right\},$$

where

$$\tilde{T}=\tilde{T}_\alpha+\tilde{T}_\beta. \tag{8.37}$$

On the other hand, the complex displacements

$$\tilde{u}=u+i\bar{u},\ \tilde{v}=v+i\bar{v},\ \tilde{w}=w+i\bar{w} \tag{8.38}$$

are obtained from the simplified system of equations containing the complex displacements

$$\begin{aligned}
&\tilde{\varepsilon}_\alpha+ic\left(\tilde{\varkappa}_\beta-\nu\tilde{\varkappa}_\alpha\right)=\frac{1}{Eh}\left\{T_\alpha^*-\nu T_\beta^*+\frac{i}{c}M_\beta^*\right\};\\
&\tilde{\varepsilon}_\beta+ic\left(\tilde{\varkappa}_\alpha-\nu\tilde{\varkappa}_\beta\right)=\frac{1}{Eh}\left\{T_\beta^*-\nu T_\alpha^*+\frac{i}{c}M_\alpha^*\right\};\\
&\frac{\tilde{\omega}}{2}-ic(1+\nu)\tilde{\tau}=\frac{1}{Eh}\left\{(1+\nu)S^*-\frac{i}{c}H^*\right\}.
\end{aligned} \tag{8.39}$$

Finally, for the complex boundary quantities

$$\begin{aligned}
&\tilde{Q}_{\nu\nu}=Q_{\nu\nu}-iEhc\varkappa_{tt}=Q_{\nu\nu}^*-iEhc\tilde{\varkappa}_{tt},\\
&\tilde{Q}_{\nu t}=Q_{\nu t}+iEhc\varkappa_{t\nu}=Q_{\nu t}^*+iEhc\tilde{\varkappa}_{t\nu},\\
&\tilde{Q}_{\nu n}=Q_{\nu n}-iEhc\varkappa_{tn}=Q_{\nu n}^*-iEhc\tilde{\varkappa}_{tn},\\
&\tilde{M}_{\nu\nu}=M_{\nu\nu}+iEhc\varepsilon_{tt}=M_{\nu\nu}^*+iEhc\tilde{\varepsilon}_{tt}
\end{aligned} \tag{8.40}$$

the following approximate expressions are valid

$$\begin{aligned}
&\tilde{Q}_{\nu\nu}=\tilde{T}_{\nu\nu},\ \tilde{Q}_{\nu t}=\tilde{T}_{\nu t};\\
&\tilde{Q}_{\nu n}+i(1+\nu)c\left\{\frac{d\tilde{T}_{\nu t}}{ds_t}-q_\nu\right\}=ic\frac{d\tilde{T}}{ds_\nu};\\
&\tilde{M}_{\nu\nu}+i(1+\nu)c\tilde{T}_{\nu\nu}=ic\tilde{T}
\end{aligned} \tag{8.41}$$

$$\left(\begin{array}{c} q_\gamma = \cos\gamma q_\alpha + \sin\gamma q_\beta, \\ \frac{d}{ds_\gamma} = \frac{\cos\gamma}{A}\frac{\partial}{\partial\alpha} + \frac{\sin\gamma}{B}\frac{\partial}{\partial\beta},\ \frac{d}{ds_t} = -\frac{\sin\gamma}{A}\frac{\partial}{\partial\alpha} + \frac{\cos\gamma}{B}\frac{\partial}{\partial\beta} \end{array}\right).$$

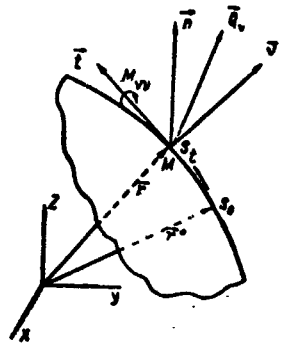

Figure 17

In concluding the section we present expressions for the deformation energy /115

$$V = \int_S\int W AB d\alpha d\beta,\ \overline{V} = \int_S\int \overline{W} AB d\alpha d\beta, \tag{8.42}$$

where

$$W = \frac{Eh}{2(1-\nu^2)}\left\{\frac{1+\nu}{2}(\varepsilon_\alpha+\varepsilon_\beta)^2 + \frac{1-\nu}{2}(\varepsilon_\alpha-\varepsilon_\beta)^2 + 2(1-\nu)\left(\frac{\omega}{2}\right)^2\right\} + \\ + \frac{Eh^3}{24(1-\nu^2)}\left\{\frac{1+\nu}{2}(\varkappa_\alpha+\varkappa_\beta)^2 + \frac{1-\nu}{2}(\varkappa_\alpha-\varkappa_\beta)^2 + 2(1-\nu)\tau^2\right\}; \tag{8.43}$$

$$\overline{W} = \frac{1}{2Eh}\left\{\frac{1-\nu}{2}(T_\alpha+T_\beta)^2 + \frac{1+\nu}{2}(T_\alpha-T_\beta)^2 + 2(1+\nu)S^2\right\} + \\ + \frac{6}{Eh^3}\left\{\frac{1-\nu}{2}(M_\alpha+M_\beta)^2 + \frac{1+\nu}{2}(M_\alpha-M_\beta)^2 + 2(1+\nu)H^2\right\}. \tag{8.44}$$

CHAPTER IX. GENERAL THEOREMS. VARIATIONAL AND EXTREMAL PRINCIPLES

In the first part of the book we obtained general relationships for the theory of shells. Since we used various simplifying assumptions in deriving them, it is desirable to see whether the resulting relationships contradict the general theorems of elasticity. In this case, of course, it is first necessary to formulate these theorems as they apply to the particular version of the theory of thin shells. Such an analysis is carried out in the present chapter. In addition, certain extremal and variational principles are formulated which are of prime interest in practice. The question concerning the boundary conditions in the theory of shells is also considered briefly. /116

1. The Clapeyron Theorem. Betti Reciprocity Theorem

Let us assume that the shell is in equilibrium and is subject to certain surface and edge loads. We compute the work done by these external forces in producing elastic displacements of the shell points (they are frequently called the statically corresponding forces for a given external load)

$$L=\int_S\int \{q_1u_1+q_2u_2+q_nw\}\,A_1A_2d\alpha_1d\alpha_2+ \\ +\int_s \{Q_{\nu\nu}u_\nu+Q_{\nu t}u_t+Q_{\nu n}w+M_{\nu\nu}\vartheta_\nu\}\,ds_t+\sum_i Q_n^i w_i. \quad (1.1)$$

*

We replace the quantities $A_1A_2q_1$, $A_1A_2q_2$, q_n, $Q_{\nu\nu}$, $Q_{\nu t}$, $Q_{\nu n}$, $M_{\nu\nu}$ in expressions under the integral signs by qualities taken from (Chapter VIII, 8.24) and (Chapter VIII, 8.25). By carrying out, in reverse order, the operations of section 2, Chapter III, we obtain

$$L=A=\int_S\int \{T_1e_1+T_2e_2+S\omega+M_1\varkappa_1+M_2\varkappa_2+2H\tau\}\,A_1A_2d\alpha_1d\alpha_2. \quad (1.2)$$

---

*We recall that in this case Q_n^i designates the normal concentrated force applied to the i-th point of the boundary contour; w_i is the corresponding displacement.

Replacing the forces and moments with expressions from (Chapter VIII, 8.26), we find that A = 2V, from which we obtain the equality /117

$$V=\frac{1}{2}L, \tag{1.3}$$

which proves the Clapeyron theorem: the work of shell deformation or, what is the same, the elastic energy accumulated by the body during the process of deformation is equal to half the work performed by the external forces during the corresponding static displacements. The presence of the factor 1/2 is explained by the fact that when we have a quasistatically loaded elastic body (for which we can assume that the external force during each instant of loading is balanced by the internal elastic forces), the external forces vary gradually from zero to their final values. In this case the true work performed by the external forces, which becomes transformed into the elastic deformation energy of the body, is equal to the work done by the average values of the external forces against the final values of elastic displacements.

The Betti theorem is closely associated with the Clapeyron theorem. Let us assume that the shell is in equilibrium under the action of some system of external forces. These forces and the corresponding displacements, forces and moments are designated by the sign $^{(1)}$. We introduce a similar system of quantities designating it with the sign $^{(2)}$. Let us compute the work done by external forces of the first system, while causing the displacements of the second

$$\begin{aligned} L_{12}=&\int\int_S \{q_1^{(1)}u_1^{(2)}+q_2^{(1)}u_2^{(2)}+q_n^{(1)}w^{(2)}\}A_1A_2d\alpha_1d\alpha_2+\\ &+\int_s\{Q_{\nu\nu}^{(1)}u_\nu^{(2)}+Q_{\nu t}^{(1)}u_t^{(2)}+Q_{\nu n}^{(1)}w^{(2)}+M_{\nu\nu}^{(1)}\vartheta_\nu^{(2)}\}ds_t+\sum_i Q_n^{i\,(1)}w_i^{(2)}. \end{aligned} \tag{1.4}$$

Repeating the transformations carried out above we obtain the following expression for L_{12}

$$\begin{aligned} L_{12}=&\frac{Eh}{1-\nu^2}\int\int_S\left\{\varepsilon_1^{(1)}\varepsilon_1^{(2)}+\nu\left(\varepsilon_2^{(1)}\varepsilon_1^{(2)}+\varepsilon_1^{(1)}\varepsilon_2^{(2)}\right)+\varepsilon_2^{(1)}\varepsilon_2^{(2)}+\right.\\ &\left.+\frac{1-\nu}{2}\omega^{(1)}\omega^{(2)}\right\}A_1A_2d\alpha_1d\alpha_2+\\ &+\frac{Eh^3}{12(1-\nu^2)}\int\int_S\left\{\varkappa_1^{(1)}\varkappa_1^{(2)}+\nu\left(\varkappa_2^{(1)}\varkappa_1^{(2)}+\varkappa_1^{(1)}\varkappa_2^{(2)}\right)+\varkappa_2^{(1)}\varkappa_2^{(2)}+\right.\\ &\left.+2(1-\nu)\tau^{(1)}\tau^{(2)}\right\}A_1A_2d\alpha_1d\alpha_2. \end{aligned} \tag{1.5}$$

It is easy to see that it is symmetric with respect to the signs (1) and /118 (2). It follows from this that by formulating the expression for the work performed by the external forces of the second system to produce displacements in the first

$$L_{21}=\int\int_S (q_1^{(2)}u_1^{(1)}+q_2^{(2)}u_2^{(1)}+q_n^{(2)}w^{(1)})\,A_1A_2\,d\alpha_1\,d\alpha_2+$$
$$+\int_s (Q_{\nu\nu}^{(2)}u_\nu^{(1)}+Q_{\nu t}^{(2)}u_t^{(1)}+Q_{\nu n}^{(2)}w^{(1)}+M_{\nu\nu}^{(2)}\vartheta_\nu^{(1)})\,ds_t+\sum_i Q_n^{i(2)}w_i^{(1)} \qquad (1.6)$$

and transforming it in a manner analogous to L_{12} we obtain the very same expression. Thus we have obtained the equality

$$L_{12}=L_{21}, \qquad (1.7)$$

which proves the Betti reciprocity theorem: the work produced by a system of external forces acting on a shell to overcome elastic displacements produced by another system of forces is equal to the work performed by these other forces to produce displacements in the first system.

We shall show that the terms which are not under the integral signs in the above expressions may be dropped, if we introduce the Dirac function $\delta(x)$ or, as it is frequently called, the delta function. This is the name given to a generalized function which associates a conventional function $f(x)$ with its value when $x = 0$, i.e., with $f(0)$. This property which determines it may be written in the following manner[1]

$$\int_{x_1}^{x_2} f(x)\delta(x-x_0)\,dx=f(x_0)\quad (x_1<x_0<x_2). \qquad (1.8)$$

By letting

$$Q'_{\nu n}=Q'_n\delta(s_t-s_i),$$

we have

$$\int_s Q'_{\nu n}w\,ds_t=Q'_n\int_s w(s_t)\delta(s_t-s_i)\,ds_t=Q'_nw(s_i)=Q'_nw_i.$$

[1] The mathematical determination of the generalized Dirac function presented here does not contradict the traditional methods used in mechanics to determine the δ-function (see, for example, ref. 32).

Thus the terms mentioned above, which are outside the integral signs, may be dropped without reducing the generality. In the future, as a rule we shall drop them. If it is necessary to consider a case when concentrated forces are applied to the edge of the shell these terms may be reinstated by introducing the δ-functions and other related generalized functions into the expressions under the integral sign.

2. Uniqueness Theorem. Basic Versions of Boundary Conditions in the Theory of Shells

Let us assume that we have two solutions of some problem in the theory of shells, pertaining to the same boundary conditions and the same surface loads. We take their difference /119

$$\dot{u}_1 = u_1^{(2)} - u_1^{(1)}, \ldots, \quad \dot{T}_1 = T_1^{(2)} - T_1^{(1)}, \ldots, \quad \dot{Q}_{\nu\nu} = Q_{\nu\nu}^{(2)} - Q_{\nu\nu}^{(1)}, \ldots \tag{2.1}$$

The difference between the solutions will satisfy the zero loading surface and the homogeneous boundary conditions. Therefore, by utilizing the Clapeyron theorem we find (8.42)

$$\begin{aligned} &\frac{Eh}{2(1-\nu^2)}\int\int_S \left\{\frac{1+\nu}{2}(\dot{\varepsilon}_1+\dot{\varepsilon}_2)^2+\frac{1-\nu}{2}(\dot{\varepsilon}_1-\dot{\varepsilon}_2)^2+\right.\\ &+2(1-\nu)\left(\frac{\dot{\omega}}{2}\right)^2\Big\}A_1A_2d\alpha_1d\alpha_2+\frac{Eh^3}{24(1-\nu^2)}\int\int_S\left\{\frac{1+\nu}{2}(\dot{\varkappa}_1+\dot{\varkappa}_2)^2+\right.\\ &+\frac{1-\nu}{2}(\dot{\varkappa}_1-\dot{\varkappa}_2)^2+2(1-\nu)\dot{\tau}^2\Big\}A_1A_2d\alpha_1d\alpha_2=\\ &=\int_s\left\{\dot{Q}_{\nu\nu}\dot{u}_\nu+\dot{Q}_{\nu t}\dot{u}_t+\dot{Q}_{\nu n}\dot{w}+\dot{M}_{\nu\nu}\dot{\vartheta}_\nu\right\}ds. \end{aligned} \tag{2.2}$$

Let us assume that the homogeneous boundary conditions of the problem are such that the contour integral is equal to zero. Then the left side of the equality is also equal to zero. Since the expressions under the integral sign are not negative, the latter is only possible if $\dot{u}_1$, $\dot{u}_2$, $\dot{w}$ represent the displacements of the middle surface as rigid whole. Thus when the contour integral is equal to zero, this provides for the unique solution of the formulated problem (with an accuracy up to the possible rigid displacement). On the other hand, let us assume that the two solutions do not differ by more than the rigid displacement. Then $\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \ldots = \dot{\tau} = 0$, and the left side of the equality and consequently the right side become equal to zero. Thus the necessary and sufficient condition that the solution of the formulated problem be unique is that the contour integral be equal to zero as a result of the homogeneous boundary conditions of the problem

$$\int_{\Sigma}\{\dot{Q}_{\nu}\dot{u}_{\nu}+\dot{Q}_{\nu t}\dot{u}_{t}+\dot{Q}_{\nu n}\dot{w}+\dot{M}_{\nu\nu}\dot{\vartheta}_{\nu}\}ds_{t}=0. \tag{2.3}$$

This condition is satisfied in particular by the ordinary boundary conditions, because in this case one of the cofactors for each term in the expression under the integral sign becomes equal to zero. We recall that in the first part of the book we referred to ordinary boundary conditions as those for which one quantity is assigned from a pair of two quantities (generalized force--generalized displacement).

Let us write condition (2.3) in the form

$$\int_{\Sigma}\{\dot{\vec{Q}}_{\nu}\cdot\dot{\vec{U}}+\dot{M}_{\nu\nu}\dot{\vartheta}_{\nu}\}ds_{t}=0. \tag{2.4}$$

Since the vector of the edge forces $\dot{\vec{Q}}_{\nu}$ is expressed in the terms of the principal vector $\dot{\vec{F}}$ by means of the relationship (Chapter VIII, 8.32) /120

$$\dot{\vec{Q}}_{\nu}=\frac{d\dot{\vec{F}}}{ds_{t}},$$
$$\int_{\Sigma}\dot{\vec{Q}}_{\nu}\cdot\dot{\vec{U}}ds_{t}=\int_{\Sigma}\frac{d\dot{\vec{F}}}{ds_{t}}\cdot\dot{\vec{U}}ds_{t}=-\int_{\Sigma}\frac{d\vec{U}}{ds_{t}}\cdot\dot{\vec{F}}ds_{t}. \tag{2.5}$$

However, it follows from (Chapter VIII, 3.13 and 3.14) that

$$-\int_{\Sigma}\left\{\frac{d\vec{U}}{ds_{t}}\cdot\dot{\vec{F}}\right\}ds_{t}=\int_{\Sigma}\{\dot{\omega}_{tt}\dot{F}_{\nu}-\dot{e}_{tt}\dot{F}_{t}+\dot{\vartheta}_{t}\dot{F}_{n}\}\,ds_{t}.$$

Substituting the resulting expression into (2.4), we arrive at the following form for the uniqueness conditions of the solution

$$\int_{\Sigma}\{\dot{\omega}_{tt}\dot{F}_{\nu}-\dot{e}_{tt}\dot{F}_{t}+\dot{\vartheta}_{t}\dot{F}_{n}+\dot{\vartheta}_{\nu}\dot{M}_{\nu\nu}\}\,ds_{t}=0. \tag{2.6}$$

Finally, by introducing the principal moment of boundary forces and moments $\dot{\vec{B}}$, we have (Chapter VIII, 8.33)

$$\frac{d\vec{B}}{ds_t} = \dot{M}_v\vec{t} - [\vec{t}, \vec{F}]. \tag{2.7}$$

Taking the scalar product of this equality and the vector for the angular rotation of the boundary element

$$\vec{\Omega}_t = -\dot{\vartheta}_t\vec{v} + \dot{\vartheta}_v\vec{t} + \dot{\omega}_{tt}\vec{n}, \qquad \text{(Chapter VIII, 8.14)}$$

we obtain

$$\frac{d\vec{B}}{ds_t}\cdot\vec{\Omega}_t = \dot{M}_v\dot{\vartheta}_v + \dot{F}_v\dot{\omega}_{tt} + \dot{F}_n\dot{\vartheta}_t.$$

By using the resulting expressions condition (2.6) may be written in the form

$$\int_s \left\{-\dot{\varepsilon}_{tt}\dot{F}_t + \vec{\Omega}_t\cdot\frac{d\vec{B}}{ds_t}\right\} ds_t = 0. \tag{2.8}$$

Integrating the second term by parts we obtain in accordance with (Chapter VIII, 8.18)

$$-\int_s \left\{\vec{B}\vec{x}_t + \dot{F}_t\dot{\varepsilon}_{tt}\right\} ds_t = 0 \tag{2.9}$$

or, in the expanded form

$$\int_s \left\{\dot{B}_v\dot{x}_{tt} - \dot{B}_t\dot{x}_{tv} + \dot{B}_n\dot{x}_{tn} - \dot{F}_t\dot{e}_{tt}\right\} ds_t = 0. \tag{2.10}$$

By examining the expressions under the integral signs in equalities (2.3), (2.6) and (2.10) we see that the pairs of quantities contained in the following three groups

$$\begin{array}{llll} \text{I}\ Q_{vv}, & Q_{vt}, & Q_{vn}, & M_{vv}; \\ u_v, & u_t, & w, & \vartheta_v \end{array} \tag{2.11}$$

$$\begin{array}{llll} \text{II}\ F_{vv}, & F_{vt}, & F_{vn}, & M_{vv}; \\ \omega_{tt}, & -\varepsilon_{tt}, & \vartheta_t, & \vartheta_v \end{array} \tag{2.12}$$

$$\text{III}\ \begin{matrix} B_\nu, & B_t, & B_n, & F_t, \\ \varkappa_{tt}, & \varkappa_{t\nu}, & \varkappa_{tn}, & \varepsilon_{tt} \end{matrix} \tag{2.13}$$

may be looked upon as the generalized forces and the corresponding generalized displacements. The first group of quantities is well known and is widely used in the theory of shells. The quantities contained in the second group are used to solve problems of the deformation of plates by the methods of the theory of a complex variable. /121

The utilization of deformation boundary conditions formulated in terms of quantities $\varkappa_{tt}$, $\varkappa_{t\nu}$, $\varkappa_{tn}$, ε_{tt} is promising.[1] As stated in the first part of the book, the advantage of deformation boundary conditions is that when they are used it is not necessary to determine the displacements initially. It follows from the same equalities (2.3), (2.6) and (2.10) that the usual boundary conditions which correspond to these three versions provide for the unique solution of the corresponding problems.

In the theory of shells we frequently encounter the conditions of elastic constraint. In the future we shall limit ourselves only to the examination of elastic supports for wnich

$$Q_\nu = -\frac{\partial R}{\partial u_\nu};\quad Q_t = -\frac{\partial R}{\partial u_t};\quad Q_n = -\frac{\partial R}{\partial w};\quad M_\nu = -\frac{\partial R}{\partial \vartheta_\nu}. \tag{2.14}$$

In this case we assume that the potential of the generalized edge forces R may be represented in the form

$$R = \frac{1}{2}\{a_{11}u_\nu^2 + 2a_{12}u_\nu u_t + 2a_{13}u_\nu w + 2a_{14}u_\nu\vartheta_\nu + a_{22}u_t^2 + \\ + 2a_{23}u_t w + 2a_{24}u_t\vartheta_\nu + a_{33}w^2 + 2a_{34}w\vartheta_\nu + a_{44}\vartheta_\nu^2\}. \tag{2.15}$$

By making use of equations (2.14) we obtain

$$\begin{aligned} -Q_\nu &= a_{11}u_\nu + a_{12}u_t + a_{13}w + a_{14}\vartheta_\nu; \\ -Q_t &= a_{12}u_\nu + a_{23}u_t + a_{23}w + a_{24}\vartheta_\nu; \\ -Q_n &= a_{13}u_\nu + a_{23}u_t + a_{33}w + a_{34}\vartheta_\nu; \\ -M_\nu &= a_{14}u_\nu + a_{24}u_t + a_{34}w + a_{44}\vartheta_\nu. \end{aligned} \tag{2.16}$$

We shall assume that this system can always be solved for the generalized displacements in such a way that /122

[1] In essence K. Z. Galimov arrived at these quantities in his work (ref. 236).

$$
\begin{aligned}
-u_\nu &= b_{11}Q_{\nu\nu} + b_{12}Q_{\nu t} + b_{13}Q_{\nu n} + b_{14}M_{\nu\nu};\\
-u_t &= b_{12}Q_{\nu\nu} + b_{22}Q_{\nu t} + b_{23}Q_{\nu n} + b_{24}M_{\nu\nu};\\
-w &= b_{13}Q_{\nu\nu} + b_{23}Q_{\nu t} + b_{33}Q_{\nu n} + b_{34}M_{\nu\nu};\\
-\vartheta_\nu &= b_{14}Q_{\nu\nu} + b_{24}Q_{\nu t} + b_{34}Q_{\nu n} + b_{44}M_{\nu\nu}
\end{aligned}
\tag{2.17}
$$

and

$$
u_\nu = -\frac{\partial \bar{R}}{\partial Q_{\nu\nu}};\quad u_t = -\frac{\partial \bar{R}}{\partial Q_{\nu t}};\quad w = -\frac{\partial \bar{R}}{\partial Q_{\nu n}};\quad \vartheta_\nu = -\frac{\partial \bar{R}}{\partial M_{\nu\nu}},
\tag{2.18}
$$

where

$$
\begin{aligned}
\bar{R} = \frac{1}{2}\{&b_{11}Q_{\nu\nu}^2 + 2b_{12}Q_{\nu\nu}Q_{\nu t} + 2b_{13}Q_{\nu\nu}Q_{\nu n} + 2b_{14}Q_{\nu\nu}M_{\nu\nu} +\\
&+ b_{22}Q_{\nu t}^2 + 2b_{23}Q_{\nu t}Q_{\nu n} + 2b_{24}Q_{\nu t}M_{\nu\nu} + b_{33}Q_{\nu n}^2 + 2b_{34}Q_{\nu n}M_{\nu\nu} + b_{44}M_{\nu\nu}^2\}.
\end{aligned}
\tag{2.19}
$$

We shall require that the quadratic forms R and $\bar{R}$ be determined positively, i.e., that the equality $R = 0$ ($\bar{R} = 0$) be satisfied only when $u_\nu = u_t = w = 0$, $\vartheta_\nu = 0$ ($Q_{\nu\nu} = Q_{\nu t} = Q_{\nu n} = 0$, $M_{\nu\nu} = 0$). The limitations imposed on the nature of the elastic supports are satisfied in most problems of practical interest.

By using these relationships it is easy to verify the validity of the following equalities

$$
Q_{\nu\nu}u_\nu + Q_{\nu t}u_t + Q_{\nu n}w + M_{\nu\nu}\vartheta_\nu = -2R; \tag{2.20}
$$

$$
Q_{\nu\nu}u_\nu + Q_{\nu t}u_t + Q_{\nu n}w + M_{\nu\nu}\vartheta_\nu = -2\bar{R}. \tag{2.21}
$$

They make it possible to prove that the solution is unique when the constraints are elastic. Indeed the difference of solutions

$$
\dot{u}_\nu = u_\nu^{(2)} - u_\nu^{(1)}, \ldots, \quad \dot{Q}_{\nu\nu} = Q_{\nu\nu}^{(2)} - Q_{\nu\nu}^{(1)}, \ldots \tag{2.22}
$$

is associated with the following valid relationship

$$
\dot{Q}_{\nu\nu}\dot{u}_\nu + \dot{Q}_{\nu t}\dot{u}_t + \dot{Q}_{\nu n}\dot{w} + \dot{M}_{\nu\nu}\dot{\vartheta}_\nu = -2R(\dot{u}_\nu, \dot{u}_t, \dot{w}, \dot{\vartheta}_\nu).
$$

The necessary and sufficient condition for the uniqueness of solution (2.3) is written in the form

$$\int_s R(\dot{u}_\nu, \dot{u}_t, \dot{w}, \dot{\vartheta}_\nu)\, ds_t = 0. \tag{2.23}$$

In view of the proposition that the determination of the quadratic form R be positive it follows that

$$\dot{u}_\nu = \dot{u}_t = \dot{w} = 0, \quad \dot{\vartheta}_\nu = 0,$$

and the uniqueness is proven.

The elastic constraint system may include both the specific case when /123 the edge is of infinite rigidity[1] (generalized displacements are assigned) or is of zero rigidity (generalized forces are assigned). In this case we must provide for a situation when this does not refer to all static (geometric) boundary conditions, but only to some of them. Let us assume for example that $u_\nu = 0$ at the boundary. Since this condition must be satisfied for all generalized forces, it is necessary to set $b_{11} = b_{12} = b_{13} = b_{14} = 0$ in relationships (2.17) and (2.19). Furthermore, since u_ν is assigned, the potential R does not depend on u_ν and we must let $a_{11} = a_{12} = a_{13} = a_{14} = 0$ in equality (2.15). We must proceed in the same manner if some other quantity is assigned (generalized force or generalized displacement).

In view of these statements the quantity

$$I = \int_s (Q_{\nu\nu} u_\nu + Q_{\nu t} u_t + Q_{\nu n} w + M_{\nu\nu} \vartheta_\nu)\, ds_t$$

may be represented in the form

$$I = \int_s' (Q_{\nu\nu} u_\nu^0 + Q_{\nu t} u_t^0 + Q_{\nu n} w^0 + M_{\nu\nu} \vartheta_\nu^0)\, ds_t + \int_s'' (Q_{\nu\nu}^0 u_\nu + Q_{\nu t}^0 u_t + Q_{\nu n}^0 w + M_{\nu\nu}^0 \vartheta_\nu)\, ds_t - \int_s''' 2R\, ds_t. \tag{2.24}$$

Here we introduce the following designations. The sign $'$ means that the expression under the integral sign retains only the terms corresponding to

[1] A. L. Gol'denveyzer terminology (ref. 11).

infinite rigidity, i.e., the pairs generalized force--generalized displacement, for which the generalized displacement is assigned on the contour. The sign '' is assigned to pairs with zero rigidity (in which the generalized forces are assigned). Finally the sign ''' means that the expression for the potential R ($\overline{R}$) retains only those terms which correspond to the nonzero finite rigidity of the support. In this case the specified integrals are not taken over the entire contour, but only along its part (which has its own location for each pair), where according to the signs the specific property of the support takes place (zero rigidity, finite rigidity, infinite rigidity).

We note that in view of equalities (2.20)-(2.21) the last integral in (2.24) may be replaced (if necessary) by the following

$$\ldots - \int_{s}^{\cdots} 2\overline{R} ds_t. \tag{2.25}$$

It is easy to see that the mixed boundary conditions, which are considered at the end of this section, also provide for the uniqueness of the solution.

3. Extremal Principles. Dual Evaluations

In considering a series of problems associated with the theory of /124 shells it is useful to utilize the so-called extremal principles which establish the extremal properties of the actual state of stress occurring in the shell. Associated with these are the dual evaluations of the shells' deformation energy. Let us obtain these. To do so we consider two systems of quantities

$$\begin{aligned} &\text{I} \quad T_1^{(1)}, T_2^{(1)}, \ldots, H^{(1)}; \varepsilon_1^{(1)}, \ldots, \tau^{(1)}; \\ &\text{II} \quad T_1^{(2)}, T_2^{(2)}, \ldots, H^{(2)}; \varepsilon_1^{(2)}, \ldots, \tau^{(2)}. \end{aligned}$$

Here we do not assume that the quantities T_1, T_2, ..., H satisfy the equilibrium conditions or that the quantities ϵ_1, ϵ_2, ..., τ satisfy the continuity equations. Thus the quantities introduced are not, generally speaking, forces-moments and deformation components. The only condition which they must satisfy is that these quantities are associated by Hooke's law (Chapter VIII, 8.26 and 8.27).

The following identity, easy to prove, exists

$$\begin{aligned} &2\left[\left(T_1^{(1)} - T_1^{(2)}\right)\varepsilon_1^{(2)} + \ldots + 2\left(H^{(1)} - H^{(2)}\right)\tau^{(2)}\right] = \\ &= \left[T_1^{(1)}\varepsilon_1^{(1)} + \ldots + 2H^{(1)}\tau^{(1)}\right] - \left[T_1^{(2)}\varepsilon_1^{(2)} + \ldots + 2H^{(2)}\tau^{(2)}\right] - \\ &- \left[T_1^{(1)}\left(\varepsilon_1^{(1)} - \varepsilon_1^{(2)}\right) + \ldots + 2H^{(1)}\left(\tau^{(1)} - \tau^{(2)}\right)\right] - \\ &- \left[\left(T_1^{(2)} - \underline{T_1^{(1)}}\right)\varepsilon_1^{(2)} + \ldots + 2\left(H^{(2)} - \underline{H^{(1)}}\right)\tau^{(2)}\right]. \end{aligned} \tag{3.1}$$

From Hooke's law we have

$$T_1^{(1)}\varepsilon_1^{(2)} + \ldots + 2H^{(1)}\tau^{(2)} = T_1^{(2)}\varepsilon_1^{(1)} + \ldots + 2H^{(2)}\tau^{(1)}. \tag{3.2}$$

Replacing the terms underlined in (3.1) with these equalities and integrating the resulting identity over region S, we obtain the integral equality

$$\begin{aligned} 2\iint_S \{(T_1^{(1)} - T_1^{(2)})\,\varepsilon_1^{(2)} + \ldots + 2(H^{(1)} - H^{(2)})\,\tau^{(2)}\}\, A_1A_2 d\alpha_1 d\alpha_2 = \\ = \iint_S \{T_1^{(1)}\varepsilon_1^{(1)} + \ldots + 2H^{(1)}\tau^{(1)}\}\, A_1A_2 d\alpha_1 d\alpha_2 - \\ - \iint_S \{T_1^{(2)}\varepsilon_1^{(2)} + \ldots + 2H^{(2)}\tau^{(2)}\}\, A_1A_2 d\alpha_1 d\alpha_2 - \\ - \iint_S \{(T_1^{(1)} - T_1^{(2)})(\varepsilon_1^{(1)} - \varepsilon_1^{(2)}) + \ldots \\ \ldots + 2(H^{(1)} - H^{(2)})(\tau^{(1)} - \tau^{(2)})\}\, A_1A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{3.3}$$

According to Hooke's law the last integral is equal to a nonpositive quantity

$$-2V\{\varepsilon_1^{(1)} - \varepsilon_1^{(2)}, \ldots, \tau^{(1)} - \tau^{(2)}\}$$

(Chapter VIII, 8.42). Dropping this integral we obtain the inequality /125

$$\begin{aligned} \frac{1}{2}\Big\{\iint_S [T_1^{(1)}\varepsilon_1^{(1)} + \ldots + 2H^{(1)}\tau^{(1)}]\, A_1A_2 d\alpha_1 d\alpha_2 - \\ - \iint_S [T_1^{(2)}\varepsilon_1^{(2)} + \ldots + 2H^{(2)}\tau^{(2)}]\, A_1A_2 d\alpha_1 d\alpha_2\Big\} \geqslant \\ \geqslant \iint_S [(T_1^{(1)} - T_1^{(2)})\,\varepsilon_1^{(2)} + \ldots + 2(H^{(1)} - H^{(2)})\,\tau^{(2)}]\, A_1A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{3.4}$$

Let the quantities $\varepsilon_1^{(2)}$, $\varepsilon_2^{(2)}$, ..., $\tau^{(2)}$ satisfy the continuity equations. Then they may be looked upon as the deformation components corresponding to certain displacements $u_1^{(2)}$, $u_2^{(2)}$, $w^{(2)}$. Therefore the integral in the right side may be transformed into the following by using procedures analogous to those presented in section 1

$$\int_{\Sigma}\{(Q_{\nu\nu}^{(1)}-Q_{\nu\nu}^{(2)})u_{\nu}^{(2)}+(Q_{\nu t}^{(1)}-Q_{\nu t}^{(2)})u_{t}^{(2)}+(Q_{\nu n}^{(1)}-Q_{\nu n}^{(2)})w^{(2)}+$$
$$+(M_{\nu\nu}^{(1)}-M_{\nu\nu}^{(2)})\vartheta_{\nu}^{(2)}\}\,ds_{t}+\iint_{S}\{(q_{1}^{(1)}-q_{1}^{(2)})u_{1}^{(2)}+ \tag{3.5}$$
$$+(q_{2}^{(1)}-q_{2}^{(2)})u_{2}^{(2)}+(q_{n}^{(1)}-q_{n}^{(2)})w^{(2)}\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}.$$

As the result of this transformation inequality (3.4) assumes the form

$$\frac{1}{2}\iint_{S}\{T_{1}^{(1)}\varepsilon_{1}^{(1)}+\ldots+2H^{(1)}\tau^{(1)}\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}-$$
$$-\int_{\Sigma}\{Q_{\nu\nu}^{(1)}u_{\nu}^{(2)}+\ldots+M_{\nu\nu}^{(1)}\vartheta_{\nu}^{(2)}\}\,ds_{t}-$$
$$-\iint_{S}\{q_{1}^{(1)}u_{1}^{(2)}+q_{2}^{(1)}u_{2}^{(2)}+q_{n}^{(1)}w^{(2)}\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}\geqslant$$
$$\geqslant\frac{1}{2}\iint_{S}\{T_{1}^{(2)}\varepsilon_{1}^{(2)}+\ldots+2H^{(2)}\tau^{(2)}\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}- \tag{3.6}$$
$$-\int_{\Sigma}\{Q_{\nu\nu}^{(2)}u_{\nu}^{(2)}+\ldots+M_{\nu\nu}^{(2)}\vartheta_{\nu}^{(2)}\}\,ds_{t}-$$
$$-\iint_{S}\{q_{1}^{(2)}u_{1}^{(2)}+q_{2}^{(2)}u_{2}^{(2)}+q_{n}^{(2)}w^{(2)}\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}.$$

We shall assume that the quantities with the sign $^{(2)}$ (which we shall drop in the future) correspond to the actual state of stress occurring in the body and corresponding to the specific external load. We shall require further that the quantities $T_1^{(1)}$, $T_2^{(1)}$, ..., $H^{(1)}$ constitute a statically permissible system of functions (satisfying the equilibrium conditions and the static boundary conditions). Then if we replace the sign $^{(1)}$ with $^{\wedge}$, inequality (3.6) may be written in the form

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$$\frac{1}{2}\iint_{S}\{T_{1}\varepsilon_{1}^{\wedge}+\ldots+2H^{\wedge}\tau^{\wedge}\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}-$$
$$-\int_{\Sigma'}\{Q_{\nu\nu}^{\wedge}u_{\nu}^{0}+\ldots+M_{\nu\nu}^{\wedge}\vartheta_{\nu}^{0}\}\,ds_{t}-\int_{\Sigma''}\{Q_{\nu\nu}^{\wedge}u_{\nu}+\ldots+M_{\nu\nu}^{\wedge}\vartheta_{\nu}\}\,ds_{t}\geqslant \tag{3.7}$$
$$\geqslant\frac{1}{2}\iint_{S}\{T_{1}\varepsilon_{1}+\ldots+2H\tau\}\,A_{1}A_{2}d\alpha_{1}d\alpha_{2}-$$

$$-\int_{s}^{s'}(Q_{\nu}u_{\nu}+\ldots+M_{\nu}\vartheta_{\nu})\,ds_{t}-$$
$$-\int_{s}^{s''}(Q_{\nu}u_{\nu}+\ldots+M_{\nu}\vartheta_{\nu})\,ds_{t}.*$$

Let us consider in detail the expressions under the integral signs in the integrals corresponding to the condition of elastic constraint. First of all, according to (2.17)

$$L=-(Q_{\nu}^{\wedge}u_{\nu}+\ldots+M_{\nu}^{\wedge}\vartheta_{\nu})=$$
$$=(b_{11}Q_{\nu}Q_{\nu}^{\wedge}+b_{12}(Q_{\nu}Q_{t}^{\wedge}+Q_{t}Q_{\nu}^{\wedge})+\ldots+b_{44}M_{\nu}M_{\nu}^{\wedge}).$$

Utilizing the following obvious identities

$$Q_{\nu}Q_{\nu}^{\wedge}=\frac{1}{2}[Q_{\nu}^{2}+Q_{\nu}^{\wedge 2}-(Q_{\nu}-Q_{\nu}^{\wedge})^{2}],$$
$$Q_{\nu}Q_{t}^{\wedge}+Q_{t}Q_{\nu}^{\wedge}=Q_{\nu}Q_{t}+Q_{\nu}^{\wedge}Q_{t}^{\wedge}-(Q_{\nu}-Q_{\nu}^{\wedge})(Q_{t}-Q_{t}^{\wedge})$$

and similar ones, we represent L by means of relationships (2.19) in the form

$$L=\overline{R}+\overline{R}^{\wedge}-\overline{R}(Q_{\nu}-Q_{\nu}^{\wedge},\ldots,M_{\nu}-M_{\nu}^{\wedge}). \tag{3.8}$$

Furthermore,

$$-(Q_{\nu}u_{\nu}+\ldots+M_{\nu}\vartheta_{\nu})=2\overline{R}. \tag{3.9}$$

We substitute the last two expressions into inequality (3.7). Since the determination of the quadratic form is positive, we only fortify the inequality, dropping the term

$$-\overline{R}(Q_{\nu}-Q_{\nu}^{\wedge},\ldots,M_{\nu}-M_{\nu}^{\wedge}).$$

---

*In view of the statements at the end of the preceding section we have broken down the contour integral into three parts. In this case the integrals with '' have become shorter because for these integrals $Q_{\nu\nu}^{\wedge}=Q_{\nu\nu}=Q_{\nu\nu}^{0}$, ...

We then multiply the left and right side of the inequality by (-1), and reverse the inequality sign. As a result we obtain the unknown inequality /127

$$-\frac{1}{2}\iint_S \{T_1^\wedge \varepsilon_1^\wedge + T_2^\wedge \varepsilon_2^\wedge + S^\wedge \omega^\wedge + M_1^\wedge \varkappa_1^\wedge + M_2^\wedge \varkappa_2^\wedge + 2H^\wedge \tau^\wedge\} A_1 A_2 d\alpha_1 d\alpha_2 + \int_\Gamma \{Q_{\nu\nu}^\wedge u_\nu^0 + Q_{\nu t}^\wedge u_t^0 + Q_{\nu n}^\wedge w^0 + M_{\nu\nu}^\wedge \vartheta_\nu^0\} ds_t - \int_\Gamma'''' \overline{R}^\wedge ds_t \leqslant -\frac{1}{2}\iint_S \{T_1\varepsilon_1 + T_2\varepsilon_2 + S\omega + M_1\varkappa_1 + M_2\varkappa_2 + 2H\tau\} A_1 A_2 d\alpha_1 d\alpha_2 + \int_\Gamma' \{Q_{\nu\nu} u_\nu^0 + Q_{\nu t} u_t^0 + Q_{\nu n} w^0 + M_{\nu\nu}\vartheta_\nu^0\} ds_t - \int_\Gamma'''' \overline{R} ds_t. \tag{3.10}$$

We introduce the concept of supplementary energy determining it from equality (Chapter VIII, 8.42)

$$\Pi^* = -\overline{V}^\wedge + \int_\Gamma' \{Q_{\nu\nu}^\wedge u_\nu^0 + Q_{\nu t}^\wedge u_t^0 + Q_{\nu n}^\wedge w^0 + M_{\nu\nu}^\wedge \vartheta_\nu^0\} ds_t - \int_\Gamma'''' \overline{R}^\wedge ds_t. \tag{3.11}$$

In conjunction with relationships (1.2)-(1.3) inequality (3.10) proves the following proposition: of all the statically permissible systems of functions the forces and moments which actually take place in the shell impart a maximum value to the expression for the supplemental energy.

Let us obtain another extremal principle which in a certain sense is opposite to the one just proved. For this purpose we consider two systems of deformations and the corresponding deformation components, forces, moments and external forces. Using exactly the same approach as above we can show that

$$\iint_S \{T_1^{(2)}(\varepsilon_1^{(1)} - \varepsilon_1^{(2)}) + \ldots + 2H^{(2)}(\tau^{(1)} - \tau^{(2)})\} A_1 A_2 d\alpha_1 d\alpha_2 = \int_\Gamma \{Q_{\nu\nu}^{(2)}(u_\nu^{(1)} - u_\nu^{(2)}) + \ldots + M_{\nu\nu}^{(2)}(\vartheta_\nu^{(1)} - \vartheta_\nu^{(2)})\} ds_t + \iint_S \{q_1^{(2)}(u_1^{(1)} - u_1^{(2)}) + q_2^{(2)}(u_2^{(1)} - u_2^{(2)}) + q_n^{(2)}(w^{(1)} - w^{(2)})\} A_1 A_2 d\alpha_1 d\alpha_2. \tag{3.12}$$

By using equalities (3.2) the left side of this equality may be represented in the form

$$\iint_S \{(T_1^{(1)} - T_1^{(2)})\, e_1^{(2)} + \ldots + 2(H^{(1)} - H^{(2)})\, \tau^{(2)}\}\, A_1 A_2 d\alpha_1 d\alpha_2.$$

Substituting relationship (3.12) transformed in this manner into inequality (3.4), we obtain another inequality /128

$$\begin{aligned}
&\frac{1}{2}\iint_S \{T_1^{(1)} e_1^{(1)} + \ldots + 2H^{(1)}\tau^{(1)}\}\, A_1 A_2 d\alpha_1 d\alpha_2 - \\
&\quad - \int_\Sigma \{Q_{\nu\nu}^{(2)} u_\nu^{(1)} + \ldots + M_{\nu\nu}^{(2)} \vartheta_\nu^{(1)}\}\, ds_t - \\
&- \iint_S \{q_1^{(2)} u_1^{(1)} + q_2^{(2)} u_2^{(1)} + q_n^{(2)} w^{(1)}\}\, A_1 A_2 d\alpha_1 d\alpha_2 > \\
&> \frac{1}{2}\iint_S \{T_1^{(2)} e_1^{(2)} + \ldots + 2H^{(2)}\tau^{(2)}\}\, A_1 A_2 d\alpha_1 d\alpha_2 - \\
&\quad - \int_\Sigma \{Q_{\nu\nu}^{(2)} u_\nu^{(2)} + \ldots + M_{\nu\nu}^{(2)} \vartheta_\nu^{(2)}\}\, ds_t - \\
&- \iint_S \{q_1^{(2)} u_1^{(2)} + q_2^{(2)} u_2^{(2)} + q_n^{(2)} w^{(2)}\}\, A_1 A_2 d\alpha_1 d\alpha_2.
\end{aligned} \tag{3.13}$$

As above, we select the second system of quantities so that they correspond to the true state of stress in the shell (dropping the signs $^{(2)}$). We shall assume that the first system of functions is geometrically permissible, which means that the displacements $u_1^{(1)} \equiv u_1^\vee$, $u_2^{(1)} \equiv u_2^\vee$, $w^{(1)} \equiv w^\vee$ satisfy the geometric boundary conditions of the problem along the edge. Then inequality (3.13) may be written in the form

$$\begin{aligned}
&\frac{1}{2}\iint_S \{T_1^\vee e_1^\vee + \ldots + 2H^\vee \tau^\vee\}\, A_1 A_2 d\alpha_1 d\alpha_2 \\
&\quad - \int_\Sigma^{\cdots} \{Q_{\nu\nu}^0 u_\nu^\vee + \ldots + M_{\nu\nu}^0 \vartheta_\nu^\vee\}\, ds_t - \\
&\quad - \int_\Sigma^{\cdots} \{Q_{\nu\nu} u_\nu^\vee + \ldots + M_{\nu\nu} \vartheta_\nu^\vee\}\, ds_t - \\
&- \iint_S \{q_1 u_1^\vee + q_2 u_2^\vee + q_n w^\vee\}\, A_1 A_2 d\alpha_1 d\alpha_2 > \\
&> \frac{1}{2}\iint_S \{T_1 e_1 + \ldots + 2H\tau\}\, A_1 A_2 d\alpha_1 d\alpha_2 - \\
&\quad - \int_\Sigma^{\cdots} \{Q_{\nu\nu}^0 u_\nu + \ldots + M_{\nu\nu}^0 \vartheta_\nu\}\, ds_t - \\
&\quad - \int_\Sigma^{\cdots} \{Q_{\nu\nu} u_\nu + \ldots + M_{\nu\nu} \vartheta_\nu\}\, ds_t - \\
&- \iint_S \{q_1 u_1 + q_2 u_2 + q_n w\}\, A_1 A_2 d\alpha_1 d\alpha_2.
\end{aligned} \tag{3.14}$$

Repeating the procedure used to obtain equalities (3.8) and (3.9) we can show that /129

$$\begin{aligned}-(Q_{\nu}u_{\nu}^{\vee}+\ldots+M_{\nu}\vartheta_{\nu}^{\vee})&=R+R^{\vee}-R(u_{\nu}-u_{\nu}^{\vee},\ldots);\\ -(Q_{\nu}u_{\nu}+\ldots+M_{\nu}\vartheta_{\nu})&=2R.\end{aligned}\tag{3.15}$$

If we substitute this expression into inequality (3.14) and drop the nonpositive term -R (u_{ν} - $u_{\nu}^{\vee}$, ...), we obtain the following unknown inequality

$$\begin{aligned}&\frac{1}{2}\iint_S\{T_1^{\vee}\varepsilon_1^{\vee}+\ldots+2H^{\vee}\tau^{\vee}\}A_1A_2d\alpha_1d\alpha_2-\\ &-\int_{\varepsilon}^{\prime\prime\prime}\{Q_{\nu}^0u_{\nu}^{\vee}+\ldots+M_{\nu}^0\vartheta_{\nu}^{\vee}\}A_1A_2d\alpha_1d\alpha_2+\int_{\varepsilon}^{\prime\prime\prime\prime}R^{\vee}ds_t-\\ &-\iint_{S_1}\{q_1u_1^{\vee}+q_2u_2^{\vee}+q_nw^{\vee}\}A_1A_2d\alpha_1d\alpha_2\geqslant\\ &\geqslant\frac{1}{2}\iint_S\{T_1\varepsilon_1+\ldots+2H\tau\}A_1A_2d\alpha_1d\alpha_2-\\ &-\int_{\varepsilon}^{\prime\prime\prime}\{Q_{\nu}^0u_{\nu}+\ldots+M_{\nu}^0\vartheta_{\nu}\}A_1A_2d\alpha_1d\alpha_2+\\ &+\int_{\varepsilon}^{\prime\prime\prime\prime}Rds_t-\iint_S\{q_1u_1+q_2u_2+q_nw\}A_1A_2d\alpha_1d\alpha_2.\end{aligned}\tag{3.16}$$

If we determine the total energy of the shell by means of relationship

$$\begin{aligned}\Pi'=V^{\vee}-\int_{\varepsilon}^{\prime\prime\prime}\{Q_{\nu}^0u_{\nu}^{\vee}+Q_{\tau}^0u_{\tau}^{\vee}+Q_{\nu n}^0w^{\vee}+M_{\nu}^0\vartheta_{\nu}^{\vee}\}ds+\\ +\int_{\varepsilon}^{\prime\prime\prime\prime}R^{\vee}ds_t-\iint_S\{q_1u_1^{\vee}+q_2u_2^{\vee}+q_nw^{\vee}\}A_1A_2d\alpha_1d\alpha_2,\end{aligned}\tag{3.17}$$

inequality (3.16) shows that among the geometrically possible displacements the true displacements (those actually occurring in the loaded shell) impart a minimum value to the expression for the total energy.

By using relationships (3.17) and (3.11) we obtain the following expression for the actual state

$$\begin{aligned}\Pi'-\Pi''=V+\overline{V}-\int_{\varepsilon}^{\prime\prime\prime}\{Q_{\nu}^0u_{\nu}+\ldots+M_{\nu}^0\vartheta_{\nu}\}ds_t-\\ -\int_{\varepsilon}^{\prime\prime}\{Q_{\nu}u_{\nu}^0+\ldots+M_{\nu}\vartheta_{\nu}^0\}ds_t-\\ -\iint_S\{q_1u_1+q_2u_2+q_nw\}A_1A_2d\alpha_1d\alpha_2+\int_{\varepsilon}^{\prime\prime\prime}\{R+\overline{R}\}ds_t.\end{aligned}\tag{3.18}$$

By taking into account equalities (2.20), (2.21), (1.3), (1.2) and (1.1) we find (noting that for the true state $\bar{V} = V$) /130

$$\Pi' = \Pi'' = \Pi, \tag{3.19}$$

i.e., for a true state the total energy of the shell coincides with the supplementary energy.

Joining inequalities (3.10) and (3.16), we obtain the following dual evaluation for the total energy of the shell

$$\begin{aligned}
&-\frac{1}{2}\iint_S \{T_1^\wedge \varepsilon_1^\wedge + T_2^\wedge \varepsilon_2^\wedge + S^\wedge \omega^\wedge + M_1^\wedge \varkappa_1^\wedge + \\
&+ M_2^\wedge \varkappa_2^\wedge + 2H^\wedge \tau^\wedge\} A_1 A_2 d\alpha_1 d\alpha_2 + \\
&+ \int_\varepsilon'' \{Q_{\nu\nu}^\wedge u_\nu^0 + Q_{\nu t}^\wedge u_t^0 + Q_{\nu n}^\wedge w^0 + M_{\nu\nu}^\wedge \vartheta_\nu^0\} ds_t - \int_\varepsilon'''' \bar{R}^\wedge ds_t \leqslant \\
&\leqslant \Pi \leqslant \\
&\leqslant \frac{1}{2}\iint_S \{T_1^\vee \varepsilon_1^\vee + T_2^\vee \varepsilon_2^\vee + S^\vee \omega^\vee + M_1^\vee \varkappa_1^\vee + \\
&+ M_2^\vee \varkappa_2^\vee + 2H^\vee \tau^\vee\} A_1 A_2 d\alpha_1 d\alpha_2 - \\
&- \int_\varepsilon''' \{Q_{\nu\nu}^0 u_\nu^\vee + Q_{\nu t}^0 u_t^\vee + Q_{\nu n}^0 w^\vee + M_{\nu\nu}^0 \vartheta_\nu^\vee\} ds_t + \int_\varepsilon'''' R^\vee ds_t - \\
&- \iint_S \{q_1 u_1^\vee + q_2 u_2^\vee + q_n w^\vee\} A_1 A_2 d\alpha_1 d\alpha_2.
\end{aligned} \tag{3.20}$$

In this case II may designate either expression (3.17) or (3.11).

4. Lagrange Variational Equation

In addition to the true displacements which take place in a loaded shell we shall consider the following which are close to them

$$u_1 + \delta u_1; \quad u_2 + \delta u_2; \quad w + \delta w.$$

In order that these be geometrically permissible, it is necessary to require that the variations of the displacements δu_1, δu_2, δw vanish along those regions of the boundary where the corresponding displacements are assigned. In the preceding section we have shown that the true displacements impart a minimum value to the expression for the total energy. All the more the total energy for them has a stationary value, i.e.,

$$\delta\Pi' = 0 \tag{4.1}$$

or according to (3.17)

$$\delta V - \int_{\Gamma}^{\prime\prime\prime} \{Q_{\nu}^{0}\delta u_{\nu} + \ldots + M_{\nu}^{0}\delta\vartheta_{\nu}\}\, ds_t + \int_{\Gamma}^{\prime\prime\prime} \delta R ds_t - \iint_{S} \{q_1\delta u_1 + q_2\delta u_2 + q_n\delta w\}\, A_1A_2 d\alpha_1 d\alpha_2 = 0. \tag{4.2}$$

This equality is usually called the Lagrange variational equation. As in /131 section 1, by using Hooke's law we can transform the variation of the deformation energy δV, thereby imparting the following form to equation (4.2)

$$\iint_{S} \{T_1\delta\varepsilon_1 + \ldots + 2H\delta\tau\}\, A_1A_2 d\alpha_1 d\alpha_2 - \int_{\Gamma}^{\prime\prime\prime} \{Q_{\nu}^{0}\delta u_{\nu} + \ldots + M_{\nu}^{0}\delta\vartheta_{\nu}\}\, ds_t + \int_{\Gamma}^{\prime\prime\prime} \delta R ds_t - \iint_{S} \{q_1\delta u_1 + q_2\delta u_2 + q_n\delta w\}\, A_1A_2 d\alpha_1 d\alpha_2 = 0. \tag{4.3}$$

Integrating the first integral in parts and utilizing the Ostrogradskiy equations (as we have already done in section 3, Chapter III when deriving the equilibrium equations), we obtain

$$\begin{aligned} -\iint_{S} \{&[L_1\{T_1, \ldots, H\} + A_1A_2q_1]\,\delta u_1 + [L_2\{\ldots\} + A_1A_2q_2]\,\delta u_2 - \\ &- A_1A_2[L_3\{\ldots\} - q_n]\,\delta w\}\, d\alpha_1 d\alpha_2 + \\ &+ \int_{\Gamma}^{\prime\prime\prime} \{[g_1\{T_1, \ldots, H\} - Q_{\nu}^{0}]\,\delta u_{\nu} + [g_2\{\ldots\} - Q_{\nu t}^{0}]\,\delta u_t + \\ &+ [g_3\{\ldots\} - Q_{\nu n}^{0}]\,\delta w + [g_4\{\ldots\} - M_{\nu}^{0}]\,\delta\vartheta_{\nu}\}\, ds_t + \\ &+ \int_{\Gamma}^{\prime\prime\prime} \left\{\left[g_1\{\ldots\} + \frac{\partial R}{\partial u_{\nu}}\right]\delta u_{\nu} + \left[g_2\{\ldots\} + \frac{\partial R}{\partial u_t}\right]\delta u_t + \right. \\ &\left. + \left[g_3\{\ldots\} + \frac{\partial R}{\partial w}\right]\delta w + \left[g_4\{\ldots\} + \frac{\partial R}{\partial\vartheta_{\nu}}\right]\delta\vartheta_{\nu}\right\} ds_t = 0 \end{aligned} \tag{4.4}$$

(the integral with $'$ has vanished, because according to the conditions of variation the terms δu_{ν}, δu_t, δw, $\delta\vartheta_{\nu}$ in this condition also vanish).

Since the variations of the displacements are arbitrary, we obtain three equilibrium equations

$$\begin{aligned} L_1\{T_1, \ldots, H\} &= -A_1A_2q_1; \\ L_2\{\ldots\} &= -A_1A_2q_2;\quad L_3\{\ldots\} = q_n; \end{aligned} \tag{4.5}$$

the static boundary conditions

$$g_1\{\ldots\}=Q^0_{\nu};\ g_2\{\ldots\}=Q^0_{\tau};\quad g_3\{\ldots\}=Q^0_{n};\ g_4\{\ldots\}=M^0_{\nu}. \tag{4.6}$$

(of course, along those regions where they actually take place) and the conditions for elastic junction

$$g_1\{\ldots\}=-\frac{\partial R}{\partial u_{\nu}};\ g_2\{\ldots\}=-\frac{\partial R}{\partial u_{\tau}};\quad g_3\{\ldots\}=-\frac{\partial R}{\partial w};\ g_4\{\ldots\}=-\frac{\partial R}{\partial \vartheta_{\nu}}. \tag{4.7}$$

Thus we have shown that the equilibrium equations, the static boundary conditions and the conditions of elastic junction follow from the variational equation of Lagrange. In other words, the geometrically permissible displacements which impart a stationary value to the expression for the total energy also satisfy the equilibrium equations, the static boundary conditions and the conditions of elastic junction, thereby representing the solution of the considered problem. /132

These remarks show that the problem of integrating the differential equations of the theory of shells with assigned boundary conditions is equivalent to the problem of finding the stationary value of the expression (function) for the total energy. The second variational approach makes it possible for us to apply direct methods to the solution of problems of the theory of shells. The term "direct methods" is used in mathematical physics (for those methods which reduce the problem to the solution of algebraic systems).

The Ritz method, one of the best known methods, consists of the following (as it applies to the Lagrange equation): the displacements are sought in the form

$$\begin{aligned} u_1&=u_1^{(0)}+\sum_{k=1}^{N}a_k u_1^{(k)}(\alpha_1,\alpha_2);\\ u_2&=u_2^{(0)}+\sum_{k=1}^{N}b_k u_2^{(k)}(\alpha_1,\alpha_2);\\ w&=w^{(0)}+\sum_{k=1}^{N}c_k w^{(k)}(\alpha_1,\alpha_2). \end{aligned} \tag{4.8}$$

Here $u_1^{(0)}$, $u_2^{(0)}$, $w^{(0)}$, $u_1^{(k)}$, $u_2^{(k)}$, $w^{(k)}$ are certain preselected functions (they are called either competing functions or coordinate functions), while a_k, b_k,

c_k are constants which have to be established. In this case $u_1^{(0)}$, $u_2^{(0)}$, $w^{(0)}$ satisfy the geometric conditions of the problem, while the remaining functions satisfy the corresponding homogeneous conditions. Thus for any values of the unknown constants the approximate expressions for the displacements (4.8) are geometrically permissible functions. Varying these functions, we obtain

$$\delta u_1 = \sum_{k=1}^{N} u_1^{(k)} \delta a_k; \quad \delta u_2 = \sum_{k=1}^{N} u_2^{(k)} \delta b_k; \quad \delta w = \sum_{k=1}^{N} w^{(k)} \delta c_k. \tag{4.9}$$

It is easy to compute the deformation components and their variations by using expressions (4.8) and (4.9).

Constants a_k, b_k, c_k are determined from the condition that the approximate expressions for the displacements (4.8) impart a stationary value to the expression for the total energy. We substitute the resulting expressions into the Lagrange equation, which constitutes the condition for the stationary value of the total energy. Since all of the quantities in the assumed approxi- /133 mate expressions, except the constants, are known functions, all of the integrals may be computed and we arrive at the algebraic relationship

$$\sum_{k=1}^{N} [A_k(a_m, b_m, c_m)\delta a_k + B_k(\ldots)\delta b_k + C_k(\ldots)\delta c_k] = 0. \tag{4.10}$$

Since the variations of the constants are arbitrary, we obtain a system of 3N equations for determining the same number of unknown constants

$$\begin{aligned} A_k(a_m, b_m, c_m) &= 0; \\ B_k(a_m, b_m, c_m) &= 0; \\ C_k(a_m, b_m, c_m) &= 0. \end{aligned} \tag{4.11}$$

In this case, since W is a quadratic form of the deformation components (Chapter VII, 8.43), δW depends in a linear manner on the constants. We note that the linear property of the characteristic algebraic systems, which are obtained by applying the direct methods, is general for the linear problems of the theory of elasticity.

Having solved the system we find the values of the constants and thereby determine completely the approximate expressions for the displacements. Using the latter it is easy to compute the deformation components and then the forces and moments by means of Hooke's law. The success in applying the Ritz method depends on how successful we are in selecting the coordinate functions. Generally speaking the accuracy of the method increases as we increase the number

of the coordinate functions. However, in this case there is a substantial increase in computational difficulties. Therefore a different method of improving the accuracy of the solution is preferred.

This method consists of a more qualitative selection of the coordinate functions. Thus it is sometimes possible to subject the coordinate functions to additional conditions and to require, for example, that they satisfy exactly some of the static boundary conditions. From the meaning of the Ritz method this need not be done, because the static boundary conditions which follow from the variational equation are approximately satisfied in any case (such boundary conditions are usually called natural conditions). However, the imposition of additional conditions decreases the arbitrariness in the selection of coordinate functions and may improve the convergence of the method. In this case, of course, there is an increase in the functions for selecting the coordinate functions.

The direct methods are particularly successful in cases when the basic part of the unknown solution is discovered by a fortunate selection of functions $u_1^{(0)}$, $u_2^{(0)}$, $w^{(0)}$ (these functions, for example, may be taken as some rough approximate solution of the same problem). In this case the remaining part of the solution will play the role of a correction. However, we should point out that the realization of the second approach requires certain /134 experience and a clear understanding of the manner in which a specific structure operates under the conditions in question. It is also known that a significant obstacle to the wide application of direct methods in the linear theory of shells is the presence of the edge effect which constitutes a rapidly varying part in the solution. At the same time the solution of the boundary effect type is relatively simple (Chapter XIII) to express in an explicit form.

Therefore, in utilizing the Ritz method (and other direct methods) we should isolate and include the edge effect in two functions $u_1^{(0)}$, $u_2^{(0)}$, $w^{(0)}$.

In practice the more general method of Bubnov-Galerkin is frequently used. It differs from the Ritz method by the fact that (as this applies to the Lagrange equation) the arbitrary constants in the approximate solution (4.8) are determined from equations (4.4).

The reader who is interested in the theoretical principles associated with the direct methods should become familiar with references 237, 238, 239 and 241. These references also contain an extensive bibliography on this subject.

5. The Castigliano Variational Equation

Together with two forces and moments T_1, ..., H we shall consider the forces and moments $T_1 + \delta T_1$, ..., $H + \delta H$ which are close to them. We shall

assume that the latter are statically permissible. Then the variations δT_1, ..., δH must satisfy the homogeneous equilibrium equations and the homogeneous static boundary conditions, if such exist. From the property established in section 3 which indicates that the supplementary energy is maximum, it also follows, of course, that it is stationary for true forces and moments, i.e.,

$$\delta II'' = 0. \tag{5.1}$$

In accordance with (3.11) this equality may be written in the form

$$\delta \bar{V} - \int_{\sigma}^{'} \{u_\nu^0 \delta Q_\nu + \ldots + \vartheta_\nu^0 \delta M_\nu\}\, ds_t + \int_{\sigma}^{''''} \delta \bar{R}\, ds_t = 0 \tag{5.2}$$

or in an expanded form

$$\begin{aligned} &\iint_S \{\varepsilon_1 \delta T_1 + \varepsilon_2 \delta T_2 + \omega \delta S + \varkappa_1 \delta M_1 + \varkappa_2 \delta M_2 + \tau \delta 2H\}\, A_1 A_2 d\alpha_1 d\alpha_2 - \\ &- \int_{\sigma}^{'} \{u_\nu^0 \delta Q_\nu + u_t^0 \delta Q_t + w^0 \delta Q_n + \vartheta_\nu^0 \delta M_\nu\}\, ds_t + \int_{\sigma}^{''''} \delta \bar{R}\, ds_t = 0. \end{aligned} \tag{5.3}$$

Equality (5.3) is called the Castigliano variational equation. Let us show that the equations of continuity, the geometric boundary conditions and the conditions of elastic junction follow from this equation.

First of all, by assuming that T_1^*, ..., H^* are known functions <u>/135</u>

(selected beforehand) we find from relationships (Chapter VIII, 8.28 and 8.29)

$$\begin{aligned} \delta T_1 &= Ehc\delta\bar{\varkappa}_2; \quad \delta M_1 = -Ehc\delta\bar{\varepsilon}_2; \\ \delta T_2 &= Ehc\delta\bar{\varkappa}_1; \quad \delta M_2 = -Ehc\delta\bar{\varepsilon}_1; \\ \delta S &= -Ehc\delta\bar{\tau}; \quad \delta H = Ehc\delta\frac{\bar{\omega}}{2}; \end{aligned} \tag{5.4}$$

$$\begin{aligned} \delta Q_\nu &= Ehc\delta\bar{\varkappa}_{tt}; \quad \delta Q_n = Ehc\delta\bar{\varkappa}_{tn}; \\ \delta Q_t &= -Ehc\delta\bar{\varkappa}_{t\nu}; \quad \delta M_\nu = -Ehc\delta\bar{\varepsilon}_{tt}. \end{aligned} \tag{5.5}$$

Substituting expressions (5.4) into the volume integral, we obtain in place of it

$$\begin{aligned} I_1 = -Ehc \iint_S \Big\{ &\varkappa_2 \delta\bar{\varepsilon}_1 + \varkappa_1 \delta\bar{\varepsilon}_2 - \tau\delta\bar{\omega} - \varepsilon_2 \delta\bar{\varkappa}_1 - \varepsilon_1 \delta\bar{\varkappa}_2 + 2\frac{\bar{\omega}}{2}\delta\bar{\tau} \Big\} \times \\ &\times A_1 A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{5.6}$$

Now integrating I_1 in parts and applying the Ostrogradskiy equations (using the same procedure as above) we reduce it to the form

$$I_1 = Ehc \int_S \int \left[L_1 \left\{ \varkappa_2, \varkappa_1, -\tau, -\varepsilon_2, -\varepsilon_1, \frac{\omega}{2} \right\} \delta \bar{u}_1 + L_2 \{\ldots\} \delta \bar{u}_2 + \right.$$
$$\left. + L_3 \{\ldots\} \delta \bar{w} \right] A_1 A_2 d\alpha_1 d\alpha_2 - Ehc \int_g [g_1 \{\ldots\} \delta \bar{u}_t + \qquad (5.7)$$
$$+ g_2 \{\ldots\} \delta \bar{u}_t + g_3 \{\ldots\} \delta \bar{w} + g_4 \{\ldots\} \delta \bar{\vartheta}_t] \, ds_t.$$

The quantities ε_1, ε_2, ..., τ are the true components of deformation, therefore, according to (Chapter II, 3.20)

$$g_1 \left\{ \varkappa_2, \ldots, \frac{\omega}{2} \right\} = \varkappa_{tt}, \; g_2 \{\ldots\} = -\varkappa_{nt}, \; g_3 \{\ldots\} = \qquad (5.8)$$
$$= \varkappa_{tn}, \; g_4 \{\ldots\} = -\varepsilon_{tt}$$

and

$$I_2 = \int_g [g_1 \{\ldots\} \delta \bar{u}_t + g_2 \{\ldots\} \delta \bar{u}_t + g_3 \{\ldots\} \delta \bar{w} + g_4 \{\ldots\} \delta \bar{\vartheta}_t] \, ds_t =$$
$$= \int_g \{\varkappa_{tt} \delta \bar{u}_t - \varkappa_{nt} \delta \bar{u}_t + \varkappa_{tn} \delta \bar{w} - \varepsilon_{tt} \delta \bar{\vartheta}_t\} \, ds_t. \qquad (5.9)$$

Transforming the resulting integral, using the same procedure as in section 2, we obtain by means of equalities (Chapter VIII, 8.14, 8.16 and 8.18)

$$I_2 = -\int_g \left(\vec{\varkappa}_t \delta \vec{\bar{U}} + \varepsilon_{tt} \delta \bar{\vartheta}_t \right) ds_t = -\int_g \left(\frac{d\vec{\Omega}_t}{ds_t} \delta \vec{\bar{U}} + \varepsilon_{tt} \delta \bar{\vartheta}_t \right) ds_t =$$
$$= \int_g \left\{ \vec{\Omega}_t \frac{d\delta \vec{\bar{U}}}{ds_t} - \varepsilon_{tt} \delta \bar{\vartheta}_t \right\} ds_t = \int_g \{\vartheta_t \delta \bar{\omega}_{tt} - \omega_{tt} \delta \bar{\vartheta}_t +$$
$$+ \vartheta_n \delta \bar{\varepsilon}_{tt} - \varepsilon_{tt} \delta \bar{\vartheta}_n\} \, ds_t = \int_g \left\{ -\frac{d\vec{U}}{ds_t} \delta \vec{\bar{\Omega}}_t + \vartheta_n \delta \bar{\varepsilon}_{tt} \right\} ds_t = \qquad (5.10)$$
$$= \int_g \left(\vec{U} \delta \vec{\bar{\varkappa}}_t + \vartheta_n \delta \bar{\varepsilon}_{tt} \right) ds_t =$$
$$= \int_g \left\{ -u_n \delta \bar{\varkappa}_{tt} + u_t \delta \bar{\varkappa}_{tn} - w \delta \bar{\varkappa}_{tn} + \vartheta_n \delta \bar{\varepsilon}_{tt} \right\} ds_t.$$

If we now take into consideration equalities (5.5) we have /136

$$-EhcI_2=\int_g \{u_\nu \delta Q_{\nu\nu}+u_t \delta Q_{\nu t}+w\delta Q_{\nu n}+\vartheta_\nu \delta M_{\nu\nu}\}\,ds_t.$$

Substituting the resulting expression into (5.7) and the latter into the variational equation (5.3), we find, by expanding $\delta\bar{R}$

$$\begin{aligned}
&Ehc\int_S\int \{L_1[\ldots]\,\delta\bar{u}_1+L_2[\ldots]\,\delta\bar{u}_2+L_3[\ldots]\,\delta\bar{w}\}\,A_1A_2\,d\alpha_1 d\alpha_2+\\
&+\int_g{}'\{(u_\nu-u_\nu^0)\delta Q_{\nu\nu}+(u_t-u_t^0)\,\delta Q_{\nu t}+(w-w^0)\delta Q_{\nu n}+\\
&+(\vartheta_\nu-\vartheta_\nu^0)\delta M_{\nu\nu}\}\,ds_t+\int_g{}''\left\{\left(u_\nu+\frac{\partial\bar{R}}{\partial Q_{\nu\nu}}\right)\delta Q_{\nu\nu}+\right.\\
&\left.+\left(u_t+\frac{\partial\bar{R}}{\partial Q_{\nu t}}\right)\delta Q_{\nu t}+\left(w+\frac{\partial\bar{R}}{\partial Q_{\nu n}}\right)\delta Q_{\nu n}+\left(\vartheta_\nu+\frac{\partial\bar{R}}{\partial M_{\nu\nu}}\right)\delta M_{\nu\nu}\right\}ds_t=0.
\end{aligned}\tag{5.11}$$

(the integral with '' vanishes because according to the specified condition the variations $\delta Q_{\nu\nu}$, $\delta Q_{\nu t}$, $\delta Q_{\nu n}$, $\delta M_{\nu\nu}$ in this integral also vanish).

The vector of the stress functions is expressed (section 8, Chapter VIII) in terms of the principal moment and the principal vector of forces and moments applied to an arbitrary line in the region.

Since the variations of the forces and moments are arbitrary, we may assume that the variations of the stress functions are also arbitrary. Then from (5.11) we have an equation for the continuity of the middle surface

$$\begin{aligned}
&L_1\left\{x_2,\ x_1,\ -\tau,\ -\varepsilon_2,\ -\varepsilon_1,\ \frac{\omega}{2}\right\}=0,\\
&L_2[\ldots]=0,\ L_3[\ldots]=0,
\end{aligned}\tag{5.12}$$

the geometric boundary conditions (along those regions where they are assumed) /137

$$u_\nu=u_\nu^0,\ u_t=u_t^0,\ w=w^0,\ \vartheta_\nu=\vartheta_\nu^0 \tag{5.13}$$

and the conditions of elastic junction

$$u_\nu=-\frac{\partial\bar{R}}{\partial Q_{\nu\nu}},\quad u_t=-\frac{\partial\bar{R}}{\partial Q_{\nu t}},\quad w=-\frac{\partial\bar{R}}{\partial Q_{\nu n}},\quad \vartheta_\nu=-\frac{\partial\bar{R}}{\partial M_{\nu\nu}}. \tag{5.14}$$

Thus we have shown that the statically permissible system of functions, which imparts a stationary value to the expression for the supplementary energy, also satisfies the equations of continuity, the geometric boundary conditions and the conditions for elastic junction. Therefore it is the solution of the problem in question.

The property which has been proven makes it possible to use direct methods similar to those described in the preceding section for the solution of problems. The attribute of the Castigliano variational principle is that when it is used we obtain the approximate values for the forces and moments directly. We recall that in utilizing methods which are based on the utilization of the Lagrange equations, to determine stresses it becomes necessary to compute the first and second derivatives for the approximate values of the displacements established. As it is known (ref. 241), the approximate values of the derivatives are reduced to their exact values at a much slower rate than the functions themselves. Therefore, in problems which do not require the determination of displacements it is desirable to use the Castigliano variational equation.

To be fair, we should note the shortcomings of the method associated with the difficulty of constructing the coordinate (competing) functions. Indeed, we must select six functions for the approximate solution (three forces and three moments) in such a way as to satisfy the static boundary conditions and the three equilibrium conditions. The following approach may be used for shells of simple form: we select three functions (for example, three forces), which satisfy the static boundary conditions. Substituting them into the equilibrium equations we find (by solving the latter) the remaining three functions, while subjecting them to the remaining static conditions. For shells of complex form or with a complex boundary the utilization of this approach is difficult.

The possibility of utilizing the stress functions is enticing. Indeed, expressions

$$\begin{aligned} T_1 &= T_1^* + Ehc\bar{\varkappa}_2(\bar{u}_1, \bar{u}_2, \bar{w}), & M_1 &= M_1^* - Ehc\bar{\varepsilon}_2, \\ T_2 &= T_2^* + Ehc\bar{\varkappa}_1, & M_2 &= M_2^* - Ehc\bar{\varepsilon}_1, \\ S &= S^* - Ehc\bar{\tau}, & H &= H^* + Ehc\frac{\bar{\omega}}{2} \end{aligned} \qquad \text{(Chapter VIII, 8.28)}$$

satisfy the equilibrium conditions for any stress functions $\bar{u}_1$, $\bar{u}_2$, $\bar{w}$. /138

Usually it is not very difficult to find a particular solution for the equilibrium equations (T_1^*, ..., H^*). By assigning the stress functions with discretion, it then becomes possible to construct the necessary number of function systems satisfying the equilibrium equations.

However, a very unpleasant question arises concerning the compliance of the unknown approximate expressions to the static boundary conditions, which are written (in the general case) in terms of the stress functions by means of rather complicated differential operations.

In the case when the static boundary conditions are complete, i.e., all four generalized forces are assigned along the contour part (''), the relationships of Chapter VIII make it possible to circumvent these difficulties. Indeed, by dropping the constants $\vec{U}^0$ and $\vec{\Omega}_t^0$ in expressions (Chapter VIII, 8.34) which are insignificant in this problem, we obtain

$$\vec{\Omega}_t = -\frac{\vec{F}-\vec{F}^*}{Ehc}; \quad \vec{U} = -\frac{\vec{B}-\vec{B}^*}{Ehc}. \tag{5.15}$$

Here $\vec{F} = \int_{s_0}^{s_t} \vec{Q}_v ds_t'$ is the principal vector, while $\vec{B}$ is the principal moment with respect to the running point on the contour and is associated with the value of the principal moment computed with respect to some fixed point (fig. 17)

$$\vec{B}_{(0)} = \int_{s_0}^{s_t} M_v \vec{t}\, ds_t' + \int_{s_0}^{s_t} [\vec{r}', \vec{Q}_v]\, ds_t'. \qquad \text{(Chapter IV, 2.4)}$$

by means of relationship

$$\vec{B} = \vec{B}_{(0)} - [\vec{r}, \vec{F}]. \tag{5.16}$$

Furthermore, from equations

$$\vec{\Omega}_t = -\bar{\vartheta}_t \vec{\nu} + \bar{\vartheta}_v \vec{t} + \bar{\omega}_{tt} \vec{n}, \quad \vec{U} = \bar{u}_v \vec{\nu} + \bar{u}_t \vec{t} + \bar{w}\vec{n} \qquad \text{(Chapter VIII, 8.30, 8.31)}$$

and

$$\bar{\vartheta}_v = -\frac{d\bar{w}}{ds_v} + \frac{\bar{u}_v}{R_v} - \frac{\bar{u}_t}{R_{vt}} \qquad \text{(Chapter VIII, 3.16-Chapter VII, 7.68)}$$

we find

$$\frac{d\bar{w}}{ds_v} = -\bar{\vartheta}_v + \frac{\bar{u}_v}{R_v} - \frac{\bar{u}_t}{R_{vt}} = -(\vec{\Omega}_t)_t + \frac{(\vec{U})_v}{R_v} - \frac{(\vec{U})_t}{R_{vt}}. \tag{5.17}$$

Thus, by using the values of forces and of the bending moment assigned along the region of the contour under investigation to compute their principal vector and principal moment, it is possible to formulate the boundary conditions in terms of the components for the vector of the stress functions /139

$$\vec{U}|_{\Gamma} = \vec{f}_1(s_t), \quad \frac{d\overline{w}}{ds_t} = f_2(s_t) \tag{5.18}$$

($\vec{f}_1(s_t)$, $f_2(s_t)$ are functions assigned along the region of the contour and are computed from (5.17) and (5.15)).

We should point out in conclusion that conditions of this type must be satisfied when geometrically permissible displacements are selected. We can see from what has been stated that under complete static conditions the selection of statically permissible functions is somewhat more complicated than the establishment of geometrically permissible displacements, when considering the Lagrange variational equations.

6. A Modified Principle for the Stability of the Total Energy. The Reissner Variational Principle

By using Hooke's law relationships it is easy to impart the following form to the specific energy of deformation (ref. 242)

$$\begin{aligned} W^R = {} & T_1\varepsilon_1 + T_2\varepsilon_2 + S\omega + M_1\varkappa_1 + M_2\varkappa_2 + 2H\tau - \\ & - \left\{ \frac{1}{2Eh} \left[T_1^2 - 2\nu T_1 T_2 + T_2^2 + 2(1+\nu) S^2 \right] + \right. \\ & \left. + \frac{6}{Eh^3} \left[M_1^2 - 2\nu M_1 M_2 + M_2^2 + 2(1+\nu) H^2 \right] \right\}. \end{aligned} \tag{6.1}$$

By using this expression we represent the total energy of shell deformation (3.17) in the following manner

$$\begin{aligned} \Pi^R = {} & \int_S \int W^R A_1 A_2 d\alpha_1 d\alpha_2 - \int_g^{\cdots} \left[Q_{\nu\nu}^0 u_\nu + Q_{\nu t}^0 u_t + Q_{\nu n}^0 w + M_{\nu\nu}^0 \vartheta_\nu \right] ds_t + \\ & + \int_g^{\cdots} R ds_t - \int_S \int (q_1 u_1 + q_2 u_2 + q_n w) A_1 A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{6.2}$$

If we now assume that the forces-moments are not associated with the deformation components, but are arbitrary, the condition for the stability of the transformed total energy

$$\delta \Pi^R = 0 \tag{6.3}$$

is written in the form of the Reissner variational equation

$$\iint_S \left\{\left[\varepsilon_1 - \frac{1}{Eh}(T_1 - \nu T_2)\right]\delta T_1 + \ldots\right\} A_1 A_2 d\alpha_1 d\alpha_2 + \\ + \iint_S \{T_1 \delta \varepsilon_1 + \ldots\} A_1 A_2 d\alpha_1 d\alpha_2 + \int_g'''' \varepsilon R ds_t - \\ - \int_g''' \{Q_\nu^0 \delta u_\nu + \ldots\} ds_t - \iint_S \{q_1 \delta u_1 + q_2 \delta u_2 + q_n \delta w\} A_1 A_2 d\alpha_1 d\alpha_2 = 0. \quad (6.4)$$

The last four integrals were contained in the Lagrange equation (4.2). Therefore, repeating the transformations given in section 4, we find that /140 the equilibrium conditions, the static boundary conditions and the conditions of elastic junction follow from the variational equation (6.3). On the other hand, the relationships of Hooke's law follow from the first integral because the variations of forces and moments are arbitrary. Thus when utilizing the direct methods in the approximate expressions for the forces and moments we need not impose any conditions. The displacements, on the other hand, must be geometrically permissible functions as before.

The method presented makes it possible simply to select the coordinate functions even for complex problems. This is its merit, but also its shortcoming, because the excess arbitrariness in the assignment of the coordinate functions usually impairs the convergence of the method when the number of such functions is small. In addition, the large number of variable functions (three displacements and six forces-moments) leads to high order algebraic systems and substantially increases the volume of computational work.

The arbitrariness noted above may be decreased, if we require that the quantities (T_1, ..., H) constitute a statically permissible system. In this case the number of independently variable functions is decreased to six. Thus the modified method can be looked upon as a modification of the Castigliano principle for problems where in addition to stresses we are also interested in displacements.

7. The Variational Principle for Boundary Conditions. The Design of Variational Equations

Let u_1, u_2, w; T_1, T_2, ..., H be a system of functions satisfying the relationships of Hooke's law and satisfying the equilibrium equations. Frequently it is easy to construct such a system by using a collection of particular solutions of the problem to represent it. In order that we can consider it as a true (complete) solution, it is necessary to subject this system to the boundary conditions of the problem. We can simplify the difficult problem of satisfying the boundary conditions by making the latter natural. For this purpose it is necessary to construct the variational equation for which the boundary conditions of the problem are natural, or in other words are the Euler equations.

Such a variational equation may be constructed, for example, as follows

$$\int_g^{'} \{(u_\nu - u_\nu^0)\delta Q_{\nu\nu} + \ldots\}\, ds_t + \int_g^{''} \{(Q_{\nu\nu} - Q_{\nu\nu}^0)\delta u_\nu + \ldots\}\, ds_t + \\ + \int_g^{'''} \left\{\left(Q_{\nu\nu} + \frac{\partial R}{\partial u_\nu}\right)\delta u_\nu + \ldots\right\} ds_t + \\ + \int_g^{''''} \left\{\left(u_\nu + \frac{\partial \bar{R}}{\partial Q_{\nu\nu}}\right)\delta Q_{\nu\nu} + \ldots\right\} ds_t = 0. \tag{7.1}$$

Indeed, since $\delta Q_{\nu\nu}$, ... are arbitrary along the region where the generalized displacements (') are assigned, the following geometric boundary conditions follow from the first integral /141

$$u_\nu = u_\nu^0;\ u_t = u_t^0;\ w = w^0;\ \vartheta_\nu = \vartheta_\nu^0. \tag{7.2}$$

The corresponding static conditions are satisfied along the regions where the generalized forces ('') are assigned

$$Q_{\nu\nu} = Q_{\nu\nu}^0;\ Q_{\nu t} = Q_{\nu t}^0;\ Q_{\nu n} = Q_{\nu n}^0;\ M_{\nu\nu} = M_{\nu\nu}^0. \tag{7.3}$$

Finally, for regions with elastic constraint (''') the conditions of elastic constraint follow from the third integral

$$-Q_{\nu\nu} = \frac{\partial R}{\partial u_\nu};\ -Q_{\nu t} = \frac{\partial R}{\partial u_t};\ -Q_{\nu n} = \frac{\partial R}{\partial w};\ -M_{\nu\nu} = \frac{\partial R}{\partial \vartheta_\nu}. \tag{7.4}$$

As we have already stated in section 2, these conditions, which are solved with respect to the generalized displacements, are written in the following manner

$$-u_\nu = \frac{\partial \bar{R}}{\partial Q_{\nu\nu}};\ -u_t = \frac{\partial \bar{R}}{\partial Q_{\nu t}};\ -w = \frac{\partial \bar{R}}{\partial Q_{\nu n}};\ -\vartheta_\nu = \frac{\partial \bar{R}}{\partial M_{\nu\nu}}. \tag{7.5}$$

From this in particular it follows that the fourth integral vanishes together with the third integral, so that we cannot get any additional conditions from it.

The variational equation (7.1) may be transformed somewhat. First of all, we can add two terms to the left side

$$\int_g^{'} (Q_{\nu\nu}\delta u_\nu + \ldots)\, ds_t + \int_g^{''} (u_\nu \delta Q_{\nu\nu} + \ldots)\, ds_t,$$

which are identically equal to zero, because along the regions (') $\delta u_\nu = 0$, ..., and along the regions ('') $\delta Q_{\nu\nu} = 0$, ... Then equation (7.1) is written in the following manner

$$\begin{aligned}\int_g (Q_{\nu\nu}\delta u_\nu + \ldots)\, ds_t + \int_g (u_\nu \delta Q_{\nu\nu} + \ldots)\, ds_t - \int_g^{'} (u_\nu^0 \delta Q_{\nu\nu} + \ldots)\, ds_t - \\ - \int_g^{''} \left(Q_{\nu\nu}^0 \delta u_\nu + \ldots\right) ds_t + \int_g^{'''} \left(\frac{\partial R}{\partial u_\nu}\delta u_\nu + \ldots\right) ds_t + \\ + \int_g^{'''} \left(\frac{\partial \bar{R}}{\partial Q_{\nu\nu}}\delta Q_{\nu\nu} + \ldots\right) ds_t = 0.\end{aligned} \tag{7.6}$$

Now the variational equation (7.6) may be represented in the form /142

$$\delta \Pi^g = 0. \tag{7.7}$$

In this case the quantity (ref. 242)

$$\begin{aligned}\Pi^g = \int_g (Q_{\nu\nu}u_\nu + Q_{\nu t}u_t + Q_{\nu n}w + M_{\nu\nu}\vartheta_\nu)\, ds_t - \\ - \int_g^{'} (u_\nu^0 Q_{\nu\nu} + u_t^0 Q_{\nu t} + w^0 Q_{\nu n} + \vartheta_\nu^0 M_{\nu\nu})\, ds_t - \\ - \int_g^{''} (Q_{\nu\nu}^0 u_\nu + Q_{\nu t}^0 u_t + Q_{\nu n}^0 w + M_{\nu\nu}^0 \vartheta_\nu)\, ds_t + \int_g^{'''} (R + \bar{R})\, ds_t\end{aligned} \tag{7.8}$$

may be appropriately called the boundary energy. However, the equality (7.7) itself may be looked upon as the principle of boundary energy stability. Equations (7.7) and (7.1) may be utilized to construct direct methods of solving the problems of the theory of shells. In essence these methods will be reduced to the situation where the coordinate functions will be taken as the particular solutions of problems which do not satisfy the boundary conditions.

The approach presented in this section is widely used in mathematical physics. By means of this approach it is possible to take some relationship (boundary condition) which is unsuitable for us and convert it into a natural condition so that it is not necessary for us to satisfy it when we select the coordinate functions. With a certain amount of experience it is easy to construct the most diverse variational equations. In the previous section we have

become only briefly familiar with several of these which are best known and of most interest from the standpoint of application. However, we should point out that we should not become too fascinated with the construction of excessively extensive variational equations (which include a large number of relationships for the Euler equations). As we have already stated above, the alleviation in the selection of the coordinate functions is obtained, as a rule, at the expense of an impairment in the convergence of the method. In practice we should strive as much as possible to satisfy the maximum number of relationships (boundary conditions), even though they may not be natural.

8. The Variational Principle of the Complex Theory of Shells

Let us formulate a variational principle which corresponds to the V. V. Novozhilov version of the complex relationships. For this purpose we examine the following variational equation (fig. 18)

$$\begin{aligned}&\int\limits_{S_1+S_2}\int \delta\tilde{W}A_1A_2d\alpha_1d\alpha_2-\int\limits_{s_1+s_2}\Big[g_1\Big\{\tilde{T}_1,\ \tilde{T}_2,\ \tilde{S},\ ic\times\\ &\times\left(\tilde{T}_2-\nu\tilde{T}_1\right),\ ic\left(\tilde{T}_1-\nu\tilde{T}_2\right),\ -ic(1+\nu)\tilde{S}\Big\}\delta\tilde{u}_1+g_2\{\ldots\}\delta\tilde{u}_2+\\ &+g_3\{\ldots\}\delta\tilde{w}+g_4\{\ldots\}\delta\tilde{\vartheta}_1\Big]ds_1-\\ &-\int\limits_{S_1+S_2}\int\left\{q_1\delta\tilde{u}_1+q_2\delta\tilde{u}_2+q_n\delta\tilde{w}\right\}A_1A_2d\alpha_1d\alpha_2=0.\end{aligned}\tag{8.1}$$

Here

$$\begin{aligned}\tilde{W}=&\left[\tilde{\varepsilon}_1+ic\left(\tilde{\varkappa}_2-\nu\tilde{\varkappa}_1\right)-\frac{1}{Eh}\left(T_1^*-\nu T_2^*+\frac{i}{c}M_2^*\right)\right]\tilde{T}_1+\\ &+\left[\tilde{\varepsilon}_2+ic\left(\tilde{\varkappa}_1-\nu\tilde{\varkappa}_2\right)-\frac{1}{Eh}\left(T_2^*-\nu T_1^*+\frac{i}{c}M_1^*\right)\right]\tilde{T}_2+\\ &+\left[\frac{\tilde{\omega}}{2}-ic(1+\nu)\tilde{\tau}-\frac{1}{Eh}\left((1+\nu)S^*-\frac{i}{c}H^*\right)\right]\tilde{S}.\end{aligned}\tag{8.2}$$

For the sake of generality we shall assume that (fig. 18) some line (which /143 divides region S into two parts) undergoes a discontinuity in certain quantities which characterize the geometry of the shell. Assuming that the complex forces are not associated with the complex displacements, we obtain

$$\delta\tilde{W}=\delta\tilde{W}_1+\delta\tilde{W}_2,\tag{8.3}$$

where

$$\delta\widetilde{W}_1 = \widetilde{T}_1\delta\tilde{\varepsilon}_1 + \widetilde{T}_2\delta\tilde{\varepsilon}_2 + \tilde{S}\delta\tilde{\omega} + \\ + ic\left(\tilde{T}_2 - \nu\tilde{T}_1\right)\delta\tilde{\varkappa}_1 + ic\left(\tilde{T}_1 - \nu\tilde{T}_2\right)\delta\tilde{\varkappa}_2 - ic(1+\nu)\tilde{S}\delta\tilde{\tau}; \tag{8.4}$$

$$\delta\widetilde{W}_2 = \left[\tilde{\varepsilon}_1 + ic\left(\tilde{\varkappa}_2 - \nu\tilde{\varkappa}_1\right) - \frac{1}{Eh}\left(T_1^* - \nu T_2^* + \frac{l}{c}M_2^*\right)\right]\delta\tilde{T}_1 + \\ + \left[\tilde{\varepsilon}_2 + ic\left(\tilde{\varkappa}_1 - \nu\tilde{\varkappa}_2\right) - \frac{1}{Eh}\left(T_2^* - \nu T_1^* + \frac{l}{c}M_1^*\right)\right]\delta\tilde{T}_2 + \\ + \left[\frac{\tilde{\omega}}{2} - ic(1+\nu)\tilde{\tau} - \frac{1}{Eh}\left((1+\nu)S^* - \frac{l}{c}H^*\right)\right]\delta\tilde{S}. \tag{8.5}$$

Substituting the resulting expressions into the variational equation (8.1) and carrying out the usual transformation, we obtain

$$\int_{S_1+S_2}\int\left\{\left[\tilde{\varepsilon}_1 + ic\left(\tilde{\varkappa}_2 - \nu\tilde{\varkappa}_1\right) - \frac{1}{Eh}\left(T_1^* - \nu T_2^* + \frac{l}{c}M_2^*\right)\right]\delta\tilde{T}_1 + \ldots\right\} \times \\ \times A_1A_2d\alpha_1d\alpha_2 - \iint_{S_1+S_2}\left\{\left[L_1\left\{\tilde{T}_1,\ \tilde{T}_2,\ \tilde{S},\ ic\left(\tilde{T}_2 - \nu\tilde{T}_1\right),\ ic\left(\tilde{T}_1 - \nu\tilde{T}_2\right),\right.\right.\right. \\ \left.\left.\left. - ic(1+\nu)\tilde{S}\right\} + A_1A_2q_1\right]\delta\tilde{u}_1 + \ldots\right\}A_1A_2d\alpha_1d\alpha_2 + \int_{l_1}\left\{\vec{\widetilde{Q}}_{\nu}^{(1)}\delta\vec{\widetilde{U}}^{(1)} + \right. \\ \left. + \widetilde{M}_{\nu\nu}^{(1)}\delta\widetilde{\vartheta}_{\nu}^{(1)}\right\}ds_{l_1} + \int_{l_2}\left\{\vec{\widetilde{Q}}_{\nu}^{(11)}\delta\vec{\widetilde{U}}^{(11)} + \widetilde{M}_{\nu\nu}^{(11)}\widetilde{\vartheta}_{\nu}^{(11)}\right\}ds_{l_2} = 0. \tag{8.6}$$

Since the variations $\delta\widetilde{T}_1$, $\delta\widetilde{T}_2$, $\delta\widetilde{S}$ are arbitrary and are independent of the arbitrary variations $\delta\widetilde{u}_1$, $\delta\widetilde{u}_2$, $\delta\widetilde{w}$, we obtain equations in terms of the complex forces for each region /144

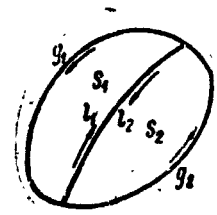

Figure 18

$$L_1\left\{\tilde{T}_1,\ \tilde{T}_2,\ \tilde{S},\ ic\left(\tilde{T}_2-\nu\tilde{T}_1\right),\ ic\left(\tilde{T}_1-\nu\tilde{T}_2\right),\ -ic(1+\nu)\tilde{S}\right\}=-A_1A_2q_1;\quad L_2\{\ldots\}=-A_1A_2q_2,\quad L_3\{\ldots\}=q_n;$$

(Chapter IV, 7.5)

equations in terms of complex displacements

$$\begin{aligned}
&\tilde{\varepsilon}_1+ic\left(\tilde{\varkappa}_2-\nu\tilde{\varkappa}_1\right)=\frac{1}{Eh}\left(T_1^*-\nu T_2^*+\frac{i}{c}M_2^*\right);\\
&\tilde{\varepsilon}_2+ic\left(\tilde{\varkappa}_1-\nu\tilde{\varkappa}_2\right)=\frac{1}{Eh}\left(T_2^*-\nu T_1^*+\frac{i}{c}M_1^*\right);\\
&\frac{\tilde{\omega}}{2}-ic(1+\nu)\tilde{\tau}=\frac{1}{Eh}\left((1+\nu)S^*-\frac{i}{c}H^*\right)
\end{aligned}$$

(Chapter IV, 7.3)

and the integral condition for the complex junction

$$\int_l\left\{\left(\vec{\tilde{Q}}_\nu^{(\mathrm{I})}\delta\vec{\tilde{U}}^{(\mathrm{I})}-\vec{\tilde{Q}}_\nu^{(\mathrm{II})}\delta\vec{\tilde{U}}^{(\mathrm{II})}\right)+\left(\tilde{M}_{\nu\nu}^{(\mathrm{I})}\delta\tilde{\vartheta}_\nu^{(\mathrm{I})}-\tilde{M}_{\nu\nu}^{(\mathrm{II})}\delta\tilde{\vartheta}_\nu^{(\mathrm{II})}\right)\right\}ds_l=0. \tag{8.7}$$

(In writing down the last relationship we have taken into account the fact that when the regions S_1 and S_2 are traced, line l is traversed in opposite directions.)

Let us recall, first of all, that

$$\delta\vec{\tilde{U}}=\delta\vec{U}+i\delta\vec{U}. \tag{8.8}$$

In addition, since the quantities T_1^*, ..., H^* represent some fixed (previously selected) system of functions, the variation of the vector for the stress functions is associated (Chapter VIII, 8.34), (Chapter VIII, 8.31) with the variations of the principal vector and of the principal moment by means of their relationship

$$\begin{aligned}
&\delta\vec{U}=-\frac{1}{Ehc}\delta\vec{B};\\
&\delta\tilde{\vartheta}_\nu=-\frac{1}{Ehc}\delta F_l\\
&\left(c=\frac{h}{\sqrt{12(1-\nu^2)}}\right).
\end{aligned} \tag{8.9}$$

From this we can see that in the case when shells of equal thickness and /145
with equal elastic constants are joined, the coincidence of variations $\delta\vec{U}$ and $\delta\tilde{\vartheta}_\nu$ along the junction means that the displacements, the angle of rotation ϑ_ν, the component of the principal vector F_t and the principal moment are continuous. Since these quantities are continuous, we should assume that

$$\delta\vec{\tilde{U}}^{(1)}=\delta\vec{\tilde{U}}^{(11)}=\delta\vec{\tilde{U}};\ \delta\tilde{\vartheta}_\nu^{(1)}=\delta\tilde{\vartheta}_\nu^{(11)}=\delta\tilde{\vartheta}_\nu. \tag{8.10}$$

Then from relationship (8.7), which is rewritten in the form

$$\int\left\{\left(\vec{\tilde{Q}}_\nu^{(1)}-\vec{\tilde{Q}}_\nu^{(11)}\right)\delta\vec{\tilde{U}}+\left(\tilde{M}_{\nu\nu}^{(1)}-\tilde{M}_{\nu\nu}^{(2)}\right)\delta\tilde{\vartheta}_\nu\right\}ds_l=0, \tag{8.11}$$

we obtain the conditions for the complex joining of shells

$$\vec{\tilde{Q}}_\nu^{(1)}=\vec{\tilde{Q}}_\nu^{(11)};\ \tilde{M}_{\nu\nu}^{(1)}=\tilde{M}_{\nu\nu}^{(11)}, \tag{8.12}$$

which are presented (for the case when the junction coincides with the coordinate line) in section 9 of Chapter IV.

Thus we have constructed a variational equation for which the equations in terms of the complex forces, the equations in terms of the complex displacements and the conditions of complex junction are the Euler equations. In seeking the approximate solution of equation (8.1) and (8.6), we approach simultaneously the solution of two (as we know from section 8, Chapter IV) equivalent system of equations: in terms of the complex forces and the complex displacements. At the same time the integral conditions for complex junction of shells (8.7) are satisfied approximately. In regard to the approximate expressions for $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{S}$, $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{w}$, they should satisfy the boundary conditions of the problem when they are assigned independently.

By using an approach analogous to the one in the preceding section the boundary conditions (or part of them) may be converted to natural conditions.

CHAPTER X. DISLOCATION DISPLACEMENTS. MULTIVALUED STRESS FUNCTIONS

In the present chapter we consider the dislocation displacements. /146
This is the term which we shall apply to multiple-valued displacements (ref. 25) which are associated with single-valued deformations. In addition, we shall consider multivalued stress functions briefly.

1. The Isolation of the Multivalued Part of the Vector for Displacements

Let us investigate the nature of the possible ambiguity in displacements when the components of deformation are single-valued. For this purpose we consider the contour Γ_j' with its origin and termination at the point M_j (fig. 19) which envelops one of the internal contours Γ_j.

As we can see from (Chapter VIII, 8.20), when we trace the contour Γ_j' the vector of displacements may undergo an increment of the form

$$\vec{U}^j + \left[\vec{\Omega}^j,\ \vec{r} - \vec{r}^j\right], \tag{1.1}$$

where

$$\vec{U}^j = \oint_{\Gamma_j'} \left\{\left[\vec{r} - \vec{r}^j,\ \vec{x}_t\right] + e_{tt}\vec{t}\right\} ds_t;\quad \vec{\Omega}^j = \oint_{\Gamma_j'} \vec{x}_t\, ds_t.$$

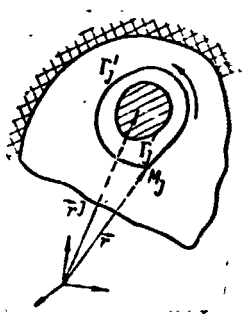

Figure 19

We note that quantities $\vec{U}^j$ and $\vec{\Omega}^j$ do not depend on the form of the contour. This is easy to establish, if we consider two contours Γ_j' and Γ_j'' which envelop Γ_j. By transforming the contour integral taken over the total contour into a volume integral limited by the contour, we find (in exactly the same manner as in section 3 of Chapter II), that the last integral vanishes due to the continuity equations. From this it follows that the quantities $\vec{U}^j$ and $\vec{\Omega}^j$ are independent of the integration path. We shall call these constants the dislocation parameters.

The analysis which has been carried out makes it possible to use a relatively simple method of constructing the dislocation (multivalued) part of the displacement vector. For this purpose we consider the expression /147

$$\left\{\vec{U}^j + \left[\vec{\Omega}^j, \vec{r} - \vec{r}^j\right]\right\} \Phi^j. \tag{1.2}$$

In regard to function $\Phi^j(\alpha, \beta)$, which we shall call the dislocation function in the future, we propose that:

(a) it produces an increment 1, when Γ_j is traced and an increment of 0 when the remaining contours are traced;

(b) the derivatives of this function in any direction are single-valued functions.

It is easy to see that the vector which has been constructed corresponds to the single-valued components of deformation. Indeed, if we compute the latter from (1.2) we obtain terms of two forms. The first contains terms in which the dislocation function was not differentiated. In these the factors before Φ^j will be the respective components of deformation constructed from expression contained in the figure brackets. Since the latter represents a vector of rigid displacement, these factors are equal to zero and all terms of the first group drop out. For the second form we consider terms which contain derivatives of Φ^j with respect to the coordinates. In view of property "b" they are all single-valued functions.

Let us assume that the dislocation parameters are known quantities. Then the vector of displacements may be written in the following form

$$\vec{U} = \sum_j \left\{\vec{U}^j + \left[\vec{\Omega}^j, \vec{r} - \vec{r}^j\right]\right\} \Phi^j + \vec{U}^0. \tag{1.3}$$

Here $\vec{U}^0$ is the single-valued part of the vector of displacements. If we use the conventional equations to compute the surface load and the boundary quantitites

corresponding to the dislocation part of the displacement vector, we shall find, as a rule, that they will not be equal to values required by the conditions of the problem. The role of the single-valued part $\vec{U}^0$ is reduced (as we shall see below) to provide for fulfillment of the boundary conditions and the satisfaction of the surface load.

As we can see from (1.3), the j-th term in the dislocation part of the displacement vector produces the following discontinuity in displacements at the section line L

$$[\vec{U}]_L = \{\vec{U}^j + [\vec{\Omega}^j, \vec{r} - \vec{r}^j]\}. \tag{1.4}$$

From this we can see that the edges of the section which correspond to the dislocation displacements must be joined with each other by means of a rigid displacement $\vec{U}^j$ and a rotation (around the point $\vec{r}^j$) by an angle $\vec{\Omega}^j$, i.e., /148 (as geometrically shown) they must be congruent. In this case as we can easily see from the presented relationships that the section itself is not localized and may be constructed arbitrarily.

It will be shown in section 3 that the dislocation displacements may be utilized in problems associated with the design of shells, when small pieces with congruent edges have been cut out of the shells.

In addition, the dislocation displacements may be used with success in the consideration of shells whose state of stress is determined by the relative displacement of the edges as a rigid whole. (See Chapter XVII.)

2. Dislocation Displacements for Shells of Revolution and for a Circular Cylindrical Shell

For a shell of revolution (fig. 20) (see also section 1, Chapter V)

$$\alpha_1 = \theta;\ \alpha_2 = \varphi;\ A_1 = R_1(\theta);$$
$$A_2 = R_2(\theta)\sin\theta = r(\theta);\ z = -\int_{\theta_0}^{\theta} R_1 \sin\theta\, d\theta; \tag{2.1}$$

$$\frac{dz}{d\theta} = z' = -R_1\sin\theta;\ \frac{dr}{d\theta} = r' = R_1\cos\theta. \tag{2.2}$$

The spherical belt which we are considering represents a doubly connected region. For this reason the sum in expression (1.3) is reduced to one term. For the dislocation function we select the following expression

$$\Phi^1 = \frac{1}{2\pi}\varphi, \tag{2.3}$$

and for the pole we select some point on the shell axis ($\theta = \theta_0$).

We assign the dislocation parameters in the following form

$$\begin{aligned} \vec{U}^1 &= U_x^1 \vec{e}_x + U_y^1 \vec{e}_y + U_z^1 \vec{e}_z; \\ \vec{\Omega}^1 &= \Omega_x^1 \vec{e}_x + \Omega_y^1 \vec{e}_y + \Omega_z^1 \vec{e}_z. \end{aligned} \tag{2.4}$$

It is easy to establish the following equalities from figure 21a and b

$$\begin{aligned} \vec{e}_x &= \cos\theta\cos\varphi\,\vec{e}_1 - \sin\varphi\,\vec{e}_2 + \sin\theta\cos\varphi\,\vec{n}; \\ \vec{e}_y &= \cos\theta\sin\varphi\,\vec{e}_1 + \cos\varphi\,\vec{e}_2 + \sin\theta\sin\varphi\,\vec{n}; \\ \vec{e}_z &= -\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{n}; \end{aligned} \tag{2.5}$$

$$\vec{r} - \vec{r}^1 = (r\cos\theta - z\sin\theta)\vec{e}_1 + (r\sin\theta + z\cos\theta)\vec{n}. \tag{2.6}$$

Let us now substitute the resulting relationships into the expression for the dislocation vector /149

$$\vec{U}g = \left\{\vec{U}^1 + \left[\vec{\Omega}^1,\ \vec{r} - \vec{r}^1\right]\right\}\frac{1}{2\pi}\varphi.$$

Noting in this connection that

$$\left[\vec{e}_1,\ \vec{e}_2\right] = \vec{n};\ \left[\vec{e}_2,\ \vec{n}\right] = \vec{e}_1;\ \left[\vec{n},\ \vec{e}_1\right] = \vec{e}_2.$$

we obtain the following expressions for its components

$$\begin{aligned} 2\pi u^g &= \{\cos\theta\,(\cos\varphi\,U_x^1 + \sin\varphi\,U_y^1) - \sin\theta\,U_z^1 + \\ &+ (r\sin\theta + z\cos\theta)\,(-\sin\varphi\,\Omega_x^1 + \cos\varphi\,\Omega_y^1)\}\varphi; \\ 2\pi w^g &= \{\sin\theta(\cos\varphi\,U_x^1 + \sin\varphi\,U_y^1) + \cos\theta\,U_z^1 - \\ &- (r\cos\theta - z\sin\theta)\,(-\sin\varphi\,\Omega_x^1 + \cos\varphi\,\Omega_y^1)\}\varphi; \\ 2\pi v^g &= \{-\sin\varphi\,U_x^1 + \cos\varphi\,U_y^1 - (\cos\varphi\,\Omega_x^1 + \sin\varphi\,\Omega_y^1)\,z + \\ &+ r\,\Omega_z^1\}\varphi. \end{aligned} \tag{2.7}$$

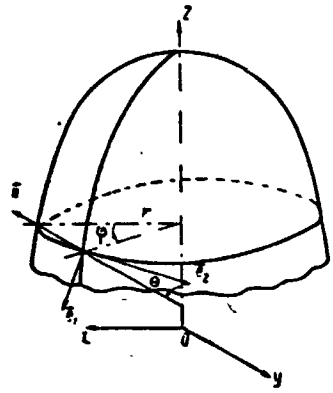

Figure 20

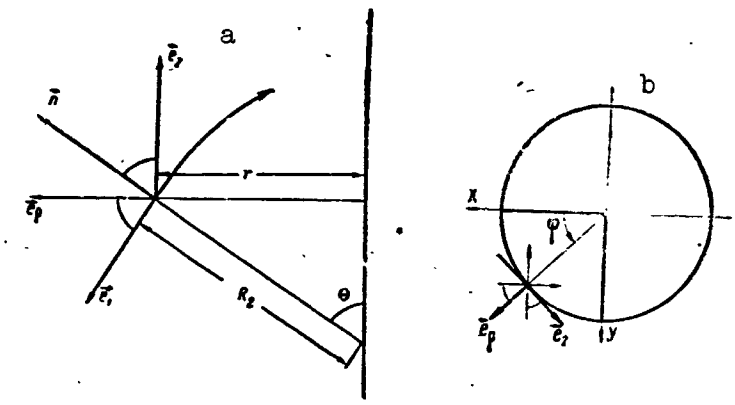

Figure 21

From this we find the horizontal and vertical displacements

$$2\pi u_\rho^g = \{\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1 + z(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}\varphi;$$
$$2\pi u_z^g = \{U_z^1 + r(\sin\varphi\, \Omega_x^1 - \cos\varphi\, \Omega_y^1)\}z. \qquad (2.8)$$

By using displacements (2.7) and equalities of section 8, Chapter VIII, it is easy to obtain the following expressions for the angles of rotation ($\vartheta_\alpha = \vartheta$, $\vartheta_\beta = \psi$)

$$2\pi\vartheta^g = \{-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1\}\varphi;$$
$$2\pi\psi^g = \{-\cos\theta(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1) + \sin\theta\, \Omega_z^1\}\varphi -$$
$$-\frac{1}{r}\{\sin\theta(\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1) + \cos\theta\, U_z^1 - (r\cos\theta - z\sin\theta)\times \qquad (2.9)$$
$$\times(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}$$

and the deformation component ($\alpha \to 1$, $\beta \to 2$)

$$
\begin{gathered}
\varepsilon_1^g = \varkappa_1^g = 0; \\
2\pi\varepsilon_2^g = \frac{1}{r}\{-\sin\varphi\, U_x^1 + \cos\varphi\, U_y^1 - z(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1) + r\,\Omega_z^1\}; \\
2\pi\omega^g = \frac{1}{r}\{\cos\theta\,(\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1) - \sin\theta\, U_z^1 + \\
+ (r\sin\theta + z\cos\theta)(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}; \\
2\pi\varkappa_2^g = \frac{1}{r^2}\{\sin\theta(\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1) + (-2r\cos\theta + z\sin\theta)\times \\
\times (\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1) + r\sin\theta\, \Omega_z^1\}; \\
2\pi\tau^g = \frac{1}{r^2}\{\sin\theta\cos\theta(\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1) + \cos^2\theta\, U_z^1 + \\
+ \sin\theta\,(r\sin\theta + z\cos\theta)(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}.
\end{gathered}
\tag{2.10}
$$

The circular cylindrical shell may be looked upon as a particular case of a shell of revolution, for which /150

$$
\theta = \frac{\pi}{2};\quad r = r_0;\quad z = -r_0\xi.
$$

Therefore it follows from the equalities obtained above that for this shell

$$
\begin{aligned}
2\pi u^g &= \{-U_z^1 + r_0(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}\varphi; \\
2\pi\omega^g &= \{\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1 - r_0\xi(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}\varphi; \\
2\pi v^g &= \{-\sin\varphi\, U_x^1 + \cos\varphi\, U_y^1 + r_0\xi(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1) + r_0\,\Omega_z^1\}\varphi; \\
2\pi\vartheta^g &= \{-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1\}\varphi;
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
2\pi\psi^g = \{\Omega_z^1\}\varphi + \frac{1}{r}\{&-(\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1) + r_0\xi(-\sin\varphi\, \Omega_x^1 + \\
&+ \cos\varphi\, \Omega_y^1)\};
\end{aligned}
\tag{2.12}
$$

$$
\begin{gathered}
\varepsilon_1^g = \varkappa_1^g = 0; \\
2\pi\varepsilon_2^g = \frac{1}{r_0}\{-\sin\varphi\, U_x^1 + \cos\varphi\, U_y^1 + r_0\xi(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1) + r_0\,\Omega_z^1\}; \\
2\pi\omega^g = \frac{1}{r_0}\{-U_z^1 + r_0(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}; \\
2\pi\varkappa_2^g = \frac{1}{r_0^2}\{\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1 - r_0\xi(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1) + r_0\,\Omega_z^1\}; \\
2\pi\tau^g = \frac{1}{r_0^2}\{r_0(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\}.
\end{gathered}
\tag{2.13}
$$

3. Mounting Stresses in a Circular Cylindrical Shell

Let us assume that a small longitudinal wedge (strip) has been either cut out of or added to a circular cylindrical shell. After this the edges are jointed. Even if external loads are absent, stresses are generated in a shell

which has been subjected to this operation. This type of problem occurs /151
for example, when we wish to evaluate the magnitude of the mounting stresses which occur when a cylindrical shell is assembled from two semitroughs. Since these are of complex form they may have to be drawn together or moved apart during welding to produce a cylindrical shell. Also during assembly the edges may be welded with misalignment. The solution of this problem may be approached by utilizing the dislocation displacements introduced above.

Let us make the cut coincide with the section $\varphi = 0$ (fig. 22). According to (2.11) and (2.12) the discontinuities in displacements and in the angle of rotation at the cut are equal to

$$\begin{aligned}
&[u^g] = u^g(\xi, 2\pi) - u^g(\xi, 0) = -U_z^1 + r_0\Omega_y^1;\\
&[v^g] = U_y^1 + r_0\xi\Omega_x^1 + r_0\Omega_z^1;\\
&[w^g] = U_x^1 - r_0\xi\Omega_y^1;\\
&[\vartheta^g] = \Omega_z^1.
\end{aligned} \tag{3.1}$$

We seek the total displacements in the form

$$u = u^g + u^0,\ v = v^g + v^0,\ w = w^g + w^0. \tag{3.2}$$

In order to avoid the occurrence of small differences between quantities which are close in value we represent u^0, v^0, w^0 as a sum of two terms

$$u^0 = u^{00} + u^k,\ v^0 = v^{00} + v^k,\ w^0 = w^{00} + w^k, \tag{3.3}$$

where the quantities u^{00}, v^{00}, w^{00} are determined from the condition that the tangential components of deformation comprised by them become equal to zero when added to the dislocation components. In other words, they represent some solution of the system

$$\begin{aligned}
&2\pi\frac{\partial u^{00}}{\partial\xi} = -2\pi r_0\varepsilon_1^g = 0;\\
&2\pi\left(\frac{\partial v^{00}}{\partial\varphi} + w^{00}\right) = -2\pi r_0\varepsilon_2^g = \sin\varphi\, U_x^1 - \cos\varphi\, U_y^1 -\\
&\qquad - r_0\xi(\cos\varphi\,\Omega_x^1 + \sin\varphi\,\Omega_y^1) - r_0\Omega_z^1;\\
&2\pi\left(\frac{\partial u^{00}}{\partial\varphi} + \frac{\partial v^{00}}{\partial\xi}\right) = -2\pi r_0\omega^g = U_z^1 + r_0(\sin\varphi\,\Omega_x^1 - \cos\varphi\,\Omega_y^1);
\end{aligned} \tag{3.4}$$

As a solution of this system we adopt the following expressions

$$
\begin{aligned}
&2\pi u^{00}=0;\\
&2\pi v^{00}=\xi U_z^1-r_0\xi(\sin\varphi\,\Omega_x^1-\cos\varphi\,\Omega_y^1);\\
&2\pi w^{00}=\sin\varphi\,U_x^1-\cos\varphi\,U_y^1-r_0\Omega_z^1.
\end{aligned}
\tag{3.5}
$$

They correspond to the following components of flexural deformation /152

$$
\begin{aligned}
&2\pi r_0^2\varkappa_1^{00}=0;\\
&2\pi r_0^2\varkappa_2^{00}=\sin\varphi\,U_x^1-\cos\varphi\,U_y^1-r_0\xi(\cos\varphi\,\Omega_x^1+\sin\varphi\,\Omega_y^1);\\
&2\pi r_0^2\tau^{00}=U_z^1-r_0(\sin\varphi\,\Omega_x^1-\cos\varphi\,\Omega_y^1).
\end{aligned}
\tag{3.6}
$$

Thus the first two terms in the expressions for displacements

$$u^g+u^{00},\ v^g+v^{00},\ w^g+w^{00}$$

correspond to the deformation components

$$
\begin{aligned}
&\varepsilon_1^g+\varepsilon_1^{00}=\omega^g+\omega^{00}=\varepsilon_2^g+\varepsilon_2^{00}=0,\ \varkappa_1^g+\varkappa_1^{00}=0;\\
&2\pi r_0^2(\varkappa_2^g+\varkappa_2^{00})=2(\sin\varphi\,U_x^1-\cos\varphi\,U_y^1)-2r_0\xi(\cos\varphi\,\Omega_x^1+\\
&\quad+\sin\varphi\,\Omega_y^1)+r_0\Omega_z^1;\\
&2\pi r_0^2(\tau^g+\tau^{00})=U_z^1-2r_0(\sin\varphi\,\Omega_x^1-\cos\varphi\,\Omega_y^1).
\end{aligned}
\tag{3.7}
$$

Furthermore, according to the generalized Hooke's law (section 8, Chapter VIII) we have the following corresponding forces-moments

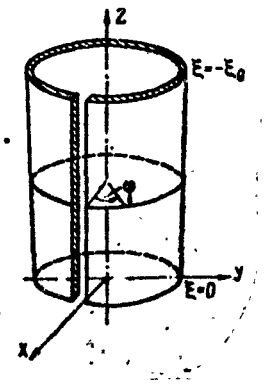

Figure 22

$$
\begin{aligned}
&T_1^{\varepsilon} + T_1^{(0)} = T_2^{\varepsilon} + T_2^{(0)} = S^{\varepsilon} + S^{(0)} = 0;\\
&2\pi\left(M_2^{\varepsilon} + M_2^{(0)}\right) = \frac{D}{r_0^2}\left\{2\left(\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1\right) - \right.\\
&\qquad \left. - 2r_0\xi\left(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1\right) + r_0\,\Omega_z^1\right\};\\
&2\pi\left(H^{\varepsilon} + H^{(0)}\right) = \frac{D}{r_0^2}(1-\nu)\left\{U_z^1 - 2r_0\left(\sin\varphi\, \Omega_x^1 - \cos\varphi\, \Omega_y^1\right)\right\};\\
&M_1^{\varepsilon} + M_1^{(0)} = \nu\left(M_2^{\varepsilon} + M_2^{(0)}\right)\\
&\qquad\qquad \left(D = Ehc^2 = \frac{Eh^3}{12(1-\nu^2)}\right).
\end{aligned}
\tag{3.8}
$$

Finally, by substituting the resulting expressions into the left side of the equilibrium equations (Chapter VIII, 8.24) we obtain the respective components of the surface loading ($\frac{1}{R_{\alpha\beta}} = \frac{1}{R_\alpha} = 0$, $R\beta = r_0$)

$$
\begin{aligned}
&2\pi\left(q_1^{\varepsilon} + q_1^{(0)}\right) = 0;\\
&2\pi\left(q_2^{\varepsilon} + q_2^{(0)}\right) = \frac{D}{r_0^4}\left\{2\left(\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1\right) - \right.\\
&\qquad \left. - 2r_0\xi\left(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1\right)\right\};\\
&2\pi\left(q_n^{\varepsilon} + q_n^{(0)}\right) = \frac{D}{r_0^4}\left\{2\left(\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1\right) - \right.\\
&\qquad \left. - 2r_0\xi\left(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1\right)\right\}.
\end{aligned}
\tag{3.9}
$$

The corresponding static boundary quantities (Chapter VIII, 8.25) have the form ($\gamma = 0$, $\gamma = \pi$)

$$
\begin{aligned}
&2\pi\left(T_2^{\varepsilon} + T_1^{(0)}\right) = 0, \qquad \underline{(\xi = \xi_0)}\\
&2\pi\left(Q_{12}^{\varepsilon} + Q_{12}^{(0)}\right) = \frac{D}{r_0^3}2(1-\nu)\left\{U_z^1 - 2r_0\left(\sin\varphi\, \Omega_x^1 - \cos\varphi\, \Omega_y^1\right)\right\};\\
&2\pi\left(M_2^{\varepsilon} + M_1^{(0)}\right) = \frac{D}{r_0^2}\nu\left\{2\left(\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1\right) - \right.\\
&\qquad \left. - 2r_0\xi_0\left(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1\right) + r_0\,\Omega_z^1\right\};\\
&2\pi\left(Q_{1n}^{\varepsilon} + Q_{1n}^{(0)}\right) = -\frac{D}{r_0^2}2(2-\nu)\left\{\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1\right\};
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&2\pi\left(T_2^{\varepsilon} + T_1^{(0)}\right) = 0, \qquad \underline{(\xi = 0)}\\
&2\pi\left(Q_{12}^{\varepsilon} + Q_{12}^{(0)}\right) = \frac{D}{r_0^3}2(1-\nu)\left\{U_z^1 - 2r_0\left(\sin\varphi\, \Omega_x^1 - \cos\varphi\, \Omega_y^1\right)\right\};\\
&2\pi\left(M_2^{\varepsilon} + M_1^{(0)}\right) = \frac{D}{r_0^2}\nu\left\{2\left(\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1\right) + r_0\,\Omega_z^1\right\};\\
&2\pi\left(Q_{1n}^{\varepsilon} + Q_{1n}^{(0)}\right) = -\frac{D}{r_0^2}2(2-\nu)\left\{\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1\right\};
\end{aligned}
\tag{3.11}
$$

In order to determine the stresses produced in the shell by the defects in the middle surface we must set the total surface loading and the static boundary conditions equal to zero. Then according to (3.9) the unknown compensating displacements u^k, v^k, w^k will be given by the surface loading /153

$$\begin{aligned}
&2\pi q_1^k = -2\pi\left(q_1^g + q_1^{00}\right) = 0;\\
&2\pi q_2^k = -2\pi\left(q_2^g + q_2^{00}\right) = \frac{D}{r_0^4}\{-2(\cos\varphi\, U_x^1 + \sin\varphi\, U_y^1) + \\
&\qquad + 2r_0\xi(-\sin\varphi\, \Omega_x^1 + \cos\varphi\, \Omega_y^1)\};\\
&2\pi q_n^k = -2\pi\left(q_n^g + q_n^{00}\right) = \frac{D}{r_0^4}\{-2(\sin\varphi\, U_x^1 - \cos\varphi\, U_y^1) + \\
&\qquad + 2r_0\xi(\cos\varphi\, \Omega_x^1 + \sin\varphi\, \Omega_y^1)\}.
\end{aligned} \tag{3.12}$$

Let us assume that the cylindrical shell in question is not too long, so that the particular solution of the problem may be taken from the membrane theory. Determining the forces we find that they have an order of magnitude equal to

$$T^k = O\left(\frac{D}{r_0^3} U^1\right). \tag{3.13}$$

Using these to determine the displacements we obtain quantities of the order of

$$u^k = O\left(\frac{h^2}{r_0^2} U^1\right).$$

The resulting displacements are negligibly small compared to u_i^g, u_i^{00} while the stresses ($\sigma_T^k = \frac{T^k}{h}$) corresponding to (3.13) are less than the stresses corresponding to M^g, M^{00} ($\sigma_M = \frac{6M}{h^2}$) by a factor h/r_0. Therefore, without exceeding the error common to the theory of shells, we can neglect the particular solution corresponding to the surface loading when we determine u^k, v^k, w^k. In this way the role of the compensating displacements is reduced, so that /154 they make it possible to satisfy the boundary conditions of the problem in question.

Since in our problem the ends of the shell are free, we have the following static boundary conditions for determining the compensating displacements

$$
\begin{aligned}
&2\pi T_1^k = -2\pi\left(T_1^f + T_1^{00}\right) = 0, \qquad \underline{(\xi=\xi_0)}\\
&2\pi Q_{12}^k = -2\pi\left(Q_{12}^f + Q_{12}^{00}\right) = \frac{D}{r_0^3}2(1-\nu)\left\{-U_z^1 + 2r_0\left(\sin\varphi\,\Omega_x^1 - \right.\right.\\
&\qquad \left.\left. - \cos\varphi\,\Omega_y^1\right)\right\};\\
&2\pi M_1^k = -2\pi\left(M_1^f + M_1^{00}\right) = \frac{D}{r_0^2}\nu\left\{-2\left(\sin\varphi\,U_x^1 - \cos\varphi\,U_y^1\right) + \right.\\
&\qquad \left. + 2r_0\xi_0\left(\cos\varphi\,\Omega_x^1 + \sin\varphi\,\Omega_y^1\right) - r_0\Omega_z^1\right\};\\
&2\pi Q_{1n}^k = -2\pi\left(Q_{1n}^f + Q_{1n}^{00}\right) = \frac{D}{r_0^2}2(2-\nu)\left\{\cos\varphi\,\Omega_x^1 + \sin\varphi\,\Omega_y^1\right\};
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&2\pi T_1^k = 0, \qquad \underline{(\xi=0)}\\
&2\pi Q_{12}^k = \frac{D}{r_0^3}2(1-\nu)\left\{-U_z^1 + 2r_0\left(\sin\varphi\,\Omega_x^1 - \cos\varphi\,\Omega_y^1\right)\right\};\\
&2\pi M_1^k = \frac{D}{r_0^2}\nu\left\{-2\left(\sin\varphi\,U_x^1 - \cos\varphi\,U_y^1\right) - r_0\Omega_z^1\right\};\\
&2\pi Q_{1n}^k = \frac{D}{r_0^2}2(2-\nu)\left\{\cos\varphi\,\Omega_x^1 + \sin\varphi\,\Omega_y^1\right\};
\end{aligned}
\tag{3.15}
$$

In line with the characteristic relationship between the boundary conditions and φ, we divide the problem into three parts (section 11, Chapter V):

(1) the first symmetric case corresponding to Ω_z^1;

(2) the second symmetric case--U_z^1;

(3) the antisymmetric case--U_x^1, U_y^1, Ω_x^1, Ω_y^1.

Let us consider the first of these in detail. According to (3.8) for this case

$$
\begin{aligned}
&T_1^f + T_1^{00} = T_2^f + T_2^{00} = 0;\\
&2\pi\left(M_2^f + M_2^{00}\right) = \frac{D}{r_0}\Omega_z^1; \quad 2\pi\left(M_1^f + M_1^{00}\right) = \frac{D}{r_0}\nu\Omega_z^1.
\end{aligned}
\tag{3.16}
$$

When $\xi = 0$ and $\xi = \xi_0$, the boundary conditions are written in the same way

$$
T_1^k = Q_{1n}^k = 0; \; 2\pi M_1^k = -\frac{D}{r_0}\nu\Omega_z^1. \tag{3.17}
$$

We shall assume that the shell is sufficiently long so that we can neglect the interaction of the edges. Utilizing expressions presented in section 11, Chapter V we obtain the following for the total stresses

$$\sigma_{M_1}=\frac{6\left(M_1^g+M_1^{r_0}+M_1^h\right)}{h^2}=\frac{6}{h^2}\frac{D}{2\pi r_0}\Omega_z^1\left\{\nu\left[1-(\cos\beta+\sin\beta)e^{-\beta}\right]\right\};$$

$$\sigma_{M_2}=\frac{6\left(M_2^g+M_2^{00}+M_2^h\right)}{h^2}=\frac{6}{h^2}\frac{D}{2\pi r_0}\Omega_z^1\left\{1-\nu^2(\cos\beta+\sin\beta)e^{-\beta}\right\};\qquad(3.18)$$

$$\sigma_{T_2}=\frac{T_2^h}{h}=\frac{6}{h^2}\frac{D}{2\pi r_0}\Omega_z^1\left\{\frac{\nu\sqrt{3(1-\nu^2)}}{3}(\cos\beta-\sin\beta)e^{-\beta}\right\}$$

$$\left(D=\frac{Eh^3}{12(1-\nu^2)},\ \beta=\sqrt{\frac{r^0}{2c}}(\xi_0-\xi),\ c=\frac{h}{\sqrt{12(1-\nu^2)}}\right).$$

(We drop the detailed calculation steps assuming that the reader who becomes familiar with Chapter V of the first part will carry these out independently, using equations (3.18) as a reference.) /155

We can see from equations (3.18) that the characteristic stresses in this case are the values σ_{M_2} remote from the edges

$$\sigma_{M_2}=\frac{E}{4\pi(1-\nu^2)}\left(\frac{h}{r_0}\right)\left(\frac{r_0\Omega_z^1}{r_0}\right)=\frac{E}{4\pi(1-\nu^2)}\left(\frac{h}{r_0}\right)^2\left(\frac{r_0\Omega_z^1}{h}\right).\qquad(3.19)$$

In the same manner we determine the states of stress corresponding to the other two cases. It is suggested that the reader use relationships presented in section 11, Chapter V to convince himself that the characteristic stresses will be as follows

$$\sigma_H=\frac{6(H^g+H^{00})}{h^2}=\frac{E}{4\pi(1+\nu)}\left(\frac{h}{r_0}\right)\left(\frac{U_z^1}{r_0}\right)=\frac{E}{4\pi(1+\nu)}\left(\frac{h}{r_0}\right)^2\left(\frac{U_z^1}{h}\right);\qquad(3.20)$$

$$(\sigma_{M_2})_{\max}=-\frac{E}{2\pi(1-\nu^2)}\left(\frac{h}{r_0}\right)\left(\frac{U_y^1}{r_0}\right)=-\frac{E}{2\pi(1-\nu^2)}\left(\frac{h}{r_0}\right)^2\left(\frac{U_y^1}{h}\right)\ (\varphi=0);$$

$$(\sigma_{M_2})_{\max}=\frac{E}{2\pi(1-\nu^2)}\left(\frac{h}{r_0}\right)\left(\frac{U_x^1}{r_0}\right)=\frac{E}{2\pi(1-\nu^2)}\left(\frac{h}{r_0}\right)^2\left(\frac{U_x^1}{r_0}\right)\quad\left(\varphi=\frac{\pi}{2}\right).\qquad(3.21)$$

(away from the edge)

$$-\sigma_{M_2}=\frac{E}{2\pi}\left(\frac{h}{r_0}\right)\left(\frac{r_0\xi_0\Omega_x^1}{r_0}\right)=\frac{E}{2\pi}\left(\frac{h}{r_0}\right)^2\left(\frac{r_0\xi_0\Omega_x^1}{h}\right)\qquad(\varphi=0);$$

$$-\sigma_{M_2}=\frac{E}{2\pi}\left(\frac{h}{r_0}\right)\left(\frac{r_0\xi_0\Omega_y^1}{r_0}\right)=\frac{E}{2\pi}\left(\frac{h}{r_0}\right)^2\left(\frac{r_0\xi_0\Omega_y^1}{h}\right)\qquad\left(\varphi=\frac{\pi}{2}\right)\qquad(3.22)$$

(at the edge $\xi=\xi_0$).

When the shells are not too thin, the dislocation stresses may reach significant values. Let us assume for example that

$$\frac{U_y^1}{r_0}=0.02;\ \frac{h}{r_0}=0.05;\ E=2.1\cdot10^6\ \text{kg/cm}^2;\ \nu=0.3.$$

Then according to $(3.21)_1$

$$(-\sigma_{M_1})_{\max}=\frac{2{,}1\cdot10^6\cdot0{,}05\cdot0{,}02}{2\cdot3{,}14\cdot0{,}91}=365\ \text{kg/cm}^2.$$

The resulting stress when superposed on the design stresses may have a certain effect on the strength of the structure. The above analysis shows that in the considered shell the state of stress is composed of a state with high moments and of the edge effect (sections 11 and 12, Chapter XIII). /156

4. Multivalued Stress Functions

Utilizing the static-geometric analogy we introduce the dislocation stress functions by means of the following relationships

$$\vec{U}=\sum_j\left\{\vec{U}^j+\left[\vec{\Omega}^j,\ \vec{r}-\vec{r}^j\right]\right\}\Phi^j(\alpha_1,\ \alpha_2)+\vec{U}^0. \tag{4.1}$$

Here $\Phi^j(\alpha_1, \alpha_2)$, as in section 1, are dislocations functions; $\vec{U}^0$ is the single-valued part of the vector for the stress functions. With regard to parameters $\vec{U}^j$ and $\vec{\Omega}^j$, they are determined quite easily from the forces and moments which compose the principal vector and the principal moment and which are applied to specific contours Γ_j. Indeed, when we trace the j-th contour, vector (4.1) undergoes an increment

$$\left[\vec{U}\right]_{\Gamma_j}=\vec{U}^j+\left[\vec{\Omega}^j,\ \vec{r}-\vec{r}^j\right]. \tag{4.2}$$

Let us turn to relationship (Chapter VIII, $8.34)_2$. Dropping quantity $\vec{B}^*$ which is insignificant for the problem in question we obtain the following expression for the increment in vector for the stress functions

$$\left[\vec{U}\right]_{\Gamma_j}=-\frac{1}{Ehc}\vec{B}^j. \tag{4.3}$$

Usually it is more convenient to deal with the principal moment $\vec{B}^j_{(j)}$, computed with respect to some fixed point $(\vec{r}^j)$. In this case it is easy to see that

$$\vec{B}^j_{(0)}=\vec{B}^j+\left[\vec{r}^j,\ \vec{F}^j\right]$$

and (4.3) assumes the form

$$[\vec{U}]_{r_j} = -\frac{1}{Ehc}\{\vec{B}_{(l)}^{j} - [\vec{r}, \vec{F}^{j}]\}. \qquad (4.3')$$

Comparing expressions (4.2) with (4.3') we find

$$Ehc\,\vec{\Omega}^{j} = -\vec{F}^{j};\ Ehc\,\vec{U}^{j} = -\vec{B}_{(l)}^{j} + [\vec{r}^{j}, \vec{F}^{j}]. \qquad (4.4)$$

Usually the values of the principal vectors and of the principal moments are known for each specific contour in problems of the theory of shells. /157 Therefore parameters $\vec{U}^{j}$, $\vec{\Omega}^{j}$ are assumed to be known. This makes it possible, when utilizing the method of stress functions, to isolate in a rather simple fashion from the unknown vector $(\vec{\bar{U}})$ for the stress functions, the multiple-valued part and to formulate the boundary problems only for the single-valued part of solution $\vec{\bar{U}}^{0}$.

On the other hand, when we use the method of complex displacements, this approach makes it possible to write out the system of functions in an explicit form T_1^*, T_2^*, ..., H^*, which is statically equivalent to the unknown solution along each contour. The latter in particular may be utilized in analyzing shells with concentrated loads (ref. 201).

CHAPTER XI. EQUATIONS IN TERMS OF COMPLEX DISPLACEMENTS

In section 8 of Chapter IV we proposed a simplified version for the complex Hooke's law, from which the following simplified system of equations in terms of complex displacements followed /158

$$\begin{gathered}\tilde{\varepsilon}_2+ic\left(\tilde{\varkappa}_1+\tilde{\varkappa}_2\right)=\frac{1}{Eh}\left(T_1^*+T_2^*+\frac{i}{c}M_1^*\right);\\ \tilde{\varepsilon}_1-\tilde{\varepsilon}_2=\frac{i}{Ehc}\left(M_2^*-M_1^*\right);\\ \tilde{\omega}=-\frac{i}{Ehc}2H^*.\end{gathered} \tag{0.1}$$

The absence of many small terms in these equations make the method, based on their utilization, competitive with the method of equations with complex forces.

Below system (0.1) is considered in detail as it applies to shells of revolution and to cylindrical shells. In particular it is shown that for a symmetrically loaded shell of revolution the well-known Meissner-Clark equations follow from system (0.1). Also an analog of the Meissner equation is obtained for antisymmetric loading. The latter is generalized in ref. 111 for the case of a shell with variable thickness.

It is shown that we can obtain from the equations with complex displacements a more consistent form for the V. Z. Vlasov equations of hollow shells than the one which is usually utilized. Another approach is the purpose of refining the Vlasov equations.

1. The Potential of Displacements for Shells of Revolution

With future applications in mind we represent the displacements in the following form

$$\tilde{u}=u^{\mathrm{d}}+\tilde{u}^{\mathrm{c}},\quad \tilde{v}=v^{\mathrm{d}}+\tilde{v}^{\mathrm{c}},\quad \tilde{w}=w^{\mathrm{d}}+\tilde{w}^{\mathrm{c}}, \tag{1.1}$$

(d = dislocation; c = correction)

where the first terms represent the dislocation displacements, while the second term represents the correction terms. Substituting these expressions into (0.1) we obtain a system of equations for determining the compensating parts of the complex displacements /159

$$\begin{gathered}\tilde{\varepsilon}_2^{\text{c}}+ic\left(\tilde{\varkappa}_1^{\text{c}}+\tilde{\varkappa}_2^{\text{c}}\right)=\frac{1}{Eh}\left\{\underline{T_1^*+T_2^*}+\frac{i}{c}M_1^*\right\}-\varepsilon_2^{\text{д}}-\underline{ic\left(\varkappa_1^{\text{д}}+\varkappa_2^{\text{д}}\right)};\\ \tilde{\varepsilon}_2^{\text{c}}-\tilde{\varepsilon}_1^{\text{c}}=\frac{i}{Ehc}\left(M_1^*-M_2^*\right)-\left(\varepsilon_2^{\text{д}}-\varepsilon_1^{\text{д}}\right);\\ \tilde{\omega}^{\text{c}}=-\frac{i}{Ehc}2H^*-\omega^{\text{д}}.\end{gathered} \tag{1.2}$$

In the future we shall use sufficiently smooth functions for the dislocation displacements and for the functions of the static system. For these functions the following relationships are valid

$$\begin{gathered}\left\{\varkappa_1^{\text{д}},\ \varkappa_2^{\text{д}},\ \tau^{\text{д}}\right\}=O\left\{\frac{\varepsilon_1^{\text{д}}}{R_0},\ \frac{\varepsilon_2^{\text{д}}}{R_0},\ \frac{\omega^{\text{д}}}{R_0}\right\};\\ \left\{T_1^*,\ T_2^*,\ S^*\right\}=O\left\{\frac{M_1^*}{R_0},\ \frac{M_2^*}{R_0},\ \frac{H^*}{R_0}\right\}.\end{gathered} \tag{1.3}$$

Therefore we can drop the small underlined terms in (1.2).

For shells of revolution (section 1, Chapter V) the third equation is written in the form

$$\frac{R_2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\tilde{v}^{\text{п}}}{R_2\sin\theta}\right)+\frac{1}{R_2\sin\theta}\frac{\partial\tilde{u}^{\text{п}}}{\partial\varphi}=-\frac{i}{Ehc}2H^*-\omega^{\text{д}}. \tag{1.4}$$

(п = correction; д = dislocation)

It can be satisfied identically if we let

$$\tilde{u}^{\text{п}}=\tilde{u}^{\text{ч}}-\frac{(R_2\sin\theta)^2}{R_1}\frac{\partial\tilde{\Phi}}{\partial\theta};\ \tilde{v}^{\text{п}}=\tilde{v}^{\text{ч}}+R_2\sin\theta\frac{\partial\tilde{\Phi}}{\partial\varphi}, \tag{1.5}$$

(п = correction; ч = particular). p in this case = particular.

where $\tilde{u}^{\text{р}}$ and $\tilde{v}^{\text{р}}$ is some particular solution of equation (1.4).

It is appropriate to call the function $\tilde{\Phi}$ the potential of complex displacements.

For the second basic function we adopt the following complex combination

$$\tilde{\mu} = \frac{\tilde{u}_\varphi^n}{R_2 \sin\theta} = \frac{\tilde{u}^n \cos\theta + \tilde{w}^n \sin\theta}{R_2 \sin\theta}. \tag{1.6}$$

Then according to (Chapter V, 1.12)

$$\begin{aligned} \tilde{\varepsilon}_1^n &= \frac{1}{R_1}\left(\frac{\partial \tilde{u}^n}{\partial\theta} + \tilde{w}^n\right) = \frac{\tilde{u}^n \cos\theta + \tilde{w}^n \sin\theta}{R_1 \sin\theta} + \frac{\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\tilde{u}^n}{\sin\theta}\right) = \\ &= \frac{R_2}{R_1}\tilde{\mu} - \frac{\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left[\frac{R_2^2 \sin\theta}{R_1}\frac{\partial\tilde{\Phi}}{\partial\vartheta}\right] + \frac{\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\tilde{u}^\varphi}{\sin\theta}\right); \\ \tilde{\varepsilon}_2^n &= \tilde{\mu} + \frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} + \frac{1}{R_2\sin\theta}\frac{\partial\tilde{v}^\varphi}{\partial\varphi}. \end{aligned} \tag{1.7}$$

By using the last equality we obtain another expression for $\tilde{\varepsilon}_1^c$ from the second equation of system (1.2) /160

$$\tilde{\varepsilon}_1^n = \tilde{\mu} + \frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} + \frac{1}{R_2\sin\theta}\frac{\partial\tilde{v}^\varphi}{\partial\varphi} + \frac{i}{Ehc}(M_2^* - M_1^*) + (\varepsilon_2^\text{I} - \varepsilon_1^\text{I}). \tag{1.8}$$

Furthermore, by using the identity

$$\tilde{\vartheta}^n = \operatorname{ctg}\theta\,\tilde{\varepsilon}_1^n - \frac{1}{R_1\sin\theta}\frac{\partial\tilde{u}_\varphi^n}{\partial\theta} \tag{1.9}$$

which can be easily verified, we find, by utilizing (1.8) and (1.6)

$$\begin{aligned} \vartheta^n = \operatorname{ctg}\theta\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} - \frac{R_2}{R_1}\frac{\partial\tilde{\mu}}{\partial\theta} + \frac{\operatorname{ctg}\theta}{R_2\sin\theta}\frac{\partial\tilde{v}^\varphi}{\partial\varphi} + \\ + \operatorname{ctg}\theta\left[\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^\text{I} - \varepsilon_1^\text{I}\right]. \end{aligned} \tag{1.10}$$

From equalities (Chapt V, 1.12) we obtain an expression for the flexural components of the complex deformation

$$\tilde{\varkappa}_1^n = -\frac{1}{R_1}\frac{\partial}{\partial\theta}\left(\frac{R_2}{R_1}\frac{\partial\tilde{\mu}}{\partial\theta}\right) + \frac{\operatorname{ctg}\theta}{R_1}\frac{\partial^3\tilde{\Phi}}{\partial\theta\partial\varphi^2} - \frac{1}{R_1\sin^2\theta}\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} + \tilde{\varkappa}_1^u;$$

$$\tilde{\varkappa}_2^n = -\frac{1}{R_2\sin^2\theta}\frac{\partial^2\tilde{\mu}}{\partial\varphi^2} - \frac{\operatorname{ctg}\theta}{R_1}\frac{\partial\tilde{\mu}}{\partial\theta} - \frac{\operatorname{ctg}\theta}{R_1}\frac{\partial^3\tilde{\Phi}}{\partial\theta\partial\varphi^2} + \frac{1}{R_2\sin^2\theta}\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} + \tilde{\varkappa}_2^u; \tag{1.11}$$

$$\tilde{\tau}^n = -\frac{\cos\theta}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{\partial^2\tilde{\Phi}}{\partial\theta\partial\varphi}\right) + \frac{1}{R_1\sin\theta}\frac{\partial^2\tilde{\Phi}}{\partial\theta\partial\varphi} - \frac{1}{R_1\sin\theta}\frac{\partial^2\tilde{\mu}}{\partial\theta\partial\varphi} + \frac{\cos\theta}{R_1\sin^2\theta}\frac{\partial\tilde{\mu}}{\partial\varphi} + \tilde{\tau}^u,$$

where

$$\tilde{\varkappa}_1^u = \frac{1}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\frac{\partial\tilde{v}^u}{\partial\varphi}\right) + \frac{1}{R_1}\frac{\partial}{\partial\theta}\operatorname{ctg}\theta\left[\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^a - \varepsilon_1^a\right];$$

$$\tilde{\varkappa}_2^u = \frac{\cos\theta}{R_2^2\sin^3\theta}\frac{\partial^2\tilde{u}^u}{\partial\varphi^2} + \frac{1}{R_2^2\sin^3\theta}\frac{\partial\tilde{v}^u}{\partial\varphi} + \frac{\cos^2\theta}{R_2\sin^2\theta}\left[\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^a - \varepsilon_1^a\right]; \tag{1.12}$$

$$\tilde{\tau}^u = \frac{1}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\frac{\partial\tilde{u}^u}{\partial\varphi}\right) + \frac{1}{R_1R_2\sin\theta}\frac{\partial\tilde{u}^u}{\partial\varphi} + \frac{\sin\theta}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\tilde{v}^u}{R_2\sin\theta}\right).$$

By substituting the resulting expressions into the first equation of (1.2) /161 we obtain

$$\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} + \tilde{\mu} + \frac{1}{R_2\sin\theta}\frac{\partial\tilde{v}^u}{\partial\varphi} + ic\left\{-G\left\{\tilde{\mu}\right\} - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2} + \underline{\left[\frac{1}{R_1}\frac{\partial}{\partial\theta}\left(\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\frac{\partial\tilde{v}^u}{\partial\varphi}\right) + \frac{1}{R_2^2\sin^3\theta}\frac{\partial\tilde{v}^u}{\partial\varphi} + \frac{\cos\theta}{R_2^2\sin^3\theta}\frac{\partial^2\tilde{u}^u}{\partial\varphi^2}\right]} + \underline{\frac{1}{R_1}\frac{\partial}{\partial\theta}\operatorname{ctg}\theta\left[\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^a - \varepsilon_1^a\right]} + \underline{\frac{\cos^2\theta}{R_2\sin^2\theta}\left[\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^a - \varepsilon_1^a\right]}\right\} =$$

$$= \frac{i}{Eh}\left(\underline{T_1^* + T_2^*} + \frac{i}{c}M_1^*\right) - \underline{\varepsilon_2^a} - \underline{ic\left(\varkappa_1^a + \varkappa_2^a\right)},$$

where

$$G = \frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{\partial}{\partial\theta}\right) + \frac{1}{R_2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

In view of the qualification made at the beginning of the section, the underlined terms which contain the factor ic are small and may be dropped. This leads us to the first equation of the following system

$$\begin{gathered} -icG\left\{\tilde{\mu}\right\}+\tilde{\mu}+\left[1-ic\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2}=f_1(\theta,\varphi); \\ G\left\{\tilde{\Phi}\right\}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\tilde{\mu}=f_2(\theta,\varphi), \end{gathered} \tag{1.13}$$

where

$$\begin{aligned} f_1(\theta,\varphi)&=-\frac{1}{R_2\sin\theta}\frac{\partial\tilde{v}^{\text{ц}}}{\partial\varphi}+\frac{i}{Ehc}M_1^*-\varepsilon_2^{\text{д}}; \\ f_2(\theta,\varphi)&=\frac{1}{R_1R_2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\tilde{u}^{\text{ц}}}{\sin\theta}\right)-\frac{1}{R_2^2\sin^3\theta}\frac{\partial\tilde{v}^{\text{ц}}}{\partial\varphi}- \\ &-\frac{1}{R_2\sin^2\theta}\left[\frac{i}{Ehc}(M_2^*-M_1^*)+\varepsilon_2^{\text{д}}-\varepsilon_1^{\text{д}}\right]. \end{aligned} \tag{1.14}$$

The second equation is obtained by equating expressions $(1.7)_1$ and (1.8).

By utilizing expressions (Chapter VIII, 8.35) the complex forces are obtained from $\tilde{\Phi}$ and $\tilde{\mu}$ which were established from the preceding system

$$\begin{aligned} \tilde{T}_1&=T_1^*-iEhc\left\{\tilde{\varkappa}_2^{\text{ц}}+\varkappa_2^{\text{д}}\right\}; \\ \tilde{T}_2&=T_2^*-iEhc\left\{\tilde{\varkappa}_1^{\text{ц}}+\varkappa_1^{\text{д}}\right\}; \\ \tilde{S}&=S^*+iEhc\left\{\tilde{\tau}^{\text{ц}}+\tau^{\text{д}}\right\}. \end{aligned} \tag{1.15}$$

Adding the first two of these and utilizing equation $(1.2)_1$ we obtain /162

$$\tilde{T}=T_1^*+T_2^*-iEhc\left(\tilde{\varkappa}_1+\tilde{\varkappa}_2\right)=Eh\left\{\tilde{\varepsilon}_2^{\text{ц}}+\varepsilon_2^{\text{д}}-\frac{i}{Ehc}M_1^*\right\}.$$

Replacing $\tilde{\varepsilon}_2^{\text{c}}$ with its expression given by $(1.7)_2$, we obtain a relationship which associates the introduced functions with the basic complex V. V. Novozhilov function

$$\tilde{T}=Eh\left\{\tilde{\mu}+\frac{\partial^2\tilde{\Phi}}{\partial\varphi^2}+\frac{1}{R^2\sin\theta}\frac{\partial\tilde{v}^{\text{ц}}}{\partial\varphi}-\frac{i}{Ehc}M_1^*+\varepsilon_2^{\text{д}}\right\}. \tag{1.16}$$

Let us now compute the complex boundary quantities. Utilizing the identity $\widetilde{u}_z^c = R_2 \cos\theta\widetilde{\mu} - \frac{\widehat{u}^c}{\sin\theta}$ which follows from (1.6) we obtain from (Chapter V, 1.21 and 1.22) after simple transformations

$$
\begin{aligned}
\widetilde{Q}_\rho &= Q_\rho^* - iEhc\,\varkappa_\rho^1 + iEhc\left\{\frac{1}{R_1\sin\theta}\frac{\partial\widetilde{\mu}}{\partial\theta} + \right.\\
&+ \frac{\cos\theta}{R_2\sin^2\theta}\left(\frac{\partial^2\widetilde{\mu}}{\partial\varphi^2} - \frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2}\right) + \frac{1}{R_1\sin\theta}\frac{\partial^2\widetilde{\Phi}}{\partial\theta\partial\varphi^2} -\\
&- \frac{1}{R_2\sin^2\theta}\left[\frac{1}{R_2\sin\theta}\left(\frac{\partial^2\widetilde{u}^*}{\partial\varphi^2} + \cos\theta\frac{\partial\widetilde{v}^*}{\partial\varphi}\right)\right] +\\
&\left.+ \cos\theta\left(\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^1 - \varepsilon_1^1\right)\right]\Bigg\};\\
\widetilde{Q}_z &= Q_z^* + iEhc\,\varkappa_z^1 + iEhc\left\{\frac{1}{R_2\sin\theta}\left(\frac{\partial^2\widetilde{\mu}}{\partial\varphi^2} - \frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2}\right) - \right.\\
&\left.- \frac{1}{(R_2\sin\theta)^2}\frac{\partial\widetilde{v}^*}{\partial\varphi}\right\};\\
\widetilde{Q}_{12} &= Q_{12}^* + iEhc\,\varkappa_{21}^1 + iEhc\left\{\frac{\cos\theta}{R_2\sin^2\theta}\frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2} + \frac{1}{R_1\sin\theta}\frac{\partial^2\widetilde{\Phi}}{\partial\theta\partial\varphi} - \right.\\
&- \frac{1}{R_1\sin\theta}\frac{\partial^2\widetilde{\mu}}{\partial\theta\partial\varphi} + \frac{\cos\theta}{R_2\sin^2\theta}\frac{\partial\widetilde{\mu}}{\partial\varphi} + \frac{\cos\theta}{R_2^2\sin^2\theta}\frac{\partial^2\widetilde{v}^*}{\partial\varphi^2} -\\
&\left.- \frac{1}{R_2^2\sin^2\theta}\frac{\partial\widetilde{u}^*}{\partial\varphi} + \frac{\cos\theta}{R_2\sin^2\theta}\frac{\partial}{\partial\varphi}\left[\frac{i}{Ehc}(M_2^* - M_1^*) + \varepsilon_2^1 - \varepsilon_1^1\right]\right\};\\
\widetilde{M} &= M_1^* + iEhc\,\varepsilon_2^1 + iEhc\left\{\widetilde{\mu} + \frac{\partial^2\widetilde{\Phi}}{\partial\varphi^2} + \frac{1}{R_1\sin\theta}\frac{\partial\widetilde{v}^*}{\partial\varphi}\right\}.
\end{aligned}
\tag{1.17}
$$

where

$$
\begin{aligned}
&Q_\rho^* = \cos\theta T_1^* + \sin\theta Q_{1n}^*;\\
&Q_z^* = \sin\theta T_1^* - \cos\theta Q_{1n};\quad Q_{12}^* = S^* + \frac{2H^*}{R_2};\\
&Q_{1n}^* = \frac{1}{R_1R_2\sin\theta}\left(\frac{\partial R_2\sin\theta M_1^*}{\partial\theta} + R_1\frac{\partial H^*}{\partial\varphi} - R_1\cos\theta M_2^*\right);
\end{aligned}
\tag{1.18}
$$

$$
\begin{aligned}
&\varkappa_\rho^1 = \cos\theta\varkappa_2^1 + \sin\theta\varkappa_{2n}^1;\\
&\varkappa_z^1 = -\sin\varkappa_2^1 + \cos\theta\varkappa_{2n}^1;\quad \varkappa_{21}^1 = \tau^1 - \frac{\omega^1}{R_2};\\
&\varkappa_{2n}^1 = -\frac{1}{R_1R_2\sin\theta}\left(\frac{\partial R_2\sin\theta\varepsilon_2^1}{\partial\theta} - R_1\frac{\partial\omega^1}{\partial\varphi} - R_1\cos\theta\varepsilon_1^1\right).
\end{aligned}
\tag{1.19}
$$

2. The Method of Separating Variables

Let us use the method of separating variables by adding the following /163 two equalities (Chapter V, 3.2)

$$\tilde{\Phi} = \tilde{\Phi}_{,k} \begin{matrix} \cos k\varphi \\ \sin k\varphi \end{matrix}; \quad \tilde{\mu} = \tilde{\mu}_{,k} \begin{matrix} \cos k\varphi \\ \sin k\varphi \end{matrix}. \tag{2.1}$$

As in section 3 of Chapter V, the upper signs in the function correspond to the first k-th cases while the lower signs pertain to the second cases.

Substituting expressions (Chapter V, 3.2), (2.1) into the relationships of the preceding section and reducing the factors (cos kφ or sin kφ) we obtain the following equalities

$$\tilde{u}^{п}_{,k} = \tilde{u}^{ч}_{,k} - \frac{(R_2 \sin\theta)^2}{R_1} \frac{d\tilde{\Phi}_{,k}}{d\theta}, \quad \tilde{v}^{п}_{,k} = \tilde{v}^{ч}_{,k} \mp kR_2 \sin\theta \tilde{\Phi}_{,k}, \tag{2.2}$$

(In the following equations п = correction; д = dislocation; ч = particular. p in this case = particular.)

where the terms $\tilde{u}^{p}_{,k}$ and $\tilde{v}^{p}_{,k}$ are now a particular solution of equation

$$-\frac{R_2 \sin\theta}{R_1} \frac{d}{d\theta}\left(\frac{\tilde{v}^{ч}_{,k}}{R_2 \sin\theta}\right) \mp \frac{k\tilde{u}^{ч}_{,k}}{R_2 \sin\theta} = -\frac{i}{Ehc} 2H^{*}_{,k} - \omega^{д}_{,k}, \tag{2.3}$$

Furthermore,

$$\begin{aligned}
\tilde{\theta}^{п}_{,k} = {} & -k^2 \operatorname{ctg}\theta \tilde{\Phi}_{,k} - \frac{R_2}{R_1} \frac{d\tilde{\mu}_{,k}}{d\theta} \pm k \frac{\operatorname{ctg}\theta}{R_2 \sin\theta} \tilde{v}^{ч}_{,k} + \\
& + \operatorname{ctg}\theta \left\{ \frac{i}{Ehc} (M^{*}_{2,k} - M^{*}_{1,k}) + \varepsilon^{д}_{2,k} - \varepsilon^{д}_{1,k} \right\}; \\
\tilde{T}_{1,k} = {} & T^{*}_{1,k} - \left\{ k^2 \left(\frac{\operatorname{ctg}\theta}{R_1} \frac{d\tilde{\Phi}_{,k}}{d\theta} - \frac{1}{R_2 \sin^2\theta} \tilde{\Phi}_{,k} + \frac{1}{R_2 \sin^2\theta} \tilde{\mu}_{,k} \right) - \right. \\
& - \frac{\operatorname{ctg}\theta}{R_1} \frac{d\tilde{\mu}_{,k}}{d\theta} - k^2 \frac{\cos\theta}{R_2^2 \sin^2\theta} \tilde{u}^{ч}_{,k} \pm \frac{k}{R_2^2 \sin^2\theta} \tilde{v}^{ч}_{,k} + \\
& \left. + \frac{\cos^2\theta}{R_2 \sin^2\theta} \left[\frac{i}{Ehc} (M^{*}_{2,k} - M^{*}_{1,k}) + \varepsilon^{д}_{2,k} - \varepsilon^{д}_{1,k} \right] + \varkappa^{д}_{,k} \right\};
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\tilde{T}_{2,k} = {} & T^{*}_{2,k} - iEhc \left\{ k^2 \left(-\frac{\operatorname{ctg}\theta}{R_1} \frac{d\tilde{\Phi}_{,k}}{d\theta} + \frac{1}{R_1 \sin^2\theta} \tilde{\Phi}_{,k} \right) - \right. \\
& - \frac{1}{R_1} \frac{d}{d\theta} \left(\frac{R_2}{R_1} \frac{d\tilde{\mu}_{,k}}{d\theta} \right) \pm \frac{k}{R_1} \frac{d}{d\theta} \left(\frac{\operatorname{ctg}\theta}{R_2 \sin\theta} \tilde{v}^{ч}_{,k} \right) + \\
& \left. + \frac{1}{R_1} \frac{d}{d\theta} \operatorname{ctg}\theta \left[\frac{i}{Ehc} (M^{*}_{2,k} - M^{*}_{1,k}) + \varepsilon^{д}_{2,k} - \varepsilon^{д}_{1,k} \right] + \varkappa^{д}_{1,k} \right\};
\end{aligned} \tag{2.5}$$

$$\tilde{S}_{,k} = S^{*}_{,k} + iEhc\left\{\pm k\left[\frac{\cos\theta}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{d\tilde{\Phi}_{,k}}{d\theta}\right)+\right.\right.$$
$$+\frac{1}{R_1\sin\theta}\left(\frac{d\tilde{\mu}_{,k}}{d\theta}-\frac{d\tilde{\Phi}_{,k}}{d\theta}\right)-\frac{\operatorname{ctg}\theta}{R_1\sin\theta}\tilde{\mu}_{,k}-$$
$$-\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{\operatorname{ctg}\theta}{R_2\sin\theta}\tilde{u}^{*}_{,k}\right)-\frac{1}{R_1R_2\sin\theta}\tilde{v}^{*}_{,k}\Big]+$$
$$\left.+\frac{\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{v}^{*}_{,k}}{R_2\sin\theta}\right)+\tilde{\tau}^{*}_{,k}\right\}.$$

The basic complex functions are determined from the systems of equations /64

$$\begin{aligned}-icG_k\left\{\tilde{\mu}_{,k}\right\}+\tilde{\mu}_{,k}-k^2\left[1-i\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\tilde{\Phi}_{,k}&=f_{1,k}(\theta);\\ G_k\left\{\tilde{\Phi}_{,k}\right\}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\tilde{\mu}_{,k}&=f_{2,k}(\theta),\end{aligned}\qquad(2.6)$$

where

$$\begin{aligned}G_k&=\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\right)-\frac{k^2}{R_2\sin^2\theta};\\ f_{1,k}(\theta)&=\mp k\frac{\tilde{v}^{*}_{,k}}{R_2\sin\theta}+\frac{i}{Ehc}M^{*}_{1,k}-s^{x}_{2,k};\\ f_{2,k}(\theta)&=\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{\tilde{u}^{*}_{,k}}{\sin\theta}\right)\mp k\frac{\tilde{v}^{*}_{,k}}{R_2^2\sin^2\theta}-\\ &-\frac{1}{R_2\sin^2\theta}\left[\frac{i}{Ehc}(M^{*}_{2,k}-M^{*}_{1,k})+s^{x}_{1,k}-s^{x}_{2,k}\right].\end{aligned}\qquad(2.7)$$

This system, like the system of equations utilizing complex forces (Chapter V, 3.3) which is characterized by the equality

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}=A=\text{const},$$

for a wide class of shells, may be reduced to two independent equations of the second order.

Finally, relationships (1.17)-(1.19) are written in the form

$$\widetilde{Q}_{\rho,k}=Q^{*}_{\rho,k}-iEhc\,\varkappa^{\lambda}_{\rho,k}+iEhc\left\{\frac{1}{R_1\sin\theta}\frac{d\widetilde{\mu}_{,k}}{d\theta}-\right.$$
$$-k^2\frac{\cos\theta}{R_2\sin^2\theta}\left(\widetilde{\mu}_{,k}-\widetilde{\Phi}_{,k}\right)-\frac{k^2}{R_1\sin\theta}\frac{d\widetilde{\Phi}_{,k}}{d\theta}-\frac{1}{R_2\sin^2\theta}\times$$
$$\times\left[\frac{1}{R_2\sin\theta}\left(-k^2\widetilde{w}^{\varkappa}_{,k}\pm k\cos\theta\widetilde{v}^{\varkappa}_{,k}\right)+\cos\theta\left(\frac{i}{Ehc}\left(M^{*}_{2,k}-M^{*}_{1,k}\right)+\right.\right.$$
$$\left.\left.+\varepsilon^{\lambda}_{2,k}-\varepsilon^{\lambda}_{1,k}\right)\right]\Bigg\};$$
$$\widetilde{Q}_{z,k}=Q^{*}_{z,k}+iEhc\,\varkappa^{\lambda}_{z,k}+iEhc\left\{-\frac{k^2}{R_2\sin\theta}\left(\widetilde{\mu}_{,k}-\widetilde{\Phi}_{,k}\right)\mp\right. \tag{2.8}$$
$$\left.\mp\frac{k}{(R_2\sin\theta)^2}\widetilde{v}^{\varkappa}_{,k}\right\};$$
$$\widetilde{Q}_{12,k}=Q^{*}_{12,k}+iEhc\,\varkappa^{\lambda}_{21,k}+iEhc\left\{\mp\frac{k^3\cos\theta}{R_2\sin^2\theta}\widetilde{\Phi}_{,k}\mp\frac{k}{R_1\sin\theta}\frac{d\widetilde{\Phi}_{,k}}{d\theta}\pm\right.$$
$$\pm\frac{k}{R_1\sin\theta}\frac{d\widetilde{\mu}_{,k}}{d\theta}\mp\frac{k\cos\theta}{R_2\sin^2\theta}\widetilde{\mu}_{,k}-k^2\frac{\cos\theta}{R_2^2\sin^3\theta}\widetilde{v}^{\varkappa}_{,k}\pm\frac{k}{R_2^2\sin^3\theta}\widetilde{u}^{\varkappa}_{,k}\mp$$
$$\left.\mp\frac{k\cos\theta}{R_2\sin^2\theta}\left[\frac{i}{Ehc}\left(M^{*}_{2,k}-M^{*}_{1,k}\right)+\varepsilon^{\lambda}_{2,k}-\varepsilon^{\lambda}_{1,k}\right]\right\};$$
$$\widetilde{M}_{1,k}=M^{*}_{1,k}+iEhc\,\varepsilon^{\lambda}_{2,k}+iEhc\left\{\widetilde{\mu}_{,k}-k^2\widetilde{\Phi}_{,k}\pm\frac{k}{R_1\sin\theta}\widetilde{v}^{\varkappa}_{,k}\right\},$$

where

$$Q^{*}_{\rho,k}=\cos\theta T^{*}_{1,k}+\sin\theta Q^{*}_{1n,k}\qquad Q^{*}_{12,k}=S^{*}_{,k}+\frac{2H^{*}_{,k}}{R_2}$$
$$Q^{*}_{z,k}=\sin\theta T^{*}_{1,k}-\cos\theta Q^{*}_{1n,k}, \tag{2.9}$$
$$Q^{*}_{1n,k}=\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2\sin\theta M^{*}_{1,k}}{d\theta}\pm 2R_1H^{*}_{,k}-R_1\cos\theta M^{*}_{2,k}\right);$$

$$\varkappa^{\lambda}_{\rho,k}=\cos\theta\varkappa^{\lambda}_{2,k}+\sin\theta\varkappa^{\lambda}_{2n,k}$$
$$\varkappa^{\lambda}_{z,k}=-\sin\theta\varkappa^{\lambda}_{2,k}+\cos\theta\varkappa^{\lambda}_{2n,k},\qquad \varkappa^{\lambda}_{21,k}=\tau^{\lambda}_{,k}-\frac{\omega^{\lambda}_{,k}}{R_2} \tag{2.10}$$
$$\varkappa^{\lambda}_{2n,k}=-\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2\sin\theta\varepsilon^{\lambda}_{2,k}}{d\theta}\mp R_1\omega^{\lambda}_{,k}-R_1\cos\theta\varepsilon^{\lambda}_{1,k}\right).$$

3. The Second Symmetric Case (Symmetric Torsion)

In the second symmetric case which pertains to the lower functions in the relationships of the preceding section $\widetilde{v}^{c}_{,0}$ is a particular solution of the equation

$$\frac{R_2 \sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{v}^{\tau}_{,0}}{R_2\sin\theta}\right) = -\frac{i}{Ehc}2H^{*}_{,0} - \omega^{z}_{,0}. \tag{3.1}$$

when k = 0. From this and from $(2.5)_3$ it follows ($\tilde{\Phi}_{,0} = \tilde{\mu}_{,0} = 0$)

$$\tilde{S}_0 = S^{*}_{,0} + \frac{2H^{*}_{,0}}{R_2} + iEhc\left(\tau_{,0} - \frac{\omega^{z}_{,0}}{R_2}\right). \tag{3.2}$$

Separating the real and imaginary parts we find /166

$$S_{,0} = \operatorname{Re}\tilde{S}_{,0} = S^{*}_{,0} + \frac{2H^{*}_{,0}}{R_2}; \tag{3.3}$$

$$H_{,0} = (1-\nu)c\operatorname{Im}\tilde{S}_{,0} = \frac{Eh^3}{12(1+\nu)}\left(\tau^{z}_{,0} - \frac{\omega^{z}_{,0}}{R_2}\right);$$

$$\sigma_{S,0} = \frac{1}{h}\left(S^{*}_{,0} + \frac{2H^{*}_{,0}}{R_2}\right); \quad \sigma_{H,0} = \frac{Eh}{2(1+\nu)}\left(\tau^{z}_{,0} - \frac{\omega^{z}_{,0}}{R_2}\right). \tag{3.4}$$

Furthermore (Chapter V, 1.12),

$$\omega_{,0} = \omega^{z}_{,0} + \frac{R_2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{v^{n}_{,0}}{R_2\sin\theta}\right) = \\ = \frac{2(1+\nu)}{Eh}S_{,0} = \frac{2(1+\nu)}{Eh}\left(S^{*}_{,0} + \frac{2H^{*}_{,0}}{R_2}\right), \tag{3.5}$$

whence

$$v^{n}_{,0} = R_2\sin\theta\left[\frac{2(1+\nu)}{Eh}\int_0^{\theta}\left(S^{*}_{,0} + \frac{2H^{*}_{,0}}{R_2}\right)\frac{R_1\,d\theta}{R_2\sin\theta} - \right. \\ \left. - \int_0^{\theta}\frac{\omega^{z}_{,0}R_1 d\theta}{R_2\sin\theta} + a_2\right]. \tag{3.6}$$

4. The First Symmetric Case

The upper functions with k = 0 in relationships of section 2 pertain to the first symmetric case. Since in this case $\tilde{v}^{c}_{,0} = \tilde{\nu}^{p}_{,0} = S^{*}_{,0} = H^{*}_{,0} = \omega^{d}_{,0} = \tau^{d}_{,0} = 0$, we can assume that $\tilde{u}^{p}_{,0} = 0$. By dropping the sign "0" we obtain, from equation $(2.6)_1$

$$-ic\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{d\tilde{\mu}}{d\theta}\right)+\tilde{\mu}=\frac{i}{Ehc}M_1^*-\varepsilon_2^1. \tag{4.1}$$

If we expand this expression, we obtain an equation which is of the same type as the one obtained by V. V. Novozhilov (Chapter V, 6.4).

The equation for the function ϑ^c, which according to (2.4) is equal to the following expression

$$\tilde{\vartheta}^n=-\frac{R_2}{R_1}\frac{d\tilde{\mu}}{d\theta}+\operatorname{ctg}\theta\left\{\frac{i}{Ehc}(M_2^*-M_1^*)+\varepsilon_2^1-\varepsilon_1^1\right\}. \tag{4.2}$$

is more convenient for us. Let us differentiate equation (4.1) with respect to θ. By eliminating the quantity $\frac{d\tilde{\mu}}{d\theta}$ from this equation by means of (4.2) we obtain

$$\begin{aligned}
&-ic\frac{R_2}{R_1^2}\left\{\frac{d^2\tilde{\vartheta}^n}{d\theta^2}+\left(\frac{R_1\cos\theta}{R_2\sin\theta}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\tilde{\vartheta}^n}{d\theta}+\right.\\
&\left.+\left[-\frac{R_1}{R_2}-\left(\frac{R_1\cos\theta}{R_2\sin\theta}\right)^2\right]\tilde{\vartheta}^n\right\}+\tilde{\vartheta}^n=-\frac{R_2}{R_1}\frac{d}{d\theta}\left[\frac{i}{Ehc}M_1^*-\varepsilon_2^1\right]+\\
&+\operatorname{ctg}\theta\left[\frac{i}{Ehc}(M_2^*-M_1^*)+\varepsilon_2^1-\varepsilon_1^1\right]-\\
&-ic\frac{R_2}{R_1}\frac{d}{d\theta}\left\{\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}R_2\cos\theta\left[\frac{i}{Ehc}(M_2^*-M_1^*)+\varepsilon_2^1-\varepsilon_1^1\right]\right\}.
\end{aligned}$$

Dropping the underlined terms because they are small we arrive at an equation which is analogous to the well-known Meissner-Clark equation (ref. 332) /167

$$\begin{aligned}
&\frac{d^2\tilde{\vartheta}^n}{d\theta^2}+\left(\frac{R_1\cos\theta}{R_2\sin\theta}-\frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\tilde{\vartheta}^n}{d\theta}+\left[-\left(\frac{R_1\cos\theta}{R_2\sin\theta}\right)^2+i\frac{R_1^2}{cR_2}\right]\tilde{\vartheta}^n=\\
&=i\frac{R_1^2}{cR_2}\left\{-R_2\left(\varkappa_{2n}^1+\frac{i}{Ehc}Q_{1n}^*\right)\right\},
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
\varkappa_{2n}^1&=-\frac{1}{R_1R_2\sin\theta}\left[\frac{dR_2\sin\theta\varepsilon_2^1}{d\theta}-R_1\cos\theta\varepsilon_1^1\right];\\
Q_{1n}^*&=\frac{1}{R_1R_2\sin\theta}\left[\frac{dR_2\sin\theta M_1^*}{d\theta}-R_1\cos\theta M_2^*\right].
\end{aligned} \tag{4.4}$$

For the sake of brevity we shall call equation (4.3) the Meissner equation and we shall call $\tilde{\vartheta}^c$ the Meissner function. This is not entirely accurate because as we can see from $(4.11)_1$ the imaginary part $\tilde{\vartheta}^c$ does not determine the cutting force as in Meissner's case, but rather a horizontal force. Associated with this fact is the situation that for toroidal shell (section 4, Chapter XV) equation (4.3) unlike the original Meissner equation does not have singularities in the right part.

Further, by utilizing relationships (4.3) and (4.2) we find

$$\begin{aligned}\tilde{T}_1 &= T_1^* - iEhc\left\{\frac{\cos\theta}{R_2\sin\theta}\tilde{\vartheta}^n + \varkappa_2^*\right\};\\ \tilde{T}_2 &= T_2^* - iEhc\left\{\frac{1}{R_1}\frac{d\tilde{\vartheta}^n}{d\theta} + \varkappa_1^*\right\},\end{aligned} \tag{4.5}$$

from which it follows that

$$\begin{aligned}T_1 &= T_1^* + Ehc\left\{\frac{\cos\theta}{R_2\sin\theta}\operatorname{Im}\tilde{\vartheta}^n\right\};\\ T_2 &= T_2^* + Ehc\left\{\frac{1}{R_1}\operatorname{Im}\frac{d\tilde{\vartheta}^n}{d\theta}\right\};\end{aligned} \tag{4.6}$$

$$\begin{aligned}M_1 &= M_1^* + Ehc^2\left\{\frac{1}{R_1}\operatorname{Re}\frac{d\tilde{\vartheta}^n}{d\theta} + \nu\frac{\cos\theta}{R_2\sin\theta}\operatorname{Re}\tilde{\vartheta}^n\right\};\\ M_2 &= M_2^* + Ehc^2\left\{\frac{\nu}{R_1}\operatorname{Re}\frac{d\tilde{\vartheta}^n}{d\theta} + \frac{\cos\theta}{R_2\sin\theta}\operatorname{Re}\tilde{\vartheta}^n\right\};\\ &\left(M_1^* = Ehc^2\left(\varkappa_1^* + \nu\varkappa_2^*\right),\quad M_2^* = Ehc^2\left(\varkappa_2^* + \nu\varkappa_1^*\right)\right).\end{aligned} \tag{4.7}$$

According to equalities (Chapter V, 3.6)

$$\begin{aligned}\varepsilon_2 &= \varepsilon_2^* + \frac{u_\rho^n}{R_2\sin\theta} = \frac{1}{Eh}\{T_2 - \nu T_1\};\\ \varepsilon_1 &= \varepsilon_1^* - \frac{1}{R_1\sin\theta}\frac{du_z^n}{d\theta} - \operatorname{ctg}\theta\vartheta^n = \frac{1}{Eh}\{T_1 - \nu T_2\}.\end{aligned} \tag{4.8}$$

From this equation by utilizing the relationships (4.6) we find /168

$$\begin{aligned}\vartheta^n &= \operatorname{Re}\tilde{\vartheta}^n;\\ u_\rho^n &= R_2\sin\theta\left\{\varepsilon_2^* - \varepsilon_2^* + c\operatorname{Im}\left(\frac{1}{R_1}\frac{d\tilde{\vartheta}^n}{d\theta} - \nu\frac{\cos\theta}{R_2\sin\theta}\tilde{\vartheta}^n\right)\right\};\end{aligned} \tag{4.9}$$

$$u_z^n = a_1 - \int_{\theta_0}^{\theta} \left\{ \sin\theta \left(\varepsilon_1^* - \varepsilon_1^n\right) - \underline{c\,\mathrm{Im}\left(\frac{\nu}{R_1}\frac{d\tilde{\vartheta}^n}{d\theta} - \frac{\cos\theta}{R_2\sin\theta}\tilde{\vartheta}^n\right)\sin\theta} + \right.$$

$$\left. + \cos\vartheta\,\mathrm{Re}\,\tilde{\vartheta}^n \right\} R_1 d\theta$$

$$\left(\varepsilon_1^* = \frac{T_1^* - \nu T_2^*}{Eh},\quad \varepsilon_2^* = \frac{T_2^* - \nu T_1^*}{Eh}\right)$$

(The underlined terms are usually small and may be dropped.)

Let us consider briefly the problem concerning the boundary quantities associated with the parallel circle $\theta = \theta_0$. If we take into account the dislocation deformations, then according to (Chapter V, 3.9)

$$x_\rho = x_\rho^n + \frac{\vartheta^n}{r};\quad \varepsilon_2 = \varepsilon_2^n + \frac{u_\rho^n}{r},\quad x_z = x_z^n. \tag{4.10}$$

Therefore the complex boundary quantities assume the form

$$\begin{aligned}
\tilde{Q}_\rho &= Q_\rho - iEhc\left(x_\rho^n + \frac{\vartheta^n}{r}\right) = Q_\rho^* - iEhc\,x_\rho^n - iEhc\,\frac{\tilde{\vartheta}^n}{r};\\
\tilde{Q}_z &= Q_z + iEhc\,x_z^n = Q_z^* + iEhc\,x_z^n;\\
\tilde{M}_1 &= M_1 + iEhc\left(\varepsilon_2^n + \frac{u_\rho^n}{r}\right) = M_1^* + iEhc\,\varepsilon_2^n + iEhc\,\frac{\tilde{u}_\rho^n}{r}.
\end{aligned} \tag{4.11}$$

We obtain the following expressions for these quantities from (2.8) /169

$$\begin{aligned}
\tilde{Q}_\rho &= Q_\rho^* - iEhc\,x_\rho^n + iEhc\left\{\frac{1}{R_1\sin\theta}\frac{d\tilde{\mu}}{d\theta} - \right.\\
&\left. - \frac{\mathrm{ctg}\,\theta}{R_2\sin\theta}\left[\frac{i}{Ehc}\left(M_2^* - M_1^*\right) + \varepsilon_2^n - \varepsilon_1^n\right]\right\};\\
\tilde{Q}_z &= Q_z^* + iEhc\,x_z^n;\\
\tilde{M}_1 &= M_1^* + iEhc\,\varepsilon_2^n + iEhc\left\{\tilde{\mu}\right\}.
\end{aligned} \tag{4.12}$$

Eliminating the quantity $-\frac{R_2}{R_1}\frac{d\tilde{\mu}}{d\theta}$ from (4.1) by means of (4.2) we obtain, after dropping the small terms

$$\tilde{\mu} = -\frac{ic}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(R_2\sin\theta\tilde{\vartheta}^n\right) + \frac{i}{Ehc}M_1^* - \varepsilon_2^1. \tag{4.13}$$

The resulting expression as well as (4.2) make it possible for us to write the complex boundary quantities in terms of the basic complex function

$$\begin{aligned}
&\tilde{Q}_\rho \equiv Q_\rho - iEhc\varkappa_\rho = Q_\rho^* - iEhc\varkappa_\rho^2 - iEhc\frac{\tilde{\xi}^n}{R_2\sin\theta};\\
&\tilde{Q}_z \equiv Q_z + iEhc\varkappa_z = Q_z^* + iEhc\varkappa_z^2;\\
&\tilde{M}_1 \equiv M_1 + iEhc\varepsilon_2 = Ehc^2\frac{1}{R_1R_2\sin\theta}\frac{d\left(R_2\sin\theta\tilde{\vartheta}^n\right)}{d\theta}.
\end{aligned} \tag{4.14}$$

We recall at this point that

$$\begin{aligned}
&Q_\rho^* = \cos\theta T_1^* + \sin\theta Q_{1n}^*,\quad Q_z^* = \sin\theta T_1^* - \cos\theta Q_{1n}^*;\\
&\varkappa_\rho^2 = \cos\theta\varkappa_2^1 + \sin\theta\varkappa_{2n}^2,\quad \varkappa_z^2 = -\sin\theta\varkappa_2^1 + \cos\theta\varkappa_{2n}^2;\\
&Q_{1n}^* = \frac{1}{R_1R_2\sin\theta}\left\{\frac{dR_2\sin\theta M_1^*}{d\theta} - R_1\cos\theta M_2^*\right\};\\
&\varkappa_{2n}^2 = -\frac{1}{R_1R_2\sin\theta}\left\{\frac{dR_2\sin\theta\varepsilon_2^1}{d\theta} - R_1\cos\theta\varepsilon_1^2\right\}.
\end{aligned} \tag{4.15}$$

From the resulting equalities it follows that the conditions at the joints of two shells of the same thickness and with equal elastic constants may be formulated in a complex form

$$\begin{aligned}
&\left\{Q_\rho^* - iEhc\varkappa_\rho^2 - iEhc\frac{\tilde{\vartheta}^n}{R_2\sin\theta}\right\}^{\mathrm{I}} = \{\ldots\}^{\mathrm{II}};\\
&\left\{\frac{1}{R_1R_2\sin\theta}\frac{d\left(R_2\sin\theta\tilde{\vartheta}^n\right)}{d\theta}\right\}^{\mathrm{I}} = \{\ldots\}^{\mathrm{II}}\ (ds_1 = R_1d\theta);\\
&\{Q_z^*\}^{\mathrm{I}} = \{Q_z^*\}^{\mathrm{II}},\quad \{\varkappa_z^1\}^{\mathrm{I}} = \{\varkappa_z^2\}^{\mathrm{II}}.
\end{aligned} \tag{4.16}$$

The first two conditions make it possible to associate the complex constants of the solutions which are joined. This places natural limitations on functions of the static system and on the dislocation displacements. Thus the static systems of both joined shells must produce the same vertical forces along the common parallel ring. The latter relationship on the other hand is one of the six conditions (associated with the first symmetric case) which provide for the equivalence of displacement discontinuities and discontinuities of the angle of rotation along the junction. It is recommended that the reader

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utilize equalities (4.15) and (Chapter X, 2.10) to become convinced that the latter equality is equivalent to the following

$$\{\Omega_z^1\}^{\mathrm{I}} = \{\Omega_z^1\}^{\mathrm{II}}. \tag{4.17}$$

5. Antisymmetric Case

We recall that the antisymmetric case corresponds to the value k = 1. In this case $\tilde{u}^{p}_{,1}$ and $\tilde{v}^{p}_{,1}$ are certain solutions of equation

$$\frac{R_2 \sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{v}^{\mathrm{II}}_{,1}}{R_2\sin\theta}\right) \mp \frac{\tilde{u}^{\mathrm{II}}_{,1}}{R_2\sin\theta} = -\frac{i}{Ehc}2H^*_{,1} - \omega^{\mathrm{I}}_{,1}. \tag{5.1}$$

As a solution of this equation it is convenient to take

$$\tilde{v}^{\mathrm{II}}_{,1} = 0,\ \tilde{u}^{\mathrm{II}}_{,1} = \pm R_2 \sin\theta\left(\frac{i}{Ehc}2H^*_{,1} + \omega^{\mathrm{I}}_{,1}\right). \tag{5.2}$$

then

$$\begin{aligned} \tilde{u}^{\mathrm{II}}_{,1} &= \pm R_2 \sin\theta\left(\frac{i}{Ehc}2H^*_{,1} + \omega^{\mathrm{I}}_{,1}\right) - \frac{(R_2\sin\theta)^2}{R_1}\frac{d\tilde{\Phi}_{,1}}{d\theta}; \\ \tilde{v}^{\mathrm{II}}_{,1} &= \mp R_2 \sin\theta\tilde{\Phi}_{,1} \end{aligned} \tag{5.3}$$

in the system of equations (2.6) assumes the form

$$\begin{aligned} &-icG_1\left\{\tilde{\mu}_{,1}\right\} + \tilde{\mu}_{,1} - \left[1 - ic\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\tilde{\Phi}_{,1} = f_{1,1}(\theta); \\ &G_1\left\{\tilde{\Phi}_{,1}\right\} - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\tilde{\mu}_{,1} = f_{2,1}(\theta), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} f_{1,1}(\theta) &= \frac{i}{Ehc}M^*_{1,1} - e^{\mathrm{I}}_{2,1}; \\ f_{2,1}(\theta) &= \frac{i}{Ehc}\left\{\frac{1}{R_2\sin^2\theta}\left(M^*_{1,1} - M^*_{2,1}\right) \pm \frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(R_2 2H^*_{,1}\right)\right\} - \\ &\quad - \left\{\frac{1}{R_2\sin^2\theta}\left(e^{\mathrm{I}}_{2,1} - e^{\mathrm{I}}_{1,1}\right) \mp \frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(R_2\omega^{\mathrm{I}}_{,1}\right)\right\}; \end{aligned} \tag{5.5}$$

$$G_1\{\ldots\}=\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\right)-\frac{1}{R_2\sin^2\theta}. \tag{5.6}$$

Furthermore it follows from $(5.3)_2$ that

$$\tilde{\Phi}_{,1}=\mp\frac{\tilde{v}^n_{,1}}{R_2\sin\theta}. \tag{5.7}$$

Let us introduce the function $\tilde{\gamma}$ which is analogous to the one used in section 9, Chapter V /171

$$\tilde{\gamma}=\tilde{\mu}_{,1}-\tilde{\Phi}_{,1}=\frac{\cos\theta\tilde{u}^n_{,1}+\sin\theta\tilde{w}^n_{,1}\pm\tilde{v}^n_{,1}}{R_2\sin\theta}=\tilde{\varepsilon}_{2,1}-\varepsilon^*_{2,1}. \tag{5.8}$$

Multiplying the second equation in (5.4) by ic and adding it to the first, we obtain a differential equation for the introduced function

$$\begin{aligned}-icG_1\{\tilde{\gamma}\}+\tilde{\gamma}-ic\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\tilde{\gamma}=\\ =f_{1,1}+icf_{2,1}\approx\frac{i}{Ehc}M^*_{1,1}-\varepsilon^*_{2,1},\end{aligned} \tag{5.9}$$

which is of the same type as the V. V. Novozhilov equation (Chapter V, 9.25). In place of $\tilde{\gamma}$ it is convenient to introduce the following function which is familiar from section 9, Chapter V

$$\tilde{\chi}=\tilde{\vartheta}^n_{,1}+\frac{\tilde{u}^n_{z,1}}{R_2\sin\theta}\left(\tilde{u}^n_{z,1}=-\sin\theta\tilde{u}^n_{,1}+\cos\theta\tilde{w}^n_{,1}\right). \tag{5.10}$$

First of all, it follows from (2.4), (1.6) and (5.3) that

$$\begin{aligned}\tilde{\vartheta}^n_{,1}=-\operatorname{ctg}\theta\tilde{\Phi}_{,1}-\frac{R_2}{R_1}\frac{d\tilde{\mu}_{,1}}{d\theta}+\\ +\operatorname{ctg}\theta\left\{\frac{i}{Ehc}(M^*_{2,1}-M^*_{1,1})+\varepsilon^*_{2,1}-\varepsilon^*_{1,1}\right\};\\ \tilde{u}^n_{z,1}=-\sin\theta\tilde{u}^n_{,1}+\cos\theta\tilde{w}^n_{,1}=\operatorname{ctg}\theta\tilde{u}^n_{\varphi}-\frac{1}{\sin\theta}\tilde{u}^n_{,1}=\\ =R_2\cos\theta\tilde{\mu}_{,1}+\frac{R_2^2\sin\theta}{R_1}\frac{d\tilde{\Phi}_{,1}}{d\theta}\mp R_2\left(\frac{i}{Ehc}2H^*_{,1}+\omega^*_{,1}\right).\end{aligned} \tag{5.11}$$

From this we obtain

$$\tilde{\chi} = -\frac{R_2^2 \sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{\gamma}}{R_2\sin\theta}\right) + \frac{i}{Ehc}\left[\operatorname{ctg}\theta\left(M^*_{2,1} - M^*_{1,1}\right) \mp \right.$$
$$\left. \mp \frac{2H^*_{,1}}{\sin\theta}\right] + \left[\operatorname{ctg}\theta\left(\varepsilon^x_{2,1} - \varepsilon^x_{1,1}\right) \mp \frac{\omega^x_{,1}}{\sin\theta}\right]. \qquad (5.12)$$

Let us expand equation (5.9)

$$-ic\left\{\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left(\frac{R_2^2\sin\theta}{R_1}\frac{d\tilde{\gamma}}{d\theta}\right) + \left(\frac{1}{R_1} - \frac{2}{R_2}\right)\frac{1}{\sin^2\theta}\tilde{\gamma}\right\} + \tilde{\gamma} =$$
$$= \frac{i}{Ehc}M^*_{1,1} - \varepsilon^x_{2,1}.$$

If we represent the first term in the form

$$\frac{1}{R_1R_2\sin\theta}\frac{d}{d\theta}\left\{(R_2\sin\theta)^2 \cdot \frac{1}{\sin\theta}\cdot\frac{1}{R_1}\frac{d\tilde{\gamma}}{d\theta}\right\}$$

and differentiate it by considering it to be a product, then we obtain the following form for the above equation /172

$$-ic\left\{\frac{1}{R_1}\frac{d}{d\theta}\left[\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{\gamma}}{R_2\sin\theta}\right)\right] + \frac{2\cos\theta}{R_2\sin\theta}\left[\frac{R_2^2\sin\theta}{R_1}\times\right.\right.$$
$$\left.\left.\times\frac{d}{d\theta}\left(\frac{\tilde{\gamma}}{R_2\sin\theta}\right)\right] - \underline{\underline{\frac{2}{R_2}\tilde{\gamma}}}\right\} + \tilde{\gamma} = \frac{i}{Ehc}M^*_{1,1} - \varepsilon^x_{2,1}. \qquad (5.13)$$

The term which is underlined twice is substantially less than the one following it and may be dropped without impairing the accuracy common to the theory of thin shells. Dropping this term and performing the operation $\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{}{R_2\sin\theta}\right)$, in the left and right sides of the equation, we obtain, after usual simplifications and after replacing $\tilde{\gamma}$ with expression (5.12), the basic complex equation for antisymmetric cases

$$\frac{d^2\tilde{\chi}}{d\theta^2} + \left(\frac{R_1\cos\theta}{R_2\sin\theta} - \frac{1}{R_1}\frac{dR_1}{d\theta}\right)\frac{d\tilde{\chi}}{d\theta} + \left[-4\left(\frac{R_1\cos\theta}{R_2\sin\theta}\right)^2 + i\frac{R_1^2}{cR_2}\right]\tilde{\chi} =$$
$$= i\frac{R_1^2}{cR_2}\left\{\chi^x + \frac{i}{Ehc}\chi^*\right\}, \qquad (5.14)$$

where

$$\chi^* = -\left(\frac{R_2}{R_1}\frac{dM^*_{1,1}}{d\theta} \pm \frac{2H^*_{,1}}{\sin\theta} - \operatorname{ctg}\theta M^*_{2,1}\right);$$
$$\chi^{\Lambda} = \frac{R_2}{R_1}\frac{d\varepsilon^{\Lambda}_{2,1}}{d\theta} \mp \frac{\omega^{\Lambda}_{,1}}{\sin\theta} - \operatorname{ctg}\theta\varepsilon^{\Lambda}_{1,1}. \tag{5.15}$$

Let us express all of the quantities of interest to us in terms of the basic complex function $\tilde{\chi}$. First of all, by using (5.12) we obtain the approximate expression for $\tilde{\gamma}$ from equality (5.13)

$$\tilde{\gamma} = -ic\left\{\frac{1}{R_1}\frac{d\tilde{\chi}}{d\theta} + 2\frac{\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\} + \frac{i}{Ehc}M^*_{1,1} - \varepsilon^{\Lambda}_{2,1}. \tag{5.16}$$

Furthermore, by assuming that k = 1 in $(2.5)_1$, we arrive at the following expression for $\tilde{T}_{1,1}$

$$\tilde{T}_{1,1} = T^*_{1,1} - iEhc\left\{-\frac{\operatorname{ctg}\theta}{R_1}\frac{d\tilde{\gamma}}{d\theta} + \frac{1}{R_2\sin^2\theta}\tilde{\gamma} \mp \right.$$
$$\mp \frac{\cos\theta}{R_2\sin^2\theta}\left(\frac{i}{Ehc}2H^*_{,1} + \omega^{\Lambda}_{,1}\right) + \frac{\cos^2\theta}{R_2\sin^2\theta}\times$$
$$\left.\times\left[\frac{i}{Ehc}(M^*_{2,1} - M^*_{1,1}) + \varepsilon^{\Lambda}_{2,1} - \varepsilon^{\Lambda}_{1,1}\right] + \varkappa^{\Lambda}_{2,1}\right\}. \tag{5.17}$$

It is easy to verify that

$$-\frac{\operatorname{ctg}\theta}{R_1}\frac{d\tilde{\gamma}}{d\theta} + \frac{1}{R_2\sin^2\theta}\tilde{\gamma} = -\frac{\cos\theta}{R_2\sin\theta}\frac{R_2^2\sin\theta}{R_1}\frac{d}{d\theta}\left(\frac{\tilde{\gamma}}{R_2\sin\theta}\right) + \frac{1}{R_2}\tilde{\gamma}.$$

Therefore, by replacing the first term with expression (5.12) and a second /173 with (5.16), we obtain the first of the following expressions after dropping the small terms

$$\tilde{T}_{1,1} = T^*_{1,1} + \frac{M^*_{1,1}}{R_2} - iEhc\left(\varkappa^{\Lambda}_{2,1} - \frac{\varepsilon^{\Lambda}_{2,1}}{R_2}\right) - iEhc\left\{\frac{\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\};$$
$$\tilde{T}_{2,1} = T^*_{2,1} + \frac{M^*_{2,1}}{R_2} - iEhc\left(\varkappa^{\Lambda}_{1,1} - \frac{\varepsilon^{\Lambda}_{1,1}}{R_2}\right) - \tag{5.18}$$
$$-iEhc\left\{\frac{1}{R_1}\frac{d\tilde{\gamma}}{d\theta} + \frac{\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\};$$

$$\tilde{S}_{,1}=S^{*}_{,1}+\frac{2H^{*}_{,1}}{R_2}+iEhc\left(\varepsilon^{\pi}_{,1}-\frac{\omega^{\pi}_{,1}}{R_2}\right)\mp iEhc\left\{\frac{1}{R_2\sin\theta}\tilde{\gamma}\right\}.$$

The second and third expressions are obtained by similar transformations (by utilizing equation $(5.4)_2$). We do not carry out the derivation (hoping that the reader will use the identities

$$\varepsilon_{1,1}-\varepsilon^{\pi}_{1,1}=\frac{1}{R_1\cos\theta}\frac{du^{\pi}_{\rho,1}}{d\theta}+\operatorname{tg}\theta\vartheta^{\pi}_{,1},$$
$$\varepsilon_{1,1}-\varepsilon^{\pi}_{1,1}=-\frac{R_2}{R_1}\frac{d}{d\theta}\left(\frac{u^{\pi}_{z,1}}{R_2\sin\theta}\right)-\operatorname{ctg}\theta\operatorname{Re}\tilde{\gamma}$$

and carry out the derivation himself), but we present the expressions for the angle of rotation and for the displacements

$$\vartheta^{\pi}_{,1}=\operatorname{Re}\tilde{\gamma}-\frac{u^{\pi}_{z,1}}{R_2\sin\theta}; \tag{5.19}$$

$$u^{\pi}_{z,1}=R_2\sin\theta\left\{\int_{\theta_0}^{\theta}\left\{-\frac{\cos\theta}{R_2\sin\theta}\operatorname{Re}\tilde{\chi}+\underline{\frac{cv}{R_1R_2}\operatorname{Im}\frac{d\tilde{\chi}}{d\theta}}+\frac{\varepsilon^{\pi}_{1,1}}{R_2}-\right.\right.$$
$$\left.\left.-\frac{1}{EhR_2}\left[T^{*}_{1,1}+\frac{M^{*}_{1,1}}{R_2}-\nu\left(T^{*}_{2,1}+\frac{M^{*}_{2,1}}{R_2}\right)\right]\right\}R_1d\theta+\begin{matrix}a_5\\a_6\end{matrix}\right\}; \tag{5.20}$$

$$u^{\pi}_{\rho,1}=\int_{\theta_0}^{\theta}\left\{-\sin\theta\vartheta^{\pi}_{,1}+\cos\theta\left(\varepsilon_{1,1}-\varepsilon^{\pi}_{1,1}\right)\right\}R_1d\theta+\begin{matrix}a_4\\a_6\end{matrix}, \tag{5.21}$$

$$\pm w^{\pi}_{,1}=R_2\sin\theta\operatorname{Re}\tilde{\gamma}-u^{\pi}_{\rho,1} \tag{5.22}$$

(The underlined term is usually small and may be dropped.) However, the cofactor Re $\tilde{\gamma}$ which is contained in the expression for $v^{c}_{,1}$ is obtained according to (5.16), from equality

$$\operatorname{Re}\tilde{\gamma}=c\operatorname{Im}\left\{\frac{1}{R_1}\frac{d\tilde{\chi}}{d\theta}+2\frac{\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\}-\varepsilon^{\pi}_{2,1}. \tag{5.23}$$

Further, by separating the real and imaginary parts of the complex forces /174 we obtain the following expressions for the forces and moments

$$T_{1,1}=T^{*}_{1,1}+\frac{M^{*}_{1,1}}{R_2}+Ehc\left\{\frac{\cos\theta}{R_2\sin\theta}\operatorname{Im}\tilde{\chi}\right\};$$
$$T_{2,1}=T^{*}_{2,1}+\frac{M^{*}_{2,1}}{R_2}+Ehc\left\{\frac{1}{R_1}\operatorname{Im}\frac{d\tilde{\chi}}{d\theta}+\frac{\cos\theta}{R_2\sin\theta}\operatorname{Im}\tilde{\chi}\right\}; \tag{5.24}$$

$$S_{,1}=S^{*}_{,1}+\frac{2H^{*}_{,1}}{R_2}\pm Ehc\left\{\frac{1}{R_2\sin\theta}\operatorname{Im}\tilde{\chi}\right\};$$

$$M_{1,1}=M'^{\pi}_{1,1}+Ehc^{3}\left\{\frac{1}{R_2}\operatorname{Re}\frac{d\tilde{\chi}}{d\theta}+(1+\nu)\frac{\cos\theta}{R_2\sin\theta}\operatorname{Re}\tilde{\chi}\right\};$$

$$M_{2,1}=M'^{\pi}_{2,1}+Ehc^{3}\left\{\frac{\nu}{R_1}\operatorname{Re}\frac{d\tilde{\chi}}{d\theta}+(1+\nu)\frac{\cos\theta}{R_2\sin\theta}\operatorname{Re}\tilde{\chi}\right\};$$

$$H_{,1}=H'^{\pi}_{,1}+Ehc^{3}\left\{(1-\nu)\frac{1}{R_2\sin\theta}\operatorname{Re}\tilde{\chi}\right\},$$

where

$$M'^{\pi}_{1,1}=Ehc^{2}\left[\varkappa^{\pi}_{1,1}-\frac{\varepsilon^{\pi}_{1,1}}{R_2}+\nu\left(\varkappa^{\pi}_{2,1}-\frac{\varepsilon^{\pi}_{2,1}}{R_2}\right)\right];$$

$$M'^{\pi}_{2,1}=Ehc^{2}\left[\varkappa^{\pi}_{2,1}-\frac{\varepsilon^{\pi}_{2,1}}{R_2}+\nu\left(\varkappa^{\pi}_{1,1}-\frac{\varepsilon^{\pi}_{1,1}}{R_2}\right)\right];$$

$$H'^{\pi}_{,1}=Ehc^{2}\left[(1-\nu)\left(\tau^{\pi}_{,1}-\frac{\omega^{\pi}_{,1}}{R_2}\right)\right]$$

$$\left(Ehc^{2}=D=\frac{Eh^{3}}{12(1-\nu^{2})}\right).$$

Adding the first two equalities of (5.18) we obtain a relationship which associates the introduced basic complex function of the antisymmetric case with the V. V. Novozhilov function

$$\tilde{T}_{,1}=T^{*}_{,1}+\frac{M^{*}_{1,1}+M^{*}_{2,1}}{R_2}-iEhc\left(\varkappa^{\pi}_{1,1}+\varkappa^{\pi}_{2,1}-\frac{\varepsilon^{\pi}_{1,1}+\varepsilon^{\pi}_{2,1}}{R_2}\right)-$$
$$-iEhc\left\{\frac{1}{R_1}\frac{d\tilde{\chi}}{d\theta}+2\frac{\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\}. \qquad (5.25)$$

Dropping the simple intermediate steps we present the following approximate expressions for the complex boundary quantities /175

$$\tilde{Q}_{z,1}=\left(Q^{*}_{z,1}+\frac{M^{*}_{1,1}}{R_2\sin\theta}\right)+iEhc\left(\varkappa^{\pi}_{z,1}+\frac{\varepsilon^{\pi}_{2,1}}{R_2\sin\theta}\right)-$$
$$-\frac{Ehc^{2}}{R_1R_2\sin\theta}\left\{\frac{d\tilde{\chi}}{d\theta}+2\frac{R_1\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\}; \qquad (5.26)$$

$$\widetilde{Q}_{\rho,1}=Q^*_{\rho,1}-iEhc\varkappa^{\lambda}_{\rho,1}-iEhc\left\{\frac{\tilde{\chi}}{R_2\sin\theta}\right\};$$

$$\widetilde{Q}_{12,1}=\widetilde{S}_{,1}=S^*_{,1}+\frac{2H^*_{,1}}{R_2}+iEhc\left(\tau^{\lambda}_{,1}-\frac{\omega^{\lambda}_{,1}}{R_2}\right)\mp iEhc\left\{\frac{1}{R_2\sin\theta}\tilde{\chi}\right\};$$

$$\widetilde{M}_{1,1}=Ehc^2\left\{\frac{1}{R_1}\frac{d\tilde{\chi}}{d\theta}+2\frac{\cos\theta}{R_2\sin\theta}\tilde{\chi}\right\}.$$

Here

$$\begin{gathered}
Q^*_{\rho,1}=\cos\theta T^*_{1,1}+\sin\theta Q^*_{1n,1};\quad Q^*_{z,1}=\sin\theta T^*_{1,1}-\cos\theta Q^*_{1n,1};\\
\varkappa^{\lambda}_{\rho,1}=\cos\theta\varkappa^{\lambda}_{2,1}+\sin\theta\varkappa^{\lambda}_{2n,1};\quad \varkappa^{\lambda}_{z,1}=-\sin\theta\varkappa^{\lambda}_{2,1}+\cos\theta\varkappa^{\lambda}_{2n,1};\\
Q^*_{1n,1}=\frac{1}{R_1R_2\sin\theta}\left(\frac{d\left(R_2\sin\theta M^*_{1,1}\right)}{d\theta}\pm R_1 2H^*_{,1}-R_1\cos\theta M^*_{2,1}\right);\\
\varkappa^{\lambda}_{2n,1}=-\frac{1}{R_1R_2\sin\theta}\left(\frac{dR_2\sin\theta\varepsilon^{\lambda}_{2,1}}{d\theta}\mp R_1\omega^{\lambda}_{,1}-R_1\cos\theta\varepsilon^{\lambda}_{1,1}\right).
\end{gathered}\tag{5.27}$$

From the relationships presented above it follows that of the four complex boundary quantities (5.26) only two are substantially different, since we have the following easily verified relationship

$$\begin{gathered}
\widetilde{Q}_{\rho,1}\mp\widetilde{Q}_{12,1}=Q^*_{\rho,1}\mp\left(S^*_{,1}+\frac{2H^*_{,1}}{R_2}\right)-\\
-iEhc\left[\varkappa^{\lambda}_{\rho,1}\pm\left(\tau^{\lambda}_{,1}-\frac{\omega^{\lambda}_{,1}}{R_2}\right)\right];\\
\widetilde{M}_{1,1}+r\widetilde{Q}_{z,1}=M^*_{1,1}+rQ^*_{z,1}+iEhc\left(r\varkappa^{\lambda}_{z,1}+\varepsilon^{\lambda}_{2,1}\right)\\
(r=R_2\sin\theta).
\end{gathered}\tag{5.28}$$

The last relationships place definite limitations on the selection of the functions for the static system and of the dislocation displacements. We can show by using relationships (Chapter V, 9.22), that their real parts determine the requirement for selecting the functions of the static system in such a way that the corresponding sheer forces and bending moments are equal along the common parallel ring. With regard to the imaginary parts of these /176
equalities, they are satisfied because the discontinuities in the displacements and in the angles of rotation corresponding to the antisymmetric cases are equal along the junction. It is recommended that the reader use equations (5.27) and (Chapter X, 2.10) to become convinced that the dislocation parameters U^1_x, U^1_y, Ω^1_x, Ω^1_y must be continuous along the junction.

To determine the complex constants we can use any two of the complex boundary quantities, for example the quantities $\widetilde{S}_{,1}$ and $\widetilde{Q}_{z,1}$. In the latter case the conditions at the joints assume the form

$$\left\{S^*_{\cdot 1}+\frac{2H^*_{\cdot 1}}{R_2}+iEhc\left(\tau^2_{\cdot 1}-\frac{\omega^{\text{II}}_{\cdot 1}}{R_2}\right)\mp iEhc\,\frac{1}{R_2\sin\theta}\tilde{\chi}\right\}^{\text{I}}=\{\ldots\}^{\text{II}};$$

$$\left\{Q^*_{z,1}+\frac{M^*_{1,1}}{R_2\sin\theta}+iEhc\left(\chi^{\text{II}}_{z,1}+\frac{e^{\text{II}}_{2,1}}{R_2\sin\theta}\right)-\right.$$

$$\left.-\frac{Ehc^2}{R_1R_2\sin\theta}\left(\frac{d\tilde{\chi}}{d\theta}+2\frac{R_1\cos\theta}{R_2\sin\theta}\tilde{\chi}\right)\right\}^{\text{I}}=\{\ldots\}^{\text{II}}. \tag{5.29}$$

During smooth joining of shells ($\theta_0^{\text{I}}=\theta_0^{\text{II}}=\theta_0$) it is usually easy to select the functions of the static system and the dislocation displacements in such a way that the equalities

$$\left\{S^*_{\cdot 1}+\frac{2H^*_{\cdot 1}}{R_2}\right\}^{\text{I}}=\{\ldots\}^{\text{II}},\quad\left\{Q^*_{z,1}+\frac{M^*_{1,1}}{R_2}\right\}^{\text{I}}=\{\ldots\}^{\text{II}};$$

$$\left\{\tau^{\text{II}}_{\cdot 1}-\frac{\omega^{\text{II}}_{\cdot 1}}{R_2}\right\}^{\text{I}}=\{\ldots\}^{\text{II}},\quad\left\{\chi^{\text{II}}_{z,1}+\frac{e^{\text{II}}_{2,1}}{R_2\sin\theta}\right\}^{\text{I}}=\{\ldots\}^{\text{II}}. \tag{5.30}$$

are satisfied.

Then the conditions at the joints (5.29) are replaced by simpler conditions

$$\{\tilde{\chi}\}^{\text{I}}=\{\ldots\}^{\text{II}},\quad\left\{\frac{d\tilde{\chi}}{ds_1}\right\}^{\text{I}}=\{\ldots\}^{\text{II}}\qquad(ds_1=R_1d\theta). \tag{5.31}$$

Comparing the resulting relationships with those presented in the preceding section for the symmetric case we see that they are quite analogous. The results presented above in ref. 111 are generalized for the case of shells of revolution with thickness varying along the meridian.

6. Cylindrical Shells

For cylindrical shells of the general form (section 1, Chapter VI) (fig. 23)

$$A_1=A_2=r_0,\quad\alpha_1=\frac{x}{r_0}=\xi,\ \alpha_2=\frac{y}{r_0}=\eta,\ \frac{1}{R_1}=0,\ R_2=r_0\rho(\eta).$$

If we do not consider the dislocation displacement and drop the insignificant term $\frac{1}{Eh} T^*$ in system (0.1), the latter will be written in the following form /177

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial \xi} - i\frac{c}{r_0}\left[\Delta'\tilde{w} - \frac{\partial}{\partial \eta}\left(\frac{\tilde{v}}{\rho}\right)\right] = i\frac{r_0}{Ehc} M_2^*;\\
&\frac{\partial \tilde{u}}{\partial \xi} - \frac{\partial \tilde{v}}{\partial \eta} - \frac{\tilde{w}}{\rho} = i\frac{r_0}{Ehc}(M_2^* - M_1^*);\\
&\frac{\partial \tilde{u}}{\partial \eta} + \frac{\partial \tilde{v}}{\partial \xi} = -i\frac{r_0}{Ehc} 2H^*\\
&\qquad\qquad \left(\Delta' = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right).
\end{aligned} \tag{6.1}$$

We introduce the potential of complex displacements by means of equalities

$$\tilde{v} = \tilde{v}^{\text{II}} - \frac{\partial \tilde{\Phi}}{\partial \eta}, \quad \tilde{u} = \tilde{u}^{\text{II}} + \frac{\partial \tilde{\Phi}}{\partial \xi}, \tag{6.2}$$

where u^p and v^p is a particular solution for the last equation of (6.1). From the second equation it follows that

$$\begin{aligned}
\tilde{w} = \rho\Big\{&\Delta'\tilde{\Phi} + \frac{\partial \tilde{u}^{\text{II}}}{\partial \xi} -\\
&- \frac{\partial \tilde{v}^{\text{II}}}{\partial \eta} - i\frac{r_0}{Ehc}(M_2^* - M_1^*)\Big\}.
\end{aligned} \tag{6.3}$$

By substituting expressions (6.2) into the first equation of (6.1) we obtain the equation

$$\frac{\partial \tilde{u}^{\text{II}}}{\partial \xi} + \frac{\partial^2 \tilde{\Phi}}{\partial \xi^2} - i\frac{c}{r_0}\left\{\Delta'\tilde{w} + \frac{\partial}{\partial \eta}\frac{1}{\rho}\frac{\partial \tilde{\Phi}}{\partial \eta} - \frac{\partial}{\partial \eta}\left(\frac{\tilde{v}^{\text{II}}}{\rho}\right)\right\} = i\frac{r_0}{Ehc} M_2^*, \tag{6.4}$$

which, with the aid of equality (6.3), is written in the following form

$$\begin{aligned}
&\Delta'\rho\Delta'\tilde{\Phi} + \frac{\partial}{\partial \eta}\frac{1}{\rho}\frac{\partial \tilde{\Phi}}{\partial \eta} + i\frac{r_0}{c}\frac{\partial^2 \tilde{\Phi}}{\partial \xi^2} = i\frac{r_0}{c}\left\{-\frac{\partial \tilde{u}^{\text{II}}}{\partial \xi} + i\frac{r_0}{Ehc} M_2^*\right\} +\\
&+ \underline{\left\{\Delta'\rho\frac{\partial \tilde{v}^{\text{II}}}{\partial \eta} + \frac{\partial}{\partial \eta}\left(\frac{\tilde{v}^{\text{II}}}{\rho}\right) - \Delta'\rho\left[\frac{\partial \tilde{u}^{\text{II}}}{\partial \xi} - i\frac{r_0}{Ehc}(M_2^* - M_1^*)\right]\right\}}
\end{aligned} \tag{6.5}$$

Since $\tilde{u}^p$, $\tilde{v}^p$, M_1^*, M_2^*, H^* are smooth, the terms underlined in the above expression are small compared to terms contained in the first figure brackets and therefore can be dropped.

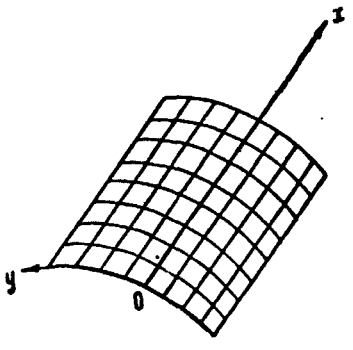

Figure 23

By expressing the components of flexural deformation in terms of the introduced functions, we obtain /178

$$\begin{aligned}
\tilde{T}_1 &= T_1^* + i\frac{Ehc}{r_0^2}\left\{\frac{\partial^2\tilde{w}}{\partial\eta^2} + \frac{\partial}{\partial\eta}\frac{1}{\rho}\frac{\partial\tilde{\Phi}}{\partial\eta} - \frac{\partial}{\partial\eta}\left(\frac{\tilde{v}^*}{\rho}\right)\right\};\\
\tilde{T}_2 &= T_2^* + i\frac{Ehc}{r_0^2}\left\{\frac{\partial^2\tilde{w}}{\partial\xi^2}\right\};\\
\tilde{S} &= S^* - i\frac{Ehc}{r_0^2}\left\{\frac{\partial^2\tilde{w}}{\partial\xi\partial\eta} + \frac{1}{\rho}\frac{\partial^2\tilde{\Phi}}{\partial\xi\partial\eta} - \frac{1}{\rho}\frac{\partial\tilde{v}^*}{\partial\xi}\right\}.
\end{aligned} \tag{6.6}$$

From these equations and from equation (6.4) we obtain relationships which associate the introduced potential of complex displacements with the basic V. V. Novozhilov complex function

$$\tilde{T} = T^* + \frac{Eh}{r_0}\left[\frac{\partial^2\tilde{\Phi}}{\partial\xi^2} + \frac{\partial\tilde{u}^*}{\partial\xi} - i\frac{r_0}{Ehc}M_2^*\right]. \tag{6.7}$$

In particular, by using this relationship it is easy to verify the equivalence of equation (6.5)

$$\Delta'\rho\Delta'\tilde{\Phi} + \frac{\partial}{\partial\eta}\frac{1}{\rho}\frac{\partial\tilde{\Phi}}{\partial\eta} + i\frac{r_0}{c}\frac{\partial^2\tilde{\Phi}}{\partial\xi^2} = i\frac{r_0}{c}\left\{-\frac{\partial\tilde{u}^*}{\partial\xi} + i\frac{r_0}{Ehc}M_2^*\right\} \tag{6.8}$$

and of the corresponding V. V. Novozhilov equation (Chapter VI, 1.6).

In the case of a circular cylindrical shell $\rho = \frac{r}{r_0} = 1$ and it is easy to obtain an equation for $\widetilde{w}$ from (6.8). Indeed, by performing the $\Delta'(\ )$ operation on the left and right parts of equation (6.8) and replacing $\Delta'\widetilde{\Phi}$ with its expression given by (6.3) we obtain, by dropping small terms

$$\Delta'\Delta'\widetilde{w} + \frac{\partial^2 \widetilde{w}}{\partial \eta^2} + i\frac{r_0}{c}\frac{\partial^2 \widetilde{w}}{\partial \xi^2} = i\frac{r_0}{c}\left\{-\frac{\partial^2}{\partial \xi \partial \eta}\left(\frac{\partial \widetilde{u}^{\text{и}}}{\partial \eta} + \frac{\partial \widetilde{v}^{\text{и}}}{\partial \xi}\right) + \right.$$
$$\left. + i\frac{r_0}{Ehc}\left[\frac{\partial^2 M_2^*}{\partial \eta^2} + \frac{\partial^2 M_1^*}{\partial \xi^2}\right]\right\}. \tag{6.9}$$

We replace the terms $\left(\frac{\partial \widetilde{u}^{\text{р}}}{\partial \eta} + \frac{\partial \widetilde{v}^{\text{р}}}{\partial \xi}\right)$ by their expressions from $(6.1)_3$ in the right side and utilize the three equilibrium equations which have the following form for a circular cylindrical shell

$$T_2^* - \frac{1}{r_0}\left\{\frac{\partial^2 M_1^*}{\partial \xi^2} + 2\frac{\partial^2 H^*}{\partial \xi \partial \eta} + \frac{\partial^2 M_2^*}{\partial \eta^2}\right\} = r_0 q_n.$$

As a result of this equation (6.9) is reduced to the form

$$\Delta'\Delta'\widetilde{w} + \frac{\partial^2 \widetilde{w}}{\partial \eta^2} + i\frac{r_0}{c}\frac{\partial^2 \widetilde{w}}{\partial \xi^2} = \frac{r_0^4}{Ehc^2}\left\{q_n - \frac{\partial^2 T_2^*}{r_0}\right\} =$$
$$= -\frac{r_0^2}{Ehc^2}\left\{\frac{\partial^2 M_1^*}{\partial \xi^2} + \frac{\partial^2 2H^*}{\partial \xi \partial \eta} + \frac{\partial^2 M_2^*}{\partial \eta^2}\right\} \tag{6.10}$$
$$\left(\Delta' = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} = r_0^2 \Delta\right).$$

7. Relationship with the V. Z. Vlasov Equation for Shallow Shells. The Refined Vlasov Equation

We generalize somewhat the results presented in section 8, Chapter IV, by introducing (in an arbitrary orthogonal system of coordinates), the complex Hooke's law in the following form /179

$$\widetilde{\varkappa}_\beta = \frac{i}{Ehc}\left\{\widetilde{T}_\alpha - T_\alpha^*\right\} - \varkappa_\beta^{\text{и}};\quad \widetilde{\varepsilon}_\beta = \frac{1}{Eh}\left\{\widetilde{T}_\alpha + \widetilde{T}_\beta + \frac{i}{c}M_\alpha^*\right\} - \varepsilon_\beta^{\text{и}};$$
$$\widetilde{\varkappa}_\alpha = \frac{i}{Ehc}\left\{\widetilde{T}_\beta - T_\beta^*\right\} - \varkappa_\alpha^{\text{и}};\quad \widetilde{\varepsilon}_\alpha = \frac{1}{Eh}\left\{\widetilde{T}_\alpha + \widetilde{T}_\beta + \frac{i}{c}M_\beta^*\right\} - \varepsilon_\alpha^{\text{и}}; \tag{7.1}$$
$$\widetilde{\tau} = -\frac{i}{Ehc}\left\{\widetilde{S} - S^*\right\} - \tau^{\text{и}};\quad \frac{\widetilde{\omega}}{2} = -\frac{i}{Ehc}H^* - \frac{\omega^{\text{и}}}{2}.$$

и = homogeneous

The rather new feature here involves the introduction of a geometric system of functions $\{\varepsilon_\alpha^h, \varepsilon_\beta^h, \ldots, \tau^h\}$ which satisfy the unique requirement: they are solutions of the continuity equations at the middle surface.

As in section 8, Chapter IV, it can be shown that the system of equations (7.1) with respect to $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ is approximately compatible and equivalent to the V. V. Novozhilov system of equations in terms of complex forces (Chapter VIII, 8.36).

Eliminating the forces from system (7.1) and dropping the small terms we obtain a simplified system of equations containing the complex displacements

$$\begin{aligned} \tilde{\varepsilon}_\alpha + ic\left(\tilde{\varkappa}_\alpha + \tilde{\varkappa}_\beta\right) &= \frac{i}{Ehc} M_\beta^* - \varepsilon_\alpha^H; \\ \tilde{\varepsilon}_\beta + ic\left(\tilde{\varkappa}_\alpha + \tilde{\varkappa}_\beta\right) &= \frac{i}{Ehc} M_\alpha^* - \varepsilon_\beta^H; \\ \tilde{\omega} &= -\frac{i}{Ehc} 2H^* - \omega^H. \end{aligned} \tag{7.2}$$

It can be written in a slightly different form

$$\begin{aligned} \tilde{\varepsilon}_\alpha + ic\left(\tilde{\varkappa}_\alpha + \tilde{\varkappa}_\beta\right) &= \frac{i}{Ehc} M_\beta^* - \varepsilon_\alpha^H; \\ \tilde{\varepsilon}_\beta - \tilde{\varepsilon}_\alpha &= \frac{i}{Ehc}\left(M_\alpha^* - M_\beta^*\right) + \left(\varepsilon_\alpha^H - \varepsilon_\beta^H\right); \\ \tilde{\omega} &= -\frac{i}{Ehc} 2H^* - \omega^H. \end{aligned} \tag{7.3}$$

(We note that the resulting system coincides with the one which was already utilized in section 1, if we take the dislocation displacements as the geometric system of functions.) /180

We recall that

$$\begin{aligned} \tilde{\varepsilon}_\alpha &= \frac{1}{A}\frac{\partial \tilde{u}}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{v} + \frac{\tilde{w}}{R_\alpha}; & \tilde{\varkappa}_\alpha &= \tilde{\varkappa}_\alpha^0 + \tilde{\gamma}_\alpha; \\ \tilde{\varepsilon}_\beta &= \frac{1}{B}\frac{\partial \tilde{v}}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{u} + \frac{\tilde{w}}{R_\beta}; & \tilde{\varkappa}_\beta &= \tilde{\varkappa}_\beta^0 + \tilde{\gamma}_\beta; \\ \tilde{\omega} &= \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{\tilde{v}}{B}\right) + \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{\tilde{u}}{A}\right) - \frac{2\tilde{w}}{R_{\alpha\beta}}; & \tilde{\tau} &= \tilde{\tau}^0 + \tilde{\gamma}_{\alpha\beta}, \end{aligned} \tag{7.4}$$

where

$$\tilde{\varkappa}_\alpha^0 = -\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{A}\frac{\partial\tilde{w}}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{1}{B}\frac{\partial\tilde{w}}{\partial\beta};$$

$$\tilde{\varkappa}_\beta^0 = -\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial\tilde{w}}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{1}{A}\frac{\partial\tilde{w}}{\partial\alpha}; \qquad (7.5)$$

$$\tilde{\tau}^0 = -\frac{1}{AB}\left(\frac{\partial^2\tilde{w}}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial\tilde{w}}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\tilde{w}}{\partial\beta}\right);$$

$$\tilde{\gamma}_\alpha = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{\tilde{u}}{R_\alpha} - \frac{\tilde{v}}{R_{\alpha\beta}}\right) + \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{\tilde{v}}{R_\beta} - \frac{\tilde{u}}{R_{\alpha\beta}}\right) + \\ + \frac{1}{2ABR_{\alpha\beta}}\left(\frac{\partial A\tilde{u}}{\partial\beta} - \frac{\partial B\tilde{v}}{\partial\alpha}\right);$$

$$\tilde{\gamma}_\beta = \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{\tilde{v}}{R_\beta} - \frac{\tilde{u}}{R_{\alpha\beta}}\right) + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{\tilde{u}}{R_\alpha} - \frac{\tilde{v}}{R_{\alpha\beta}}\right) - \\ - \frac{1}{2ABR_{\alpha\beta}}\left(\frac{\partial A\tilde{u}}{\partial\beta} - \frac{\partial B\tilde{v}}{\partial\alpha}\right); \qquad (7.6)$$

$$\tilde{\gamma}_{\alpha\beta} = \frac{1}{R_\alpha}\left(\frac{1}{B}\frac{\partial\tilde{u}}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{u}\right) + \frac{1}{R_\beta}\left(\frac{1}{A}\frac{\partial\tilde{v}}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{v}\right) - \\ - \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{R_{\alpha\beta}}\right)\tilde{u} - \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{R_{\alpha\beta}}\right)\tilde{v}.$$

Substituting expressions (7.4)-(7.6) into system (7.3) we obtain

$$\frac{1}{A}\frac{\partial\tilde{u}}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{v} + \frac{\tilde{w}}{R_\alpha} + ic\left\{-\Delta\tilde{w} + \tilde{\gamma}_\alpha + \tilde{\gamma}_\beta\right\} = \frac{l}{Ehc}M_\beta^* - \varepsilon_\alpha^n;$$

$$\frac{1}{B}\frac{\partial\tilde{v}}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{u} - \frac{1}{A}\frac{\partial\tilde{u}}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{v} - \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\tilde{w} = \frac{l}{Ehc}\times \\ \times\left(M_\alpha^* - M_\beta^*\right) + \varepsilon_\alpha^n - \varepsilon_\beta^n; \qquad (7.7)$$

$$\frac{1}{A}\frac{\partial\tilde{v}}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{v} + \frac{1}{B}\frac{\partial\tilde{u}}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{u} - \frac{2}{R_{\alpha\beta}}\tilde{w} = \\ = -\frac{l}{Ehc}2H^* - \omega^n.$$

Now let us perform the following operations on the left and right sides of the first, second and third equation, respectively, /181

$$\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial}{\partial\beta}\right\};$$

$$\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha}\left[\frac{1}{A}\frac{\partial B}{\partial\alpha}(\ )\right] - \frac{\partial}{\partial\beta}\left[\frac{1}{B}\frac{\partial A}{\partial\beta}(\ )\right]\right\};$$

$$-\frac{1}{AB}\left\{\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{\partial}{\partial\alpha}\left[\frac{1}{A}\frac{\partial A}{\partial\beta}(\ )\right] + \frac{\partial}{\partial\beta}\left[\frac{1}{B}\frac{\partial B}{\partial\alpha}(\ )\right]\right\}.$$

Then by adding the left and right parts of the resulting equalities and utilizing the Codazzi-Gauss relationships we obtain, after simple but rather awkward transformations, the following equation

$$\Delta\Delta\tilde{w}-\Delta(\tilde{\gamma}_\alpha+\tilde{\gamma}_\beta)+\frac{i}{c}\left[D\tilde{w}+\tilde{\Gamma}(\tilde{u},\ \tilde{v})\right]=-\frac{q_n^*}{Ehc^2}-\frac{i}{c}\,q_n^{\text{н}}, \tag{7.8}$$

where

$$\Delta=\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial}{\partial\beta}\right\}; \tag{7.9}$$

$$D=\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{B}{AR_\beta}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\alpha}\frac{1}{R_{\alpha\beta}}\frac{\partial}{\partial\beta}+\frac{\partial}{\partial\beta}\frac{1}{R_{\alpha\beta}}\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{A}{BR_\alpha}\frac{\partial}{\partial\beta}\right\};$$

$$\tilde{\Gamma}(\tilde{u},\ \tilde{v})=-\frac{1}{AB}\left\{\frac{\partial KB\tilde{u}}{\partial\alpha}+\frac{\partial KA\tilde{v}}{\partial\beta}\right\}; \tag{7.10}$$

$$q_n^*=\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{1}{A}\left(\frac{\partial BM_\alpha^*}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H^*}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta^*\right)+\right.$$
$$\left.+\frac{\partial}{\partial\beta}\frac{1}{B}\left(\frac{\partial AM_\beta^*}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H^*}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha^*\right)\right\};$$

$$q_n^{\text{н}}=\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{1}{A}\left(\frac{\partial B\varepsilon_\beta^{\text{н}}}{\partial\alpha}-\frac{1}{A}\frac{\partial A^2\frac{\omega^{\text{н}}}{2}}{\partial\beta}-\frac{\partial B}{\partial\alpha}\varepsilon_\alpha^{\text{н}}\right)+\right. \tag{7.11}$$
$$\left.+\frac{\partial}{\partial\beta}\frac{1}{B}\left(\frac{\partial A\varepsilon_\alpha^{\text{н}}}{\partial\beta}-\frac{1}{B}\frac{\partial B^2\frac{\omega^{\text{н}}}{2}}{\partial\alpha}-\frac{\partial A}{\partial\beta}\varepsilon_\beta^{\text{н}}\right)\right\}.$$

Equation (7.8) is a corollary of equation (7.7). Therefore it can be used to replace the first equation from system (7.7), and the latter can now be written in the form

$$\Delta\Delta\tilde{w}-\Delta\underline{\left(\tilde{\gamma}_\alpha+\tilde{\gamma}_\beta\right)}+\frac{i}{c}\underline{\left[D\tilde{w}+\tilde{\Gamma}\left(\tilde{u},\ \tilde{v}\right)\right]}=-\frac{q_n^*}{Ehc^2}-\frac{i}{c}q_n^{\text{н}};$$

$$\frac{1}{B}\frac{\partial\tilde{v}}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{u}-\frac{1}{A}\frac{\partial\tilde{u}}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{v}-\left(\frac{1}{R_\alpha}-\frac{1}{R_\beta}\right)\tilde{w}=$$
$$=\frac{i}{Ehc}\left(M_\alpha^*-M_\beta^*\right)+\varepsilon_\alpha^{\text{н}}-\varepsilon_\beta^{\text{н}}; \tag{7.12}$$

$$\frac{1}{A}\frac{\partial\tilde{v}}{\partial\alpha}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{v}+\frac{1}{B}\frac{\partial\tilde{u}}{\partial\beta}-\frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{u}-\frac{2}{R_{\alpha\beta}}\tilde{w}=-\frac{i}{Ehc}2H^*-\omega^{\text{н}}.$$

In this case according to (7.1) the complex forces are computed by means of the following equations /182

$$\tilde{T}_\alpha = T_\alpha^* - iEhc\varkappa_\beta^n - iEhc\left\{-\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial\tilde{w}}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{1}{A}\frac{\partial\tilde{w}}{\partial\alpha} + \underline{\tilde{\gamma}_\beta}\right\};$$

$$\tilde{T}_\beta = T_\beta^* - iEhc\varkappa_\alpha^n - iEhc\left\{-\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{A}\frac{\partial\tilde{w}}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{1}{B}\frac{\partial\tilde{w}}{\partial\beta} + \underline{\tilde{\gamma}_\alpha}\right\}; \qquad (7.13)$$

$$\tilde{S} = S^* + iEhc\tau^n + iEhc\left\{-\frac{1}{AB}\left(\frac{\partial^2\tilde{w}}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial\tilde{w}}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\tilde{w}}{\partial\beta}\right) + \underline{\tilde{\gamma}_{\alpha\beta}}\right\}.$$

We drop the underlined terms in the resulting relationships. Then

$$\Delta\Delta\tilde{w} + \frac{i}{c}D\tilde{w} = -\frac{q_n^*}{Ehc^2} - \frac{l}{c}q_n^n \qquad (7.14)$$

and

$$\tilde{T}_\alpha = T_\alpha^* - iEhc\varkappa_\beta^n + iEhc\left\{\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial\tilde{w}}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{1}{A}\frac{\partial\tilde{w}}{\partial\alpha}\right\};$$

$$\tilde{T}_\beta = T_\beta^* - iEhc\varkappa_\alpha^n + iEhc\left\{\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{A}\frac{\partial\tilde{w}}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{1}{B}\frac{\partial\tilde{w}}{\partial\beta}\right\}; \qquad (7.15)$$

$$\tilde{S} = S^* + iEhc\tau^n + iEhc\left\{-\frac{1}{AB}\left(\frac{\partial^2\tilde{w}}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial\tilde{w}}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\tilde{w}}{\partial\beta}\right)\right\}.$$

If we use the membrane solution ($M_\alpha^* = M_\beta^* = H^* = 0$) as the static system of functions and use the purely flexural displacements ($\epsilon_\alpha^h = \epsilon_\beta^h = \omega^h = 0$) as the geometric system, equation (7.14) will differ from the complex analog of the V. Z. Vlasov equations for shallow shells (ref. 33, p. 95) by the absence of the term q_n/Ehc^2. This is explained by the fact that in our case the complex first function is used to determine the forces and moments by relationships (7.15). In these the "carriers of nonhomogeneity" for the problem are the first terms T_α^*, T_β^*, S^*, $\varkappa_\alpha^n$, $\varkappa_\beta^n$, τ^h which are absent in Vlasov.

Our form of presentation is more coherent. Indeed, when the Vlasov equations are derived (ref. 33, p. 95), it is usually assumed that $q_1 = q_2 = 0$. /183

Furthermore in case of shells which are not shallow it is assumed that a rapidly varying state of stress exists in the shell, whereas the nature of the particular solution is determined primarily by the surface load and may be quite smooth (for example when q_n = const).

The form (7.14)-(7.15) is more flexible, because it is based on the separation of those parts of the general solution which are heterogeneous in their properties: the particular solution (usually a membrane solution) of the nonhomogeneous problem and the general (most frequently pertaining to the edge effect) solution of the homogeneous problem.

Equation $(7.12)_1$ may be looked upon in another way if we consider it as an equation for the unknown function $\widetilde{w}$. In this case it is assumed that the terms depending on $\widetilde{u}$ and $\widetilde{v}$ contained in the equation are expressed implicitly in terms of $\widetilde{w}$ by means of the second and third equation of (7.12). Equation $(7.12)_1$ interpreted in this way may be looked upon as the refined Vlasov equation.

In some cases the refined equation is written in an explicit form. Thus for a circular cylindrical shell

$$\begin{gathered}\frac{1}{R_\alpha}=\frac{1}{R_{\alpha\beta}}=0;\ R_\beta=A=B=r_0;\ \alpha=\xi,\ \beta=\varphi;\\ D=\frac{1}{r_0^3}\frac{\partial^2}{\partial\xi^2},\quad \Delta=\frac{1}{r_0^2}\left\{\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\varphi^2}\right\},\quad K=0,\ \widetilde{\Gamma}(\widetilde{u},\ \widetilde{v})=0;\\ \widetilde{\gamma}_\alpha+\widetilde{\gamma}_\beta=\frac{1}{r_0^2}\frac{\partial\widetilde{v}}{\partial\varphi},\ \Delta\left(\widetilde{\gamma}_\alpha+\widetilde{\gamma}_\beta\right)=\frac{1}{r_0^4}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2\widetilde{v}}{\partial\xi^2}+\frac{\partial^2\widetilde{v}}{\partial\varphi^2}\right);\\ q_n^*=\frac{1}{r_0^2}\left\{\frac{\partial^2M_1^*}{\partial\xi^2}+\frac{\partial^2 2H^*}{\partial\xi\partial\varphi}+\frac{\partial^2M_2^*}{\partial\varphi^2}\right\}.\end{gathered}\tag{7.16}$$

When

$$\begin{gathered}\varepsilon_1^N=\varepsilon_2^N=\omega^N=0,\quad \varkappa_1^N=\varkappa_2^N=\tau^N=0\\ \frac{\partial\widetilde{u}}{\partial\xi}-\frac{\partial\widetilde{v}}{\partial\varphi}=\widetilde{w}+i\frac{r_0}{Ehc}\left(M_2^*-M_1^*\right);\\ \frac{\partial\widetilde{u}}{\partial\varphi}+\frac{\partial\widetilde{v}}{\partial\xi}=-i\frac{r_0}{Ehc}2H^*.\end{gathered}\tag{7.17}$$

From this system we find

$$\frac{\partial^2\widetilde{v}}{\partial\xi^2}+\frac{\partial^2\widetilde{v}}{\partial\varphi^2}=-\frac{\partial\widetilde{w}}{\partial\varphi}-i\frac{r_0}{Ehc}\left\{\frac{\partial M_2^*}{\partial\varphi}-\frac{\partial M_1^*}{\partial\varphi}+2\frac{\partial H^*}{\partial\xi}\right\}.$$

Substituting the resulting expressions into $(7.12)_1$ we see that it coincides with equation (6.10). It differs from the Vlasov equation by the underlined terms which are significant for long shells (section 6, Chapter VI). /184

Similarly for a spherical shell the proposed procedure leads to a well-known general complex equation of a spherical shell.

To solve the refined Vlasov equation it is easy to construct the following method of successive approximations: for the initial approximation we take the solution of equation (7.14). Then we determine $\tilde{u}$ and $\tilde{v}$ from the second and third equation of (7.12). Using these to compute $\tilde{\Gamma}(\tilde{u}, \tilde{v})$, $\Delta(\tilde{\nabla}_\alpha + \tilde{\nabla}_\beta)$ and transposing these into the right side we obtain an equation whose solution yields the next approximation for $\tilde{w}$, etc.

The example considered above shows that the proposed method of refining the Vlasov equation may give us substantial corrections, and for shells of simple form (cylindrical and spherical) it leads to the well-known equations.

The question concerning the applicability of the modified V. Z. Vlasov equation (7.14) and its error is of considerable interest. The asymptotic analysis conducted at the end of section 1, Chapter XIII, for a homogeneous equation shows that the error, incurred when we drop the terms depending on $\tilde{u}$ and $\tilde{v}$ in $(7.12)_1$, increases as the variability index decreases. Thus for a slowly variable state (in particular for a membrane state and for a state with high moments) the accuracy of the homogeneous equation is small. The proposed modified equation makes it possible to circumvent this difficulty. Indeed, let us assume that the unknown solution is close to a membrane solution or to a purely flexural solution. Since the functions $\{T_\alpha^*, T_\beta^*, S^*; u^h, v^h, w^h\}$ are arbitrary we can obtain a good approximate expression for the unknown functions. In this case $\tilde{w}$ plays the role of a correction and the relatively large error incurred in its determination can no longer have a substantial effect on the total value of the established solution.

We may proceed in an analogous manner for other forms of the basic (slowly varying) part of the solution.

Let

$$\{A, B\} \sim L; \{R_\alpha, R_\beta, R_{\alpha\beta}\} \sim R_0 \tag{7.18}$$

(the sign $\sim$ shows that the compared quantities are of the same order of magnitude). Examining relationships (7.6), (7.10), $(7.12)_1$ and (7.13), we see that the terms contained in them which depend on $\tilde{u}$ and $\tilde{v}$ have a factor of the order L/R_0. For shallow shells $L/R_0 \ll 1$. Therefore the shallower the shell, the less is the role of the underlined terms in the refined Vlasov equation $(7.12)_1$. Vlasov himself called his equations either the equations of shallow

shells or the equations of the engineering theory of shells. The remarks made above show that the truth lies somewhere between these two extreme determinations: the Vlasov equations can also be applied to nonshallow shells, but are not universal. /185

In addition, there is a more or less proven conclusion that all practical cases of shell deformation may be handled by the modified Vlasov equation (7.14), if the selection of the static and geometric system of functions is successful.

CHAPTER XII. MEMBRANE THEORY

In the present chapter we shall consider several basic problems asso- /186
ciated with the application of the membrane theory of shells from a more general point of view than in Chapter V.

The membrane theory is substantially simpler than the moment theory. In addition, for a sufficiently large class of shells (and loads), it gives a correct representation concerning the operation of the structure. It is precisely this fact which explains the great deal of attention devoted in the past and at the present time to the membrane theory. Historically the theory preceded the moment theory. Therefore, for a long time it was developed independently of the latter. At the present time the moment theory has been sufficiently developed, and it is more or less common to regard the membrane theory as an approximate approach to find the solution of the general (moment) theory.

The most difficult problem to date, and one which has not been completely clarified, concerns the conditions under which the membrane solution is applicable and the evaluation of errors introduced by the membrane solution. It appears to us that in this connection we should clearly separate two questions:

1. When is it possible (and necessary) to utilize the membrane solution as a particular solution of the moment problem?

2. What are the conditions (in the case when the answer to the first question is positive) under which it can be considered as a total (general) solution of a moment problem?

In other words, can we use the limitation inherent in the membrane solution to satisfy the boundary conditions of the problem at hand? The second question is also associated with the inverse problem of the membrane theory (the selection of the middle surface form, of the law governing the variation in thickness and of the edge elements).

The proposed breakdown of the question concerning the conditions /187
under which the membrane theory is applicable makes it possible to investigate more deeply the nature of these two questions, which are entirely different in their content, and to formulate criteria for the applicability of the membrane theory which are more complete than the existing ones.

To avoid confusion we shall first agree on a terminology.

(a) We shall say that a membrane state of stress exists in a shell if the stresses due to forces exceed substantially the stresses produced by moments, i.e., if

$$\left.\begin{aligned} \sigma_{T_\alpha} &= \frac{T_\alpha}{h} \\ \sigma_{T_\beta} &= \frac{T_\beta}{h} \\ \sigma_S &= \frac{S}{h} \end{aligned}\right\} \gg \left\{\begin{aligned} \sigma_{M_\alpha} &= \frac{6M_\alpha}{h^2} \\ \sigma_{M_\beta} &= \frac{6M_\beta}{h^2} \\ \sigma_H &= \frac{6H}{h^2}. \end{aligned}\right. \tag{0.1}$$

Thus, in a membrane state of stress we permit the existence of small moments.

(b) We shall isolate the state of stress without flexure in a special way assuming that it pertains to the case when

$$\varkappa_\alpha = \varkappa_\beta = \tau = 0 \tag{0.2}$$

and consequently

$$M_\alpha = M_\beta = H = 0. \tag{0.3}$$

(c) We shall call the state of stress mixed (sometimes it is called the moment state of stress) when the stresses due to the forces and the moments are quantities of the same order

$$\left\{\sigma_{T_\alpha}, \sigma_{T_\beta}, \sigma_S\right\} \sim \left\{\sigma_{M_\alpha}, \sigma_{M_\beta}, \sigma_H\right\}. \tag{0.4}$$

(d) A state of stress is said to be due to large moments when it is characterized by the strong inequalities

$$\left\{\sigma_{T_\alpha}, \sigma_{T_\beta}, \sigma_S\right\} \ll \left\{\sigma_{M_\alpha}, \sigma_{M_\beta}, \sigma_H\right\}. \tag{0.5}$$

(e) An important particular case of the latter is a purely flexural state of stress for which

$$\varepsilon_\alpha = \varepsilon_\beta = \omega = 0 \tag{0.6}$$

and consequently

$$T_\alpha = T_\beta = S = 0. \tag{0.7}$$

1. The Membrane Solution as an Approach in Finding the Particular Solution of the General (Moment) Theory

The following is the initial system of equations for the membrane theory (section 8, Chapter VIII) /188

$$\begin{aligned}
&\frac{\partial BT_\alpha^*}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2S^*}{\partial\beta}-\frac{\partial B}{\partial\alpha}T_\beta^*=-ABq_\alpha;\\
&\frac{\partial AT_\beta^*}{\partial\beta}+\frac{1}{B}\frac{\partial B^2S^*}{\partial\alpha}-\frac{\partial A}{\partial\beta}T_\alpha^*=-ABq_\beta;\\
&k_\alpha T_\alpha^*-2k_{\alpha\beta}S^*+k_\beta T_\beta^*=q_n,
\end{aligned} \tag{1.1}$$

which is obtained by dropping the moments in the equilibrium equations

$$\begin{aligned}
&\frac{\partial BT_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2S}{\partial\beta}-\frac{\partial B}{\partial\alpha}T_\beta+k_\alpha\left(\frac{\partial BM_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta\right)-\\
&-k_{\alpha\beta}\left(\frac{\partial AM_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\right)+\frac{\partial}{\partial\beta}\left(k_\alpha AH-k_{\alpha\beta}BM_\beta\right)+\\
&+\frac{\partial A}{\partial\beta}\left(k_\beta H-k_{\alpha\beta}M_\alpha\right)=-ABq_\alpha\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\begin{pmatrix}\alpha\rightleftarrows\beta\\ A\to B\end{pmatrix};\\
&k_\alpha T_\alpha-2k_{\alpha\beta}S+k_\beta T_\beta-\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{1}{A}\left(\frac{\partial BM_\alpha}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta\right)+\right.\\
&\left.+\frac{\partial}{\partial\beta}\frac{1}{B}\left(\frac{\partial AM_\beta}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha\right)\right\}+\\
&+k_{\alpha\beta}\left[-(k_\alpha+k_\beta)H+k_{\alpha\beta}(M_\alpha+M_\beta)\right]=q_n\\
&\left(k_\alpha=\frac{1}{R_\alpha},\ k_{\alpha\beta}=\frac{1}{R_{\alpha\beta}},\ k_\beta=\frac{1}{R_\beta}\right).
\end{aligned} \tag{1.2}$$

The displacements corresponding to the membrane forces T_α^*, T_β^*, S^* determined from system (1.1) are established from the relationships of Hooke's law

$$\begin{aligned}
&\frac{1}{A}\frac{\partial u^*}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}v^*+k_\alpha w^*\equiv\varepsilon_\alpha^*=\frac{1}{Eh}\left(T_\alpha^*-\nu T_\beta^*\right);\\
&\frac{1}{B}\frac{\partial v^*}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u^*+k_\beta w^*\equiv\varepsilon_\beta^*=\frac{1}{Eh}\left(T_\beta^*-\nu T_\alpha^*\right);\\
&\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v^*}{B}\right)+\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u^*}{A}\right)-2k_{\alpha\beta}w^*\equiv\omega^*=\frac{2(1+\nu)}{Eh}S^*.
\end{aligned} \tag{1.3}$$

The displacements u*, v*, w* correspond to the components of flexural deformation

$$\varkappa_\alpha^* = -\frac{\partial}{A\partial\alpha}\frac{\partial w^*}{A\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial w^*}{B\partial\beta} + \frac{\partial}{A\partial\alpha}\left(k_\alpha u^* - k_{\alpha\beta}v^*\right) + \\ + \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(k_\beta v^* - k_{\alpha\beta}u^*\right) + \frac{k_{\alpha\beta}}{2AB}\left(\frac{\partial Au^*}{\partial\beta} - \frac{\partial Bv^*}{\partial\alpha}\right) \quad (\alpha \rightleftarrows \beta);$$

$$\tau^* = -\frac{1}{AB}\left(\frac{\partial^2 w^*}{\partial\alpha\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w^*}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w^*}{\partial\beta}\right) + \\ + k_\alpha\left(\frac{\partial u^*}{B\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u^*\right) + k_\beta\left(\frac{\partial v^*}{A\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v^*\right) - \\ - \frac{\partial k_{\alpha\beta}}{A\partial\alpha}u^* - \frac{\partial k_{\alpha\beta}}{B\partial\beta}v^*. \quad (1.4)$$

The following moments correspond to the latter

$$M_\alpha^* = \frac{Eh^3}{12(1-\nu^2)}\left(\varkappa_\alpha^* + \nu\varkappa_\beta^*\right); \quad M_\beta^* = \frac{Eh^3}{12(1-\nu^2)}\left(\varkappa_\beta^* + \nu\varkappa_\alpha^*\right); \\ H^* = \frac{Eh^3}{12(1+\nu)}\tau^*. \quad (1.5)$$

We shall show that solutions constructed in this manner make it possible to give a positive (generally speaking) answer to the first question (i.e., they can be taken as the particular solutions of the membrane theory), if the following conditions are satisfied /189

(a) the curvatures k_α, k_β, $k_{\alpha\beta}$ and the thickness h (α, β) are smooth functions which do not increase substantially during differentiation;

(b) the surface loading is smooth;

(c) the shell is not too shallow, and its linear dimensions are quantities of the same order of magnitude as the radii of curvature (at least of the same order of magnitude as one of them).

Indeed, let

$$\left\{A, B, \frac{1}{k_\alpha}, \frac{1}{k_\beta}, \frac{1}{k_{\alpha\beta}}\right\} \sim R_0, \quad \{q_\alpha, q_\beta, q_n\} \sim p, \quad (1.6)$$

where R_0 is some characteristic dimension of the middle surface. From relationships (1.1) and (1.3) we see that

$$\{T_\alpha^*,\ T_\beta^*,\ S^*\} \sim R_0 p,\ \{\varepsilon_\alpha^*,\ \varepsilon_\beta^*,\ \omega^*\} \sim \frac{R_0 p}{Eh}. \qquad (1.7)$$

The solution of system (1.3) may be represented in the form

$$u^* = u' + u''; \quad v^* = v' + v''; \quad w = w' + w'', \qquad (1.8)$$

p in this case = particular;

where u^p, v^p, w^p is some particular solution while u^h, v^h, w^h is the solution of the corresponding homogeneous system

$$\varepsilon_\alpha^* = \varepsilon_\beta^* = \omega^* = 0. \qquad (1.9)$$

The solution of the homogeneous system gives a purely flexural displacement of the shell and its displacement as a rigid body. The true boundary conditions usually limit the value of the purely flexural displacements, so that their order of magnitude does not exceed the order of magnitude of quantities u^p, v^p, w^p. In any case, in considering the problem of interest to us (particular solution) we can always limit u^h, v^h, w^h and assume that (as follows from (1.3)) /190

$$\{u^*,\ v^*,\ w^*\} \sim \frac{R_0^2 p}{Eh}. \qquad (1.10)$$

Furthermore, from (1.4) and (1.5) we find

$$\{\varkappa_\alpha^*,\ \varkappa_\beta^*,\ \tau^*\} \sim \frac{p}{Eh};\ \{M_\alpha^*,\ M_\beta^*,\ H^*\} \sim h^2 p. \qquad (1.11)$$

From the relationships obtained above it follows, first of all, that

$$\left\{\sigma_{T_\alpha^*},\ \sigma_{T_\beta^*},\ \sigma_{S^*}\right\} \sim \frac{R_0}{h} p,\ \left\{\sigma_{M_\alpha^*},\ \sigma_{M_\beta^*},\ \sigma_{H^*}\right\} \sim 1p. \qquad (1.12)$$

Since for thin shells $\frac{R_0}{h} \gg 1$ the last equality shows that if the method presented above is used, we arrive at a membrane state of stress (see criterion (0.1)).

We must now verify the degree to which the constructed solution satisfies the equations of the general (moment) theory. First of all, the continuity equations are exactly satisfied since we have in fact established the displacement $u = u^*$, $v = v^*$, $w = w^*$. Let us now verify the fulfillment of the equilibrium equations (1.2). For this purpose we substitute the computed forces and moments into them. In this case the equilibrium equations will assume the following form

$$\left\{\frac{\partial BT_\alpha^*}{\partial \alpha}+\ldots\right\}=-ABq_\alpha-\underline{k_\alpha\left\{\frac{\partial BM_\alpha^*}{\partial \alpha}+\ldots\right\}}. \qquad (1.13)$$

According to the estimates obtained for (1.6), (1.7) and (1.11), the terms contained in the left side as well as the first terms in the right side have an order of magnitude $\sim R_0^2 p$. The underlined correction terms associated with the "membrane" moments are of the order $h^2 p$. Thus, equalities (1.13) show that the equilibrium conditions of the general (moment) theory are satisfied at the expense of the meager variation in the surface load (by a quantity of the order $(h/R_2)^2$ compared with 1).

We note that since we are speaking of a particular solution, the requirement for smooth shell geometry and surface loading does not exclude possible discontinuities in these along individual lines. In this case it is necessary to break the shell up into parts where these quantities are continuous and to consider each of such pieces separately.

2. Singular Cases

The discussions in the preceding section have shown that the requirements "a", "b" and "c" formulated there with respect to shell geometry and surface loading make it possible, generally speaking, to borrow the particular solution from the membrane theory. As we can see from the analysis which has been carried out above, an exception to this may be the case when the constructed displacements and the corresponding components of flexural deformation and the moments assume infinite values for certain values of α and β. If conditions "a", "b" and "c" are satisfied, the latter may be produced only by singularities in the geometry of the shell. /191

Looking over the relationships of the preceding section we see that they contain the curvatures k_α, k_β, $k_{\alpha\beta}$. These cannot become infinite because when the theory of thins shells was constructed it was assumed that $\{hk_\alpha, hk_\beta, hk_{\alpha\beta}\} \ll 1$. Therefore, we must exclude from our consideration, first of all, the singular points of the type associated with the apex of a conic shell, etc. A three-dimensional state of stress occurs in the neighborhood of such points,

and this state is not described by the equation from the theory of thin shells.

Therefore, only the neighborhoods of the points at which the normal curvatures of the shell become equal to zero should be considered (the neighborhoods of asymptotic lines). At such points some of the terms in the equations drop out which (as we shall see below) may lead to infinite values for the displacements. Infinitely removed points, if they exist, should also be referred to singular points.

Taking into account what has been said above, we refer, initially, the following points to the singular points of the middle surface[1]

(1) points at which one of the normal curvatures of the middle surface becomes equal to zero;

(2) the infinitely removed points of a region.

The term "neighborhoods of singular points" will be used to refer to points of the middle surface where one of the normal curvatures assumes a small value and also points which are sufficiently removed.

We shall consider initially several simple, but informative examples.

EXAMPLE 1. Let us consider a plate. Since in this case $k_\alpha = k_{\alpha\beta} = k_\beta = 0$, the third equation in (1.1) shows that for a plate the transverse load ($q_n \neq 0$) cannot be balanced in a membrane fashion. This well-known fact illustrates the singular point of type "1". It is clear that the proposition is also valid for very shallow shells, all of whose points may be assumed to be the neighborhood of a singular point (it is appropriate to call it the planar neighborhood). /192

EXAMPLE 2. Let us consider a closed cylindrical shell of arbitrary configuration $\alpha = \xi$, $\beta = \eta$, $R_\beta = r(\eta)$, $A = B = r_0$, $1/R_\alpha = 0$. If the surface load is a uniform normal pressure, the relationships of the membrane theory assume the form

$$\frac{\partial T_1^*}{\partial \xi} + \frac{\partial S^*}{\partial \eta} = 0; \quad \frac{\partial S^*}{\partial \xi} + \frac{\partial T_2^*}{\partial \eta} = 0; \quad T_2^* = p r_0 \rho; \tag{2.1}$$

$$\frac{\partial u^*}{\partial \xi} = r_0 \varepsilon_1^* = \frac{r_0}{Eh}(T_1^* - \nu T_2^*);$$

$$\frac{\partial v^*}{\partial \eta} + \frac{w^*}{\rho} = r_0 \varepsilon_2^* = \frac{r_0}{Eh}(T_2^* - \nu T_1^*);$$

$$\frac{\partial u^*}{\partial \eta} + \frac{\partial v^*}{\partial \xi} = r_0 \omega^* = \frac{2(1+\nu) r_0}{Eh} S^*; \tag{2.2}$$

[1] In section 6 we shall refine the determination of singular points.

$$r_0^2 \varkappa_1^* = -\frac{\partial^2 w^*}{\partial \xi^2};$$
$$r_0^2 \varkappa_2^* = -\frac{\partial^2 w^*}{\partial \eta^2} + \frac{\partial}{\partial \eta}\left(\frac{v^*}{\rho}\right), \qquad (2.3)$$
$$r_0^2 \tau^* = -\frac{\partial^2 w^*}{\partial \xi \partial \eta} + \frac{1}{\rho}\frac{\partial v^*}{\partial \xi}.$$

Here $\rho(\eta) = \frac{r(\eta)}{r_0}$, where r_0 is the mean radius of the tube.

The integration of this system is not a very large task and is reduced to a four-stage successive determination of quadratures over ξ. The four arbitrary functions of integration (which depend on the variable η) are fixed by satisfying the four boundary conditions at the edges of the tube.

Thus, for the following version of boundary conditions

$$T_1 = 0, \; v = 0 \text{ when } \xi = 0 \text{ and } \xi = \xi_0 = \frac{L}{r_0} \qquad (2.4)$$

(L is the length of the shell) we obtain the following expressions for forces and displacements

$$T_2^* = pr_0[\rho(\eta)]; \; T_1^* = pr_0\left\{\xi\frac{\xi - \xi_0}{2}\rho''(\eta)\right\};$$
$$S^* = pr_0\left\{\left(\frac{\xi_0}{2} - \xi\right)\rho'(\eta)\right\}; \qquad (2.5)$$
$$u^* = \frac{pr_0^2}{Eh}\left\{\nu\left(\frac{\xi_0}{2} - \xi\right)\rho(\eta) + \frac{1}{6}\left(\xi^3 - \frac{3}{2}\xi_0\xi^2 + \frac{1}{4}\xi_0^3\right)\rho''(\eta)\right\};$$
$$v^* = \frac{pr_0^2}{Eh}\left\{-\frac{2+\nu}{2}\xi(\xi - \xi_0)\rho'(\eta) - \frac{1}{24}\xi(\xi^3 - 2\xi_0\xi^2 + \xi_0^3)\rho'''(\eta)\right\}; \qquad (2.6)$$
$$w^* = \frac{pr_0^2\rho}{Eh}\left\{\rho + \xi(\xi - \xi_0)\rho''(\eta) + \frac{1}{24}\xi(\xi^3 - 2\xi_0\xi^2 + \xi_0^3)\rho^{(IV)}(\eta)\right\}.$$

From these relationships we can see that for noncircular cross sections the formulated solution becomes unsuitable when the shells are very long. /193
In order to understand the process we shall first compute all the components of the tangential or flexural deformation corresponding to the established displacements

$$\frac{Eh}{pr_0}\varepsilon_1^* = \frac{\rho''}{2}(\xi^2 - \xi_0\xi) - \nu\rho;$$
$$\frac{Eh}{pr_0}\varepsilon_2^* = -\frac{\nu\rho''}{2}(\xi^2 - \xi_0\xi) + \rho; \qquad (2.7)$$
$$\frac{Eh}{pr_0}\omega^* = 2(1+\nu)\rho'\left(\frac{\xi_0}{2} - \xi\right);$$
$$\frac{Eh}{pr_0}r_0\varkappa_1^* = -2\rho\rho'' - \frac{1}{2}\xi(\xi - \xi_0)\rho\rho^{(IV)};$$

$$\frac{Eh}{pr_0}r_0\varkappa_2^*=-(\rho^2)''-\xi(\xi-\xi_0)\left[(\rho\rho'')''+\frac{2+\nu}{2}\left(\frac{\rho'}{\rho}\right)'\right]-$$
$$-\frac{1}{24}\xi(\xi^3-2\xi_0\xi^2+\xi_0^3)\left[(\rho\rho^{(IV)})''+\left(\frac{\rho'''}{\rho}\right)'\right];$$
$$\frac{Eh}{pr_0}r_0\tau^*=-(2\xi-\xi_0)\left[(\rho\rho'')'+\frac{2+\nu}{2}\frac{\rho'}{\rho}\right]- \tag{2.8}$$
$$-\left(\frac{1}{6}\xi^3-\frac{1}{4}\xi_0\xi^2+\frac{1}{24}\xi_0^3\right)\left[(\rho\rho^{(IV)})'+\frac{\rho'''}{\rho}\right].$$

Comparing expressions (2.7) and (2.8), we see that as the length of the shell is increased (increase in ξ_0), the deformation assumes a more and more pronounced flexurable nature in spite of the fact that the purely flexural components of the deformation are limited by the boundary conditions (2.4).

If we compute the moments by means of equations (1.5) and substitute them into the equilibrium equation

$$\frac{\partial T_1}{\partial\xi}+\frac{\partial S}{\partial\eta}=0,$$
$$\frac{\partial S}{\partial\xi}+\frac{\partial T_2}{\partial\eta}+\frac{1}{r_0\rho}\left(\frac{\partial M_2}{\partial\eta}+2\frac{\partial H}{\partial\xi}\right)=0, \tag{2.9}$$
$$\frac{T_2}{\rho}-\frac{1}{r_0}\left(\frac{\partial^2M_1}{\partial\xi^2}+2\frac{\partial^2H}{\partial\xi\,\partial\eta}+\frac{\partial^2M_2}{\partial\eta^2}\right)=r_0p,$$

it is easy to see that when the shell is elongated, the role of the moment terms increases more and more. In this case, since the term underlined in (2.8) becomes the dominating one, the determining moments will be M_2 and

$$M_1\approx\nu M_2. \tag{2.10}$$

The last equality is characteristic for the semimembrane theory of cylindrical shells, as formulated in the version by V. V. Novozhilov (ref. 33).

Thus the membrane theory leads to wrong relationships in this particular case, but at the same time gives the correct qualitative indication that a semimembrane state of stress exists in the shell. The latter is in complete agreement with our concepts concerning the operation of a long cylindrical /194 shell. Indeed, there are no boundary conditions (including nontangential ones) which can have a serious effect on the state of stress far away from the edges. Therefore, at a sufficiently remote distance from the edges a state of stress-strain is established (which is determined entirely by the load and by the form of the middle surface) which is similar to the one taking place in a ring under

the action of a uniform load, normal to the axis. If the axis of the ring is different from the arc of a circle, the load (since the flexural rigidity of the ring is substantially "less" than its tensional rigidity) will bend the ring and a state of stress with large moments will occur in it (see criterion (0.5)).

From what has been stated above, it follows that we obtained the wrong result because we ignored the semimembrane state of stress actually taking place in the shell. This, in turn, made us neglect without proper basis, certain significant moment terms in the equilibrium equations.

EXAMPLE 3. Finally, let us consider a symmetrically loaded circular toroidal shell (fig. 33). For this shell the membrane equations (see section 2, Chapter XV)

$$\frac{d(1+\alpha\sin\theta)T_1^*}{d\theta}-\alpha\cos\theta\,T_2^*=-(1+\alpha\sin\theta)bq_1,$$
$$T_1^*+\frac{\alpha\sin\theta}{1+\alpha\sin\theta}T_2^*=bq_n \qquad (\alpha=\frac{b}{a}) \tag{2.11}$$

have the following solution which can easily be verified

$$T_1^*=\frac{C+ab\int_0^\theta(q_n\cos\theta-q_1\sin\theta)(1+\alpha\sin\theta)\,d\theta}{a(1+\alpha\sin\theta)\sin\theta};$$
$$T_2^*=\frac{1+\alpha\sin\theta}{\sin\theta}aq_n-\frac{C+ab\int_0^\theta(q_n\cos\theta-q_1\sin\theta)(1+\alpha\sin\theta)\,d\theta}{b\sin^2\theta}. \tag{2.12}$$

If the surface loading is a uniform normal pressure

$$q_1=0,\ q_n=p=\text{const}, \tag{2.13}$$

then the last expression is substantially simplified

$$T_1^*=\frac{bp}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}+\frac{C}{a(1+\alpha\sin\theta)\sin\theta};$$
$$T_2^*=\frac{bp}{2}-\frac{C}{b\sin^2\theta}. \tag{2.14}$$

From the relationships which have been obtained we see, first of all, the obvious necessary condition for the applicability of the membrane theory /195

$$C = 0, \tag{2.15}$$

which provides for the finite value of the forces. This condition has a clear static meaning. Indeed,

$$C = -\frac{1}{2\pi} F_z^0, \qquad \text{(Chapter V, 6.3)}$$

where F_z^0 is the total axial force acting at the parallel ring $\theta = 0$. From figure 33 we see that in this section the vertical force cannot be taken up by force T_1. It can be balanced only by cutting force Q_{1n}. In other words, when the axial force is present, the tangential membrane forces cannot provide for the equilibrium of the shell as a whole. The appearance of infinite forces is a reflection of this situation. From this it follows that conditions (2.15) must be satisfied. These conditions provide for the absence of the axial force in the cross section $\theta = 0$ $(\theta = \pi)$.

However, the fulfillment of this condition far from satisfies the existence of a membrane solution. We become convinced of this immediately by computing the components of flexural deformation. Indeed, for a symmetrically loaded toroidal shell (section 4, Chapter XV)

$$\varepsilon_1^* \equiv \frac{1}{b\cos\theta}\frac{du_\rho^*}{d\theta} + \operatorname{tg}\theta\,\vartheta^* = \frac{1}{Eh}(T_1^* - \nu T_2^*); \tag{2.16}$$

$$\varepsilon_2^* \equiv \frac{u_\rho^*}{a(1+\alpha\sin\theta)} = \frac{1}{Eh}(T_2^* - \nu T_1^*);$$

$$\varkappa_1^* \equiv \frac{1}{b}\frac{d\vartheta^*}{d\theta}; \quad \varkappa_2^* = -\frac{\cos\theta}{a(1+\alpha\sin\theta)}\vartheta^*. \tag{2.17}$$

Here u_ρ^* is the horizontal displacement,

$$\vartheta^* = -\frac{1}{b}\left(\frac{dw^*}{d\theta} - u^*\right) \tag{2.18}$$

is the angle of rotation of the tangent towards the meridian.

We find from relationships (2.16), by using the first of equations (2.11)

$$\vartheta^* = -\frac{1+\alpha\sin\theta}{Eh\alpha\sin\theta}\left[\frac{dT_1^*}{d\theta} + (1+\nu)bq_1\right]. \tag{2.19}$$

In the case of uniform pressure under consideration

$$\vartheta^* = \frac{pb}{2Eh} \frac{\operatorname{ctg}\theta}{1+\alpha\sin\theta}. \tag{2.20}$$

From this and from (2.17) we see that when $\theta \to 0$ ($\theta \to \pi$) the angle of rotation computed by means of the membrane theory together with the components of the flexural deformation increase without limit.

Thus in the neighborhood of the singular points of a toroid the flexural deformation plays a substantial role (for the considered surface loading). /196
Therefore, it becomes illegal to borrow the particular solution from the membrane theory.

We note that this conclusion is valid for another case of surface loading which is of practical importance--the case of the centrifugal force

$$q_\rho = Ba(1+\alpha\sin\theta),\ q_z = 0. \tag{2.21}$$

It is easy to see this if we compute, by means of relationships (2.19), the angle of rotation for the corresponding membrane forces

$$T_1^* = 0,\ T_2^* = Ba^2(1+\alpha\sin\theta)^2. \tag{2.22}$$

An impression may now be reached that in a toroidal shell, in general, there can be no membrane state of stress. However, this is not the case. It is easy to indicate a whole class of surface loads for which the shell does not bend. Indeed, to achieve this it is necessary, for example, in accordance with (2.17) and (2.18), to require that the following equalities be satisfied

$$\vartheta^* = 0,\ \text{i.e.,}\ u^* = \frac{dw^*}{d\theta}. \tag{2.23}$$

In particular, assuming that

$$\begin{gathered} w^* = c\sin\theta,\ u^* = c\cos\theta \\ (u_\rho^* = u^*\cos\theta + w^*\sin\theta = c,\ u_z^* = w^*\cos\theta - u^*\sin\theta = 0), \end{gathered} \tag{2.24}$$

i.e., assuming that during deformation the meridian cross section does not become distorted but experiences a radial displacement, we find

$$\varepsilon_1^*=0;\ \varepsilon_2^*=\frac{c}{a(1+\alpha\sin\theta)};\ \varkappa_1^*=\varkappa_2^*=0. \tag{2.25}$$

By using Hooke's law and the equilibrium equations we become convinced that the given deformation occurs under the action of the following, slightly exotic loading

$$q_1=\frac{Eh}{1-\nu^2}\frac{c}{ab}\frac{\alpha\cos\theta}{(1+\alpha\sin\theta)^2};\quad q_n=\frac{Eh}{1-\nu^2}\frac{c}{ab}\frac{\nu+(1+\nu)\alpha\sin\theta}{(1+\alpha\sin\theta)^2}. \tag{2.26}$$

This loading not only provides for a membrane shell, but provides considerably more--the existence of a state of stress without flexure (see (0.2)).

These examples confirm the proposition expressed above that the occurrence of singular points may be due both to the local properties of the middle surface geometry (asymptotic lines) as well as to the infiniteness of the region.

3. "Local" Singular Points

In Example 3 the singular points are the points on the parallel ring /197 which separates the part of the toroid with a positive Gaussian curvature from the part with the negative Gaussian curvature. Let us consider the neighborhood of asymptotic lines of this type in more detail.

To do this we introduce a parallel system of coordinates (section 8, Chapter VII) and select for the base curve the one on which the Gaussian curvature changes its sign. In the orthogonal system of coordinates obtained in this manner

$$ds^2=ds_\nu^2+B^2ds_t^2. \tag{3.1}$$

In this case

$$A\equiv 1,\ \lim_{s_\nu\to 0}B(s_\nu,\ s_t)=1. \tag{3.2}$$

On the other hand, the Codazzi-Gauss relationships assume the following form (Chapter VII, 7.78-7.79)

$$\begin{aligned}&\frac{\partial k_{\nu t}}{\partial s_\nu}=-\frac{\partial k_\nu}{B\partial s_t}-2\varkappa_t k_{\nu t};\\ &\frac{\partial k_t}{\partial s_\nu}=-\frac{\partial k_{\nu t}}{B\partial s_t}+\varkappa_t(k_\nu-k_t);\\ &\frac{\partial\varkappa_t}{\partial s_\nu}=-\varkappa_t^2-K.\end{aligned} \tag{3.3}$$

Here

$$\varkappa_t = \frac{1}{B}\frac{\partial B}{\partial s_\nu} \tag{3.4}$$

is the so-called geodetic curvature of the coordinate lines; s_ν = const (section 8, Chapter VII), while

$$K = k_\nu k_t - k_{\nu t}^2 \tag{3.5}$$

is the Gaussian curvature of the surface.

Since we are interested in the local properties of the unknown solutions, we shall represent all of the quantities (both the unknown and the assigned) in the form of expansions

$$\Phi(s_\nu, s_t) = \Phi_0 + \Phi_1 \frac{s_\nu}{1!} + \Phi_2 \frac{s_\nu^2}{2!} + \dots, \tag{3.6}$$

and shall use primes to indicate derivatives with respect to s_t.

First of all, assuming that $s_\nu = 0$ in (3.5), we have

$$K_0 = k_{\nu 0} k_{t0} - k_{\nu t 0}^2 = 0. \tag{3.7}$$

Since, in the present case, $k_{t0} = 0$, in order that the Gaussian curvature on the considered asymptotic line be 0, it is necessary and sufficient that $k_{\nu t0} = 0$. Thus, taking

$$k_{t0} = k_{\nu t0} = 0, \tag{3.8}$$

we obtain, by assuming that $s_\nu = 0$ in relationships (3.3) /198

$$k_{\nu 1} = -k'_{\nu 0};\ k_{t1} = \varkappa_{t0} k_{\nu 0};\ \varkappa_{t1} = -\varkappa_{t0}^2. \tag{3.9}$$

Next, differentiating the same relationships by means of (3.4) and setting $s_\nu = 0$ in the resulting equalities, we find by utilizing (3.9) that

$$k_{\nu t2} = -k'_{\nu 1} + 3\varkappa_{t0} k'_{\nu 0}; \quad \varkappa_{t2} = 2\varkappa_{t0}^3 - \varkappa_{t0} k_{\nu 0}^2;$$
$$k_{t2} = \varkappa_{t0} k_{\nu 1} + k'_{\nu 0} - 2\varkappa_{t0}^2 k_{\nu 0}. \tag{3.10}$$

Let us return to expression (3.5). Differentiating it with respect to s_ν we obtain, with the aid of (3.9)

$$K_1 = k_{\nu 0} k_{t1} = \varkappa_{t0} k_{\nu 0}^2. \tag{3.11}$$

Let us first consider the most interesting case when

$$\varkappa_{t0} \neq 0, \; k_{\nu 0} \neq 0 \; (K_1 \neq 0). \tag{3.12}$$

For this case the Gaussian curvature in the neighborhood of the asymptotic line under consideration can be represented in the form

$$K(s_\nu, s_t) \approx K_1(s_t) \cdot s_\nu \tag{3.13}$$

and changes its sign when it passes through the line.

As we have already stated above, all unknown quantities will be sought in the form of expansions of type (3.6). In this case we shall assume that the membrane solution exists if it is possible to contruct a formal solution, i.e., to determine successively the expansion coefficients $\Phi_0(s_t)$, $\Phi_1(s_t)$, ...

In considering the symmetrically loaded toroidal shell in Example 3 we encountered a case when it was impossible to determine the displacements (finite) when the surface loading was arbitrary. Therefore, in this case also we shall begin by considering system (1.3), which can be described in the parallel system of coordinates in the following manner

$$\begin{aligned}
&\frac{\partial u_\nu^*}{\partial s_\nu} + k_\nu w^* = \frac{1}{Eh}(T_\nu^* - \nu T_t^*), \\
&\frac{1}{B}\frac{\partial u_t^*}{\partial s_t} + \varkappa_t u_\nu^* + k_t w^* = \frac{1}{Eh}(T_t^* - \nu T_\nu^*), \\
&\frac{\partial u_t^*}{\partial s_\nu} - \varkappa_t u_t^* + \frac{1}{B}\frac{\partial u_\nu^*}{\partial s_t} - 2k_{\nu t} w^* = \frac{2(1+\nu)}{Eh} S^*.
\end{aligned} \tag{3.14}$$

Since according to the proposition $k_{\nu 0} \neq 0$, it is possible to determine w^* from the first equation. Eliminating it from the remaining two equations we obtain the system

$$-k_t\frac{\partial u_\nu^*}{\partial s_\nu}+\frac{k_\nu}{B}\frac{\partial u_t^*}{\partial s_t}+\varkappa_t k_\nu u_\nu^*=f,$$
$$k_\nu\frac{\partial u_t^*}{\partial s_\nu}-\varkappa_t k_\nu u_t^*+\frac{k_\nu}{B}\frac{\partial u_\nu^*}{\partial s_t}+2k_{\nu t}\frac{\partial u_\nu^*}{\partial s_\nu}=g, \tag{3.15}$$

where

$$Ehf=-k_t(T_\nu^*-\nu T_t^*)+k(T_t^*-\nu T_\nu^*);$$
$$Ehg=2k_{\nu t}(T_\nu^*-\nu T_t^*)+2(1+\nu)k_\nu S^*. \tag{3.16}$$

Assuming that $s_\nu = 0$ in relationships (3.15) and also in those obtained by differentiating with respect to s_ν, we find, taking into account equalities (3.4), (3.6), (3.9) and (3.10) /199

$$k_{\nu 0}u_{t0}^{*\prime}+\varkappa_{t0}k_{\nu 0}u_{\nu 0}^*=f_0;$$
$$k_{\nu 0}u_{t1}^*-\varkappa_{t0}k_{\nu 0}u_{t0}^*+k_{\nu 0}u_{\nu 0}^{*\prime}=g_0; \tag{3.17}$$
$$\varkappa_{t0}(k_{\nu 1}-\varkappa_{t0}k_{\nu 0})u_{\nu 0}^*+k_{\nu 0}u_{t1}^{*\prime}+(k_{\nu 1}-\varkappa_{t0}k_{\nu 0})u_{t0}^{*\prime}=f_1.$$

Eliminating $u_{\nu 0}$ and u_{t1} from these three equations we obtain an equation which determines u_{t0}

$$\left(\frac{u_{t0}^{*\prime}}{\varkappa_{t0}}\right)''+(\varkappa_{t0}u_{t0}^*)'=\frac{f_1}{k_{\nu 0}}-\frac{(k_{\nu 1}-\varkappa_{t0}k_{\nu 0})}{k_{\nu 0}^2}f_0+\left(\frac{f_0}{\varkappa_{t0}k_{\nu 0}}\right)''-$$
$$-\left(\frac{g_0}{k_{\nu 0}}\right)'\equiv\frac{F}{Eh}. \tag{3.18}$$

Continuing the differentiation of equations (3.15) we shall obtain systems of the form

$$k_{\nu 0}u_{tn}^*+\chi_{(n-1)}(u_{\nu(n-1)}^*, u_{t(n-1)}^*, u_{\nu(n-2)}^*, \ldots)=g_{(n-1)};$$
$$-(n-1)\varkappa_{t0}k_{\nu 0}u_{\nu n}^*+\psi_{(n)}(\ldots\ldots)=f_n \qquad (n>2). \tag{3.19}$$

From this we can see that the constructed solution meets no obstacles as long as $u_{\nu 0}^*$, u_{t0}^*, u_{t1}^* are known. The latter, however, are determined only in the case when u_{t0}^* satisfies equality (3.18), which can therefore be appropriately called the condition for the existence of the membrane solution for

singular points on a line characterized by equalities (3.7), (3.8) and (3.12). We shall call such lines transient asymptotic lines. The characteristic property of these curves is that the Gaussian curvature on them, by changing its sign, has a zero of the first order. We note that the function $u^*_{\nu 1}(s_t)$ is arbitrary.

In order to impart a more convenient form to condition (3.18) we express $T^*_{\nu i}$, T^*_{ti}, S^*_i entering into F in terms of $q_{\nu i}$, q_{ti}, q_{ni}. Equations (1.1) may be written in the following manner, using a parallel system of coordinates

$$\begin{aligned}
&\frac{\partial T^*_\nu}{\partial s_\nu}+\frac{\partial S^*}{B\partial s_t}+\varkappa_t(T^*_\nu-T^*_t)=-q_\nu;\\
&\frac{\partial T^*_t}{B\partial s_t}+\frac{\partial S^*}{\partial s_\nu}+2\varkappa_t S^*=-q_t;\\
&k_\nu T^*_\nu-2k_{\nu t}S^*+k_t T^*_t=q_n.
\end{aligned}\tag{3.20}$$

Proceeding as above we successively determined the quantities $T^*_{\nu i}$, T^*_{ti}, S^*_i which enter into F. Eliminating some of the cumbersome, although not complicated steps, we write the final expression for F /200

$$\begin{aligned}
F=&-\frac{1}{6\varkappa_{t0}}\left(\frac{1}{\varkappa_{t0}}S_0^{*\prime}\right)''+\frac{1}{3\varkappa_{t0}}\left(\frac{k'_{\nu 0}}{\varkappa_{t0}k_{\nu 0}}S_0^*\right)''+\frac{2}{3}\frac{k'_{\nu 0}}{\varkappa_{t0}k_{\nu 0}}\left(\frac{1}{\varkappa_{t0}}S_0^{*\prime}\right)'-\\
&-\frac{4}{3}\frac{k'_{\nu 0}}{\varkappa_{t0}k_{\nu 0}}\left(\frac{k'_{\nu 0}}{\varkappa_{t0}k_{\nu 0}}S_0^*\right)'-\frac{2}{3\varkappa_{t0}}(\varkappa_{t0}S_0^*)'+\left(\frac{1}{2\varkappa_{t0}^2}S_0^{*\prime}-\frac{k'_{\nu 0}}{\varkappa_{t0}^2k_{\nu 0}}S_0^*\right)''+\\
&+\left(\frac{k_{\nu 1}}{6\varkappa_{t0}k_{\nu 0}}-\frac{k''_{\nu 0}}{6\varkappa_{t0}^2k_{\nu 0}}-\frac{3}{2}-\nu\right)S_0^{*\prime}+\left(\frac{1}{3}\frac{k'_{\nu 0}k''_{\nu 0}}{\varkappa_{t0}^2k_{\nu 0}}-\frac{2}{3}\frac{k'_{\nu 1}}{\varkappa_{t0}k_{\nu 0}}+\right.\\
&\left.+\frac{k_{\nu 1}k'_{\nu 0}}{\varkappa_{t0}k_{\nu 0}^2}+\frac{7}{3}\frac{k'_{\nu 0}}{k_{\nu 0}}\right)S_0^*-\frac{1}{6\varkappa_{t0}}\left[\frac{(\sigma q_b)_1}{\varkappa_{t0}^2k_{\nu 0}}-\frac{k_{\nu 1}}{\varkappa_{t0}k_{\nu 0}^2}\right]''+\\
&+\frac{2}{3}\frac{k'_{\nu 0}}{\varkappa_{t0}k_{\nu 0}}\left[\frac{(\hat\sigma q_b)_1}{\varkappa_{t0}^2k_{\nu 0}}-\frac{k_{\nu 1}}{\varkappa_{t0}k_{\nu 0}^2}q_{n0}\right]'+\frac{(\hat\sigma q_b)_2}{3\varkappa_{t0}^2k_{\nu 0}}-\frac{1}{3\varkappa_{t0}}\left[\left(\frac{q_{n0}}{k_{\nu 0}}\right)''+\right.\\
&\left.+\left(\frac{q_{\nu 0}}{\varkappa_{t0}}\right)''+q'_{t0}\right]+\frac{4k'_{t0}}{3\varkappa_{t0}k_{\nu 0}}\left[\left(\frac{q_{n0}}{k_{\nu 0}}\right)'+\left(\frac{q_{\nu 0}}{\varkappa_{t0}}\right)'+q_{t0}\right]+\\
&+\left\{\frac{1}{\varkappa_{t0}}\left[\frac{(\hat\sigma q_b)_1}{2\varkappa_{t0}^2k_{\nu 0}}+\left(-\frac{1}{2}\frac{k_{\nu 1}}{\varkappa_{t0}k_{\nu 0}^2}+\frac{1}{k_{\nu 0}}\right)q_{n0}+\frac{q_{\nu 0}}{\varkappa_{t0}}\right]\right\}''+2q_{\nu 0}+\\
&+\frac{q_{\nu 1}}{\varkappa_{t0}}+\left\{-\frac{1}{6}\frac{k''_{\nu 0}}{\varkappa_{t0}^3k_{\nu 0}^2}-\frac{1}{2}\frac{k_{\nu 1}}{\varkappa_{t0}^2k_{\nu 0}^2}+\frac{11}{6}\frac{1}{\varkappa_{t0}k_{\nu 0}}\right\}(\hat\sigma q_b)_1+\\
&+\left\{-\frac{k_{\nu 2}}{3\varkappa_{t0}k_{\nu 0}^2}+\frac{1}{2}\frac{k_{\nu 1}^2}{\varkappa_{t0}^3k_{\nu 0}^2}+\frac{1}{6}\frac{k''_{\nu 0}k_{\nu 1}}{\varkappa_{t0}^2k_{\nu 0}^3}-\frac{k''_{\nu 0}}{3\varkappa_{t0}k_{\nu 0}^2}-\frac{3}{2}\frac{k_{\nu 1}}{k_{\nu 0}^2}+\right.\\
&\left.+\frac{1}{3}\frac{k_{\nu 0}}{\varkappa_{t0}}+\frac{2}{3}\frac{\varkappa_{t0}}{k_{\nu 0}}\right\}q_{n0}+\nu\left\{-\left(\frac{q_{n0}}{k_{\nu 0}\varkappa_{t0}}\right)''+q_{\nu 0}\right\},
\end{aligned}\tag{3.21}$$

where

$$(\hat{\sigma} q_b)_1 = \varkappa_{t0}(-k_{\nu 0} q_{\nu 0} + q_{n1} - \varkappa_{t0} q_{n0});$$
$$(\hat{\sigma} q_b)_2 = -2\varkappa_{t0} k_{\nu 0} q_{\nu 1} - (\varkappa_{t0} k_{\nu 1} + k'_{\nu 0} - 2\varkappa_{t0}^2 k_{\nu 0}) q_{\nu 0} + \\ + \varkappa_{t0} q_{n2} - 2\varkappa_{t0}^2 q_{n1} + (2\varkappa_{t0}^3 - \varkappa_{t0} k_{\nu 0}^2) q_{n0}, \qquad (3.22)$$

while S_0^* is an arbitrary function.

The arbitrary functions (u_{t0}^*, S_0^*) which enter into conditions (3.18)-(3.21) make it possible "almost always" to satisfy this condition, and this, as we have mentioned above, means that we can borrow the particular solution from the membrane theory. Exceptions to this may be cases when due to the meaning of the problem u_{t0}^* and S_0^* are dropped from condition (3.18)-(3.21).

Then the latter transforms into a rather restrictive limitation on the components of the surface load and one which is difficult to summarize. Unfortunately these rare (if we speak of all possible shells and loads of the considered type) exceptions include classes of shells (and loads) which are /201 of practical importance.

For example, let us consider a circular toroidal shell (Chapter XV). For this shell (fig. 33)

$$\varkappa_t = \frac{1}{a}; \quad (') = \frac{\partial}{\partial s_t} = \frac{1}{a}\frac{\partial}{\partial \varphi}; \quad \alpha = \frac{b}{a};$$
$$k_\nu = k_{\nu 0} = \frac{1}{b}; \quad \Phi_k = \frac{1}{b^k}\left(\frac{\partial^k \Phi}{\partial \theta^k}\right)_0; \quad u_t^* \equiv v^*; \quad q_\nu = q_1, \quad q_t = q_2 \qquad (3.23)$$

and condition (3.18)-(3.21) takes the form

$$\frac{1}{3}\frac{\partial^3 S_0^*}{\partial \varphi^3} - \left(\frac{13}{6} + \nu\right)\frac{\partial S_0^*}{\partial \varphi} + \frac{a^3 b}{3}\frac{\partial^2 (\hat{\sigma} q_b)_1}{\partial \varphi^2} + \frac{a^3 l}{3}(\hat{\sigma} q_b)_2 + \\ + \frac{11}{6} a^2 b (\hat{\sigma} q_b)_1 + \left(\frac{2}{3} - \nu\right) b \frac{\partial^2 q_{n0}}{\partial \varphi^2} + \frac{2a}{3}\frac{\partial^2 q_{10}}{\partial \varphi^2} - \frac{a}{3}\frac{\partial q_{20}}{\partial \varphi} + \\ + (2 + \nu) a q_{10} + \frac{a^2}{b}\left(\frac{\partial q_1}{\partial \theta}\right)_0 + \left(\frac{a^2}{3b} + \frac{2b}{3}\right) q_{n0} = \\ = \frac{Eh}{a}\frac{\partial}{\partial \varphi}\left(\frac{\partial^2 v_0^*}{\partial \varphi^2} + v_0^*\right), \qquad (3.24)$$

where now

$$\left(\hat{\sigma}q_b\right)_1 = \frac{1}{ab}\left\{-q_{10} + \left(\frac{\partial q_n}{\partial\theta}\right)_0 - \alpha q_{n0}\right\};$$
$$\left(\hat{\sigma}q_b\right)_2 = \frac{1}{ab^2}\left\{-2\left(\frac{\partial q_1}{\partial\theta}\right)_0 + 2\alpha q_{10} + \right. \tag{3.25}$$
$$\left. + \left(\frac{\partial^2 q_n}{\partial\theta^2}\right)_0 - 2\alpha\left(\frac{\partial q_n}{\partial\theta}\right)_0 + (2\alpha^2 - 1)q_{n0}\right\}.$$

From this we can see that this condition can always be satisfied by selecting the function v_0^* in the proper manner, if there are no symmetric and antisymmetric deformations in the torus. In the latter cases we have, respectively,

$$\Phi(\theta, \varphi) = \Phi(\theta); \quad \Phi(\theta, \varphi) = \Phi_{,1}(\theta)\begin{matrix}\sin\varphi \\ \cos\varphi\end{matrix}, \tag{3.26}$$
$$v^* = 0$$

and the first part of condition (3.24) becomes equal to zero. Let us consider both cases in more detail.

Antisymmetric case

For this case

$$-\frac{\partial^3 S_0^*}{\partial\varphi^3} = \frac{\partial S_0^*}{\partial\varphi} = S_{0,1}^*\begin{matrix}\cos\varphi \\ (-\sin\varphi)\end{matrix} \tag{3.27}$$

and condition (3.24) after reduction by $\begin{matrix}\cos\varphi \\ -\sin\varphi\end{matrix}$ takes the form

$$-\left(\frac{5}{2} + \nu\right)S_{0,1}^* + \Psi\left(q_{n,1},\ q_{\tau,1},\ q_{t,1}\right) = 0. \tag{3.28}$$

From this we see that in an antisymmetrically loaded torus the particular /202 solution which corresponds to the surface load (when S_0^* is determined by equality (3.28)) may be borrowed from the membrane theory.

Symmetric case

Since in this case none of the quantities depend on φ, condition (3.24) is transformed into the following rather rigid limitation on the components of surface loading

$$\frac{1}{3}\left(\frac{\partial^4 q_n}{\partial\theta^2}\right)_0+\frac{7a}{6}\left(\frac{\partial q_n}{\partial\theta}\right)_0+$$
$$+\frac{1}{3}\left(\frac{\partial q_1}{\partial\theta}\right)_0-\frac{a^2}{2}q_{n0}+\left(\frac{5}{6}+\nu\right)aq_{10}=0. \qquad (3.29)$$

It is easy to verify that such loads as uniform pressure (2.13) and centrifugal force (2.21) which are important in applications do not satisfy this condition, while the load (2.26) (when we know that it corresponds to a state of stress without flexure) does satisfy it.

Above we have assumed that conditions (3.12) are satisfied. Although cases when these conditions are not satisfied are of less interest, we shall, nevertheless, consider them for the sake of generality associated with the criteria formulated below for the applicability of the membrane theory. Actually it is necessary to consider two cases.

$$\text{(a)}\ k_{\nu 0}=0, \qquad \varkappa_{t0}\neq 0. \qquad (3.30)$$

Adding this condition to (3.8)-(3.10), we can establish that

$$K_1=0;\ K_2=0;\ K_3=3\varkappa_{10}k_{\nu 1}^2, \qquad (3.31)$$

i.e., the Gaussian curvature is equal to 0 at least with an accuracy up to S_ν^2. Furthermore by computing the mean curvature $H=\frac{1}{2}(k_\nu+k_t)$, we find $H_0=0$. The points at which the Gaussian curvatures and the mean curvatures are equal to 0 are called planar (ref. 17). They can also be determined as points where all normal curvatures become equal to zero. The case considered at this time corresponds precisely to such points.

Turning to the third equation of (3.20) we arrive at the condition $q_{n0}=0$, which is analogous to the one obtained earlier when we considered plates in Example 1. It follows from this that in the neighborhood of the planar point the shell operates like a plate, and the normal component of the surface /203 load cannot be balanced in a membrane fashion.

$$\text{(b)}\ k_{\nu 0}\neq 0, \qquad \varkappa_{t0}=0. \qquad (3.32)$$

When the normal and the geodetic curvatures are equal to 0 at the same time, this means that the spacial curvature of the considered line is also equal to zero (Chapter VII, 7.36)

$$\hat{\sigma}_0 = \sqrt{k_{10}^2 + \varkappa_{10}^2} = 0, \tag{3.33}$$

i.e., the latter is a straight line. Analysis analogous to the one conducted above shows that in this case there is a membrane solution which, however, does not contain arbitrary elements.

4. Continuation

First we obtain relationships of the membrane theory in terms of general coordinates. For this purpose we neglect the cutting forces in the equilibrium equations (Chapter VIII, 4.20), assume that $T_*^{12} = T_*^{21}$ and expand the covariant derivatives by means of relationships (Chapter VII, 5.45). We obtain

$$\begin{aligned}
&\frac{\partial T_*^{11}}{\partial \alpha^1} + \frac{\partial T_*^{12}}{\partial \alpha^2} + (2\Gamma_{11}^1 + \Gamma_{12}^2) T_*^{11} + (\Gamma_{12}^1 + \Gamma_{22}^2) T_*^{12} + \\
&\qquad + \Gamma_{22}^1 T_*^{22} + q^1 = 0; \\
&\frac{\partial T_*^{22}}{\partial \alpha^2} + \frac{\partial T_*^{12}}{\partial \alpha^1} + (2\Gamma_{22}^2 + \Gamma_{12}^1) T_*^{22} + (3\Gamma_{12}^2 + \Gamma_{11}^1) T_*^{12} + \\
&\qquad + \Gamma_{11}^2 T_*^{11} + q^2 = 0; \\
&b_{11} T_*^{11} + 2b_{12} T_*^{12} + b_{22} T_*^{22} + q_n = 0.
\end{aligned} \tag{4.1}$$

In the same way from (Chapter VIII, 2.20 and 6.15) we find

$$\begin{aligned}
&\frac{\partial u_1}{\partial \alpha^1} - \Gamma_{11}^1 u_1 - \Gamma_{11}^2 u_2 - b_{11} w = \frac{1}{Eh} \{(1+\nu) T_{11}^* - \nu a_{11} T^*\}; \\
&\frac{\partial u_2}{\partial \alpha^2} - \Gamma_{22}^2 u_2 - \Gamma_{22}^1 u_1 - b_{22} w = \frac{1}{Eh} \{(1+\nu) T_{22}^* - \nu a_{22} T^*\}; \\
&\frac{\partial u_2}{\partial \alpha^1} + \frac{\partial u_1}{\partial \alpha^2} - 2(\Gamma_{12}^1 u_1 + \Gamma_{12}^2 u_2) - 2b_{12} w = \\
&\qquad = \frac{2}{Eh} \{(1+\nu) T_{12}^* - \nu a_{12} T^*\} \\
&\qquad (T^* = a^{\alpha\beta} T_{\alpha\beta}^* = a_{\alpha\beta} T_*^{\alpha\beta}).
\end{aligned} \tag{4.2}$$

Let us pause briefly on shells with negative Gaussian curvature and refer their middle surfaces to the asymptotic lines. According to (Chapter VII, 7.72-7.73), this means, first of all, that

$$b_{11} = b_{22} = 0. \qquad (4.3)_{1,2}$$

Furthermore, we obtain the following expression for the Gaussian curvature /204 from (Chapter VII, 5.27)

$$K = -\frac{b_{12}^2}{a} = -\frac{1}{\rho^2}. \qquad (4.4)$$

From this we can see that $b_{12} \neq 0$, because we are considering a shell with a negative Gaussian curvature everywhere. In order to be specific we assume that $b_{12} > 0$, and obtain

$$b_{12} = \frac{\sqrt{a}}{\rho} \qquad \left(\rho = \frac{1}{\sqrt{-K}}\right). \qquad (4.3)_3$$

Let us compute the coefficients contained in equations (4.1)-(4.2). First of all, it follows from (Chapter VII, 5.40) that

$$\begin{aligned} \Gamma_{11}^1 + \Gamma_{12}^2 &= \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial \alpha^1} = \frac{\partial \ln \sqrt{a}}{\partial \alpha^1}; \\ \Gamma_{22}^2 + \Gamma_{12}^1 &= \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial \alpha^2} = \frac{\partial \ln \sqrt{a}}{\partial \alpha^2}. \end{aligned} \qquad (4.5)$$

Furthermore, by expanding the Codazzi relationships (Chapter VII, 5.56) with the aid of (4.3) and (Chapter VII, 5.45), we obtain

$$\begin{aligned} \Gamma_{11}^1 - \Gamma_{12}^2 &= \frac{1}{b_{12}} \frac{\partial b_{12}}{\partial \alpha^1} = \frac{\partial \ln b_{12}}{\partial \alpha^1} = \frac{\partial \ln \left(\frac{\sqrt{a}}{\rho}\right)}{\partial \alpha^1}; \\ \Gamma_{22}^2 - \Gamma_{12}^1 &= \frac{1}{b_{12}} \frac{\partial b_{12}}{\partial \alpha^2} = \frac{\partial \ln b_{12}}{\partial \alpha^2} = \frac{\partial \ln \left(\frac{\sqrt{a}}{\rho}\right)}{\partial \alpha^2}. \end{aligned} \qquad (4.6)$$

From this and from (4.5) we find

$$\Gamma_{12}^2 = \frac{\partial \ln \sqrt{\rho}}{\partial \alpha^1} = \frac{1}{\sqrt{\rho}} \frac{\partial \sqrt{\rho}}{\partial \alpha^1}, \quad \Gamma_{12}^1 = \frac{\partial \ln \sqrt{\rho}}{\partial \alpha^2} = \frac{1}{\sqrt{\rho}} \frac{\partial \sqrt{\rho}}{\partial \alpha^2}. \qquad (4.7)$$

By taking into account now relationships (4.3), we obtain from $(4.1)_3$

$$T_*^{12} = -\frac{\rho q_n}{2\sqrt{a}}. \tag{4.8}$$

Next, by substituting the resulting expressions into the first two equations of (4.1), we obtain the system of equations

$$\begin{aligned} \frac{\partial}{\partial \alpha^1}\left(\frac{a}{\sqrt{\rho}} T_*^{11}\right) + \Gamma_{22}^1 \frac{a}{\sqrt{\rho}} T_*^{22} &= X; \\ \frac{\partial}{\partial \alpha^2}\left(\frac{a}{\sqrt{\rho}} T_*^{22}\right) + \Gamma_{11}^2 \frac{a}{\sqrt{\rho}} T_*^{11} &= Y, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} X &= \frac{a}{\sqrt{\rho}}\left[-q^1 + \frac{1}{2\sqrt{a\rho}} \frac{\partial \rho^2 q_n}{\partial \alpha^2}\right]; \\ Y &= \frac{a}{\sqrt{\rho}}\left[-q^2 + \frac{1}{2\sqrt{a\rho}} \frac{\partial \rho^2 q_n}{\partial \alpha^1}\right]. \end{aligned} \tag{4.10}$$

In the same way system (4.2) is reduced to the form

$$\begin{aligned} &\frac{\partial}{\partial \alpha^1}\left(\sqrt{\frac{\rho}{a}} u_1\right) - \Gamma_{11}^2 \sqrt{\frac{\rho}{a}} u_2 = \\ &\quad = \frac{1}{Eh}\left\{(1+\nu) T_{11}^* - \nu a_{11} T^*\right\} \sqrt{\frac{\rho}{a}}; \\ &\frac{\partial}{\partial \alpha^2}\left(\sqrt{\frac{\rho}{a}} u_2\right) - \Gamma_{22}^1 \sqrt{\frac{\rho}{a}} u_1 = \\ &\quad = \frac{1}{Eh}\left\{(1+\nu) T_{22}^* - \nu a_{22} T^*\right\} \sqrt{\frac{\rho}{a}}; \\ &-2\frac{\sqrt{a}}{\rho} w = \frac{2}{Eh}\left\{(1+\nu) T_{12}^* - \nu a_{12} T^*\right\} - \\ &\quad - \rho \frac{\partial}{\partial \alpha^1}\left(\frac{u_2}{\rho}\right) - \rho \frac{\partial}{\partial \alpha^2}\left(\frac{u_1}{\rho}\right). \end{aligned} \tag{4.11}$$

We shall first consider a narrower class of surfaces, the so-called ruled surfaces for which at least one of the families of asymptotic lines consists of straight lines. Let this be the α^1-lines. As we have stated in section 8, Chapter VII, the straight lines are geodetic lines on any surface to which they belong. However, the condition that the lines be geodetic has the form /205

$$\Gamma_{11}^2 = 0. \qquad \text{(Chapter VII, 8.10)}$$

Taking into account this equality we obtain successively from (4.9) and (4.11)

$$T_*^{22}=\frac{\sqrt{\rho}}{a}\left\{\int Y d\alpha^2+\varphi_1(\alpha^1)\right\};$$
$$T_*^{11}=\frac{\sqrt{\rho}}{a}\left\{\int X d\alpha^1+\int \Gamma_{22}^1\left(\int Y d\alpha^2+\varphi_1(\alpha^1)\right)d\alpha^1+\psi_1(\alpha^2)\right\}; \quad (4.12)$$
$$u_1=\sqrt{\frac{a}{\rho}}\left\{\frac{1}{Eh}\int\sqrt{\frac{\rho}{a}}\left[(1+\nu)T_{11}^*-\nu a_{11}T^*\right]d\alpha^1+\psi_2(\alpha^2)\right\};$$
$$u_2=\sqrt{\frac{a}{\rho}}\left\{\frac{1}{Eh}\int\sqrt{\frac{\rho}{a}}\left[(1+\nu)T_{22}^*-\nu a_{22}T^*\right]d\alpha^2+\right. \quad (4.13)$$
$$\left.+\int\Gamma_{22}^1\left[\frac{1}{Eh}\int\sqrt{\frac{\rho}{a}}\left((1+\nu)T_{11}^*-\nu a_{11}T^*\right)d\alpha^1+\psi_2(\alpha^2)\right]d\alpha^2+\right.$$
$$\left.+\varphi_2(\alpha^1)\right\}.$$

Thus, for ruled surfaces with negative Gaussian curvature the integration of the equations from the membrane theory is reduced to a successive calculation of four quadratures. In this case, as we can easily see, the expressions which are obtained are valid for any surface loading.

However, in the case when condition (Chapter VII, 8.10) is not satisfied, the analysis of equations (4.9)-(4.11), which is analogous to the one presented in section 3, leads to the following conclusion: for a shell with a negative Gaussian curvature, the asymptotic lines are not singular.

Now let us consider shells of the general type with zero Gaussian curvature. These shells possess a property that (section 7, Chapter VII) their asymptotic directions blend into one which coincides with one of the principal directions. Let this direction determine the α^1-lines. Then /206

$$\frac{1}{R_\alpha}=\frac{1}{R_{\alpha\beta}}=0; \quad R_\beta=R_2. \quad (4.14)$$

In selecting the Lamé parameters, it is necessary to take into account the Codazzi-Gauss relationships (Chapter VII, 7.78-7.79) which according to (4.14) assume the following form

$$\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right)=0, \quad 0=\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2};$$
$$\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right)+\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right)=0. \quad (4.15)$$

The following equalities follow from the first two relationships

$$A_2=f_1(\alpha_2)R_2; \quad A_1=A_1(\alpha_1), \quad (4.16)$$

and by using these we obtain, from the third relationship

$$A_2 = f_2(\alpha_2)\int A_1(\alpha_1)\,d\alpha_1 + f_3(\alpha_2). \tag{4.17}$$

The functions $f_1(\alpha_2)$, $f_2(\alpha_2)$, $f_3(\alpha_2)$, $A_1(\alpha_1)$ are arbitrary. The simplest coordinate system is the one in which

$$A_1 \equiv 1. \tag{4.18}_1$$

In this case

$$A_2 = \alpha_1 f_2(\alpha_2) + f_3(\alpha_2); \quad R_2 = \frac{A_2}{f_1(\alpha_2)}. \tag{4.18}_{2,3}$$

(Assuming that here $f_2(\alpha_2) = 0$ and $f_3(\alpha_2) = 0$, we arrive, respectively, at the cylindrical and conic shells.)

The relationships which have been obtained make it possible to represent systems (1.1) and (1.3) in the following form

$$\begin{aligned}
&\frac{\partial A_2 T_1^*}{\partial \alpha_1} + \frac{\partial S^*}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1} T_2^* = -A_2 q_1;\\
&\frac{\partial T_2^*}{\partial \alpha_2} + \frac{1}{A_2}\frac{\partial A_2^2 S^*}{\partial \alpha_1} = -A_2 q_2; \quad \frac{T_2^*}{R_2} = q_n;\\
&\frac{\partial u^*}{\partial \alpha_1} = \frac{1}{Eh}(T_1^* - \nu T_2^*);
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&A_2 \frac{\partial}{\partial \alpha_1}\left(\frac{v^*}{A_2}\right) + \frac{1}{A_2}\frac{\partial u^*}{\partial \alpha_2} = \frac{2(1+\nu)}{Eh} S^*;\\
&\frac{1}{A_2}\frac{\partial v^*}{\partial \alpha_2} + \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1} u^* + \frac{w^*}{R_2} = \frac{1}{Eh}(T_2^* - \nu T_1^*).
\end{aligned} \tag{4.20}$$

It is easy to see that as in the case of the ruled surfaces with negative Gaussian curvature considered above, the integration of the specified systems is reduced (for any surface loading) to a successive computation of four quadratures. Thus, on the shells of zero Gaussian curvature, the asymptotic lines are also not singular. /207

In regard to shells with a positive Gaussian curvature, a question of this type does not arise since the asymptotic lines are absent.

5. Infinitely Removed Singular Points

Example 2 (section 2) has shown that infinitely removed points may indeed be singular for shells with zero Gaussian curvature. From relationships (4.12) the same follows for shells with a negative Gaussian curvature. Indeed, let us assume for simplicity that

$$\Gamma_{22}^{1}=0; \quad X=p=\text{const}; \quad Y=q=\text{const}; \quad q_n=0. \tag{5.1}$$

Then we obtain a particular solution from (4.12)

$$T_*^{22}=\frac{\sqrt{\rho}}{a}qa^2; \quad T_*^{11}=\frac{\sqrt{\rho}}{a}pa^1; \quad T_*^{12}=0, \tag{5.2}$$

which corresponds to a finite surface loading and to a loading which increases without limit when $\alpha^1 \to \pm\infty$ or $\alpha^2 \to \pm\infty$.

Let us show that shells with a positive Gaussian curvature possess these properties. For this purpose we consider a spiral surface of a particular type (section 10, Chapter VII) for which

$$R(\theta)=R_0=\text{const}; \quad l=R_0; \quad \rho=R_0\sin\theta, \tag{5.3}$$

i.e., a spiral surface whose generatrix is the arc of a circle with radius R_0 (with its center on the axis of rotation), while the pitch is equal to the diameter of the circle ($L = 2\pi l$) (fig. 24).

Introducing expressions (5.3) into the relationships of section 10, Chapter VII, we obtain

$$\Gamma_{11}^{1}=-\sin\theta\cos\theta; \quad \Gamma_{12}^{1}=\sin^2\theta\cos\theta;$$
$$\Gamma_{22}^{1}=-\sin\theta\cos\theta(1+\sin^2\theta); \tag{5.4}$$

$$\Gamma_{22}^{2}=-\sin^3\theta\cos\theta; \quad \Gamma_{12}^{2}=\sin\theta\cos\theta; \quad \Gamma_{11}^{2}=-\cos\theta; \tag{5.5}$$

$$b_{11}=-R_0\sin\theta; \quad b_{12}=-R_0\cos^2\theta; \quad b_{22}=-R_0\sin^2\theta;$$

$$K=\frac{b}{a}=\frac{\sin^2\theta-\cos^2\theta}{R_0^2}. \tag{5.6}$$

We can see from the last expression that the shell under consideration, when $\frac{\pi}{4} < \theta < \frac{3}{4}\pi$, has a positive Gaussian curvature. Also it contains infinitely

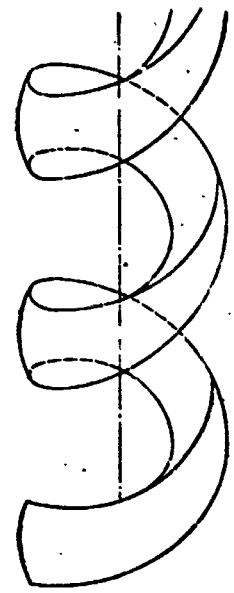

Figure 24

removed points (when $\varphi \to \pm \infty$). Substituting (5.4)-(5.5) into system (4.1) we obtain /208

$$\begin{aligned}
&\frac{\partial T_*^{11}}{\partial \theta} + \frac{\partial T_*^{1\cdot}}{\partial \varphi} - \sin\theta\cos\theta T_*^{11} + 2\sin^2\theta\cos\theta T_*^{12} - \\
&\quad - \sin\theta\cos\theta(1+\sin^2\theta)\,T_*^{22} + q^1 = 0; \\
&\frac{\partial T_*^{22}}{\partial \varphi} + \frac{\partial T_*^{12}}{\partial \theta} - \sin^2\theta\cos\theta T_*^{22} + 2\sin\theta\cos\theta T_*^{12} - \\
&\quad - \cos\theta T_*^{11} + q^2 = 0; \\
&- R_0(\sin\theta T_*^{11} + 2\cos^2\theta T_*^{12} + \\
&\quad + \sin^2\theta T_*^{22}) + q_n = 0.
\end{aligned} \tag{5.7}$$

From the third equation we have

$$T_*^{11} = \frac{q_n}{R_0\sin\theta} - \sin^2\theta T_*^{22} - 2\frac{\cos^2\theta}{\sin\theta}\,T_*^{12}. \tag{5.8}$$

By eliminating T_*^{11} from the first two equations we reduce these to the form

$$-\frac{1}{\sin\theta}\frac{\partial \sin^3\theta T_*^{22}}{\partial\theta} + \frac{\partial T_*^{12}}{\partial\varphi} - \tag{5.9}$$

$$-\frac{2}{\cos\theta}\frac{\partial}{\partial\theta}\left(\frac{\cos^3\theta}{\sin\theta}T_*^{12}\right)=X;$$
$$\frac{\partial T_*^{22}}{\partial\varphi}+\frac{1}{\sin^2\theta}\frac{\partial\sin^2\theta T_*^{12}}{\partial\theta}=Y,$$

where

$$X=-q^1-\frac{1}{R_0}\left[\frac{\partial}{\partial\theta}\left(\frac{q_n}{\sin\theta}\right)-\cos\theta q_n\right];$$
$$Y=-q^2+\operatorname{ctg}\theta\frac{q_n}{R_0}. \tag{5.10}$$

It is significant that in all of the relationships which have been obtained the coefficients do not depend on φ (as in the case of the shells of revolution). This makes it possible to utilize the method of separating the variables (with respect to φ). In particular, when we have a surface load which does not depend on φ we can introduce the concept of a quasisymmetric deformation, interpreting this to mean a deformation state which does not depend on φ. For this state (5.9) is written in the following manner

$$\frac{d\sin^2\theta T_*^{12}}{d\theta}=\sin^2\theta Y;$$
$$\frac{d\sin^3\theta T_*^{22}}{d\theta}=-\sin\theta X-2\operatorname{tg}\theta\frac{d}{d\theta}\left(\frac{\cos^3\theta}{\sin\theta}T_*^{12}\right). \tag{5.11}$$

The solution of the homogeneous system has the form /209

$$T_0^{12}=\frac{A}{\sin^2\theta};\quad T_0^{22}=-3A\frac{\cos^2\theta}{\sin^5\theta}+\frac{B}{\sin^3\theta};$$
$$T_0^{11}=A\frac{\cos^2\theta}{\sin^3\theta}-\frac{B}{\sin\theta}. \tag{5.12}$$

We multiply these expressions by φ and compute the surface load corresponding to the solution

$$T_*^{11}=\left\{A\frac{\cos^2\theta}{\sin^3\theta}-\frac{B}{\sin\theta}\right\}\varphi;\quad T_*^{12}=\frac{A}{\sin^2\theta}\varphi;$$
$$T_*^{22}=\left\{-3A\frac{\cos^2\theta}{\sin^5\theta}+\frac{B}{\sin^3\theta}\right\}\varphi. \tag{5.13}$$

Substituting it, for this purpose, into the left side of equations in system (5.7), we obtain

$$q^1 = -\frac{A}{\sin^2\theta}; \quad q^2 = \frac{3A\cos^2\theta}{\sin^5\theta} - \frac{B}{\sin^3\theta}; \quad q_n = 0 \qquad (5.14)$$

The solution (5.13) which we have constructed has the same properties as the one considered in Example 2: it corresponds to the finite load (5.14) and increases without limit when the shell is elongated.

The examples considered in the present section show that the presence of singularities in the infinitely removed point is not associated with the sign of the Gaussian curvature of its surface.

6. Criteria for the Membrane Property of a Particular Solution

The analysis conducted above (which is not claimed to be mathematically rigorous) makes it possible to refine the conditions given in section 1 for the applicability of the membrane solution as a particular solution of the general (moment) problem:

(a) the shell geometry is smooth (the curvatures of the middle surface and the thickness of the shell are smooth and do not increase substantially when the function is differentiated);

(b) the surface load is smooth;

(c) the middle surface does not contain neighborhoods of singular points.

As it follows from this presentation, the singular points of the middle shell surface contain the following:

1. Planar points, i.e., points at which all of the normal curvatures become equal to zero ($k_{\alpha 0} = k_{\beta 0} = k_{\alpha\beta 0} = 0$). In the neighborhood of a planar point the normal component of the surface load cannot be balanced in a membrane fashion. Very shallow shells may be considered as the neighborhood of a planar point.

2. Transient points, i.e., points on the asymptotic line. When crossing this line the Gaussian curvature changes it sign and has a zero or first /210 order on the line itself (see (3.13)).

3. Infinitely removed points; in this case the presence of such points is not associated with the sign of the Gaussian curvature for the middle surface. It is only important that it be of infinite extent in some direction.

Since we have clearly specified what we mean by singular points, the formulated conditions may be looked upon as sufficient conditions.

Of course they cannot be considered necessary (in the sense adopted in mathematical analysis). Thus, for example, a shell in the neighborhood of a planar point may operate in a membrane state of stress when the normal surface load is absent. In the case of transient points it is possible to borrow the particular solution from the membrane theory, if conditions (3.18)-(3.21) are satisfied. In Example 2 the infinitely long cylindrical shell may be computed for a uniform pressure so long as it has a circular configuration. In the same way, very long shells of other types may operate in a membrane state of stress when their surface loads are selected in a special manner.

7. The Possibility of Utilizing a Membrane Solution as the General Solution of the Moment Theory. Uniqueness Theorem

Let us assume that the criteria of the preceding section are satisfied for a particular problem which we are considering. In other words, the particular solution of the problem at hand may be borrowed from the membrane theory. The question immediately appears as to whether we can satisfy the boundary conditions of the moment problem by having at our disposal a rather limited arbitrary version of the membrane solution. In order to clarify this problem, which is rather complicated in its general formulation, we shall consider first of all the boundary conditions which cannot be satisfied in the membrane fashion.

1. We limit ourselves to the ordinary boundary conditions for which (section 2, Chapter IX) one quantity is assigned from each pair of quantities generalized force--generalized displacement

$$\underset{u_\nu,}{Q_{\nu\nu} = T_{\nu\nu} - \frac{M_{\nu t}}{R_{\nu t}}}; \quad \underset{u_t,}{Q_{\nu t} = T_{\nu t} + \frac{M_{\nu t}}{R_t}}, \quad \underset{w,}{Q_{\nu n} = T_{\nu n} + \frac{dM_{\nu t}}{ds_t}}, \quad \underset{\vartheta_\nu.}{M_{\nu\nu}}, \tag{7.1}$$

In section 1 we became convinced that the membrane solution is associated with negligibly small moments and cutting forces. It follows from this that the membrane solution approximately satisfies the following forced static boundary condition /211

$$Q^*_{\nu n} = 0, \quad M^*_{\nu\nu} = 0. \tag{7.2}$$

Thereby, for ordinary boundary conditions, we are denied the possibility of assigning the generalized displacements w^*, ϑ^*_ν corresponding to these quantities at the edge.

Thus, within the framework of the membrane theory we can satisfy only the tangential static and geometric boundary conditions, assigning one quantity from each pair

$$\begin{matrix} Q_{\nu\nu} = T^*_{\nu\nu}, & Q_{\nu t} = T^*_{\nu t}, \\ u^*_{\nu}, & u^*_{t}. \end{matrix} \tag{7.3}$$

However, even these boundary conditions (as we shall see below) frequently cannot be satisfied.

2. As we have stated in section 1, two arbitrary functions in the membrane solution appear when system (1.1) is integrated and when (1.3) is integrated. The latter are a solution of the homogeneous system

$$\epsilon_\alpha = \epsilon_\beta = \omega = 0 \tag{7.4}$$

and thus represent purely flexural displacements. In particular, they may be the displacements of a shell as a rigid whole. It follows from these statements that if we have at our disposal the arbitrary function of the membrane solution we may satisfy four geometric boundary conditions and only two static conditions.

"The forced assignment of half the boundary conditions has the following physical meaning in terms of the displacements: ... the shell which does not resist flexure is not a 'rigid' body, but rather is a mechanism which freely permits displacements corresponding to pure flexure. By properly fixing the edges, these displacements as a rule may be removed, i.e., the shell can be transformed into a 'rigid' system. The 'forced' boundary conditions are designed for this purpose and must be utilized ..." (ref. 33, p. 118).

3. The problem just considered is closely associated with the uniqueness of the solution for the membrane problem. Indeed, if we neglect the moments and the cutting forces in relationships (Chapter IX, 2.2) we arrive at the integral equality

$$\frac{Eh}{2(1-\nu^2)} \int\int_S \left\{ \frac{1+\nu}{2} (\dot{e}^*_\alpha + \dot{e}^*_\beta)^2 + \frac{1-\nu}{2} (\dot{e}^*_\alpha - \dot{e}^*_\beta)^2 + 2(1-\nu) \left(\frac{\dot{\omega}^*}{2} \right)^2 \right\} \times$$
$$\times AB d\alpha d\beta = \int_s \left\{ \dot{Q}^*_{\nu\nu} \dot{u}^*_\nu + \dot{Q}^*_{\nu t} \dot{u}^*_t \right\} ds_t. \tag{7.5}$$

Here, as in section 2 of Chapter IX, the sign ($\dot{}$) means that the symbols which are marked with it correspond to the difference of two solutions for the membrane theory, for example, $Q^*_{\nu\nu} = Q^{*(1)}_{\nu\nu} - Q^{*(2)}_{\nu\nu}$. /212

Let us assume that we have ordinary boundary conditions. Then it is obvious that the contour integral and the volume integral are equal to zero. Since the expressions under the integral sign are not negative, it follows that

$$\dot{\varepsilon}_\alpha^* = \dot{\varepsilon}_\beta^* = \dot{\omega}^* = 0, \text{ т. е. } \varepsilon_\alpha^{*(1)} = \varepsilon_\alpha^{*(2)}, \ \varepsilon_\beta^{*(1)} = \varepsilon_\beta^{*(2)}, \ \omega^{*(1)} = \omega^{*(2)} \tag{7.6}$$

and

$$T_\alpha^{*(1)} = T_\alpha^{*(2)}, \quad T_\beta^{*(1)} = T_\beta^{*(2)}, \quad S^{*(1)} = S^{*(2)}. \tag{7.7}$$

As far as the displacements are concerned, according to (7.6) they may differ from each other only by a purely flexural displacement. If the geometric conditions are such that pure flexure is eliminated, the displacements coincide. Then the solution is unique.

4. The variation of boundary conditions considered above is not unique. If we introduce the vectors

$$\vec{Q}_\nu^* = Q_{\nu\nu}^* \vec{\nu} + Q_{\nu t}^* \vec{t}, \quad \vec{U}^* = u_\nu^* \vec{\nu} + u_t^* \vec{t}, \tag{7.8}$$

the contour integral in (7.5) may be written in the following manner

$$J^* = \int_\xi \vec{Q}_\nu^* \vec{U}^* ds_t. \tag{7.9}$$

It follows from this that the uniqueness of the solution (with the qualifications made above) guarantees the boundary conditions

$$(\vec{Q}_\nu^*)_\xi = (\vec{Q}_\nu^*)_\xi^0, \quad (\vec{U}^*)_\eta = (\vec{U}^*)_\eta^0 \tag{7.10}$$

where ξ and η are mutually orthogonal directions.

We can also obtain other versions of boundary conditions. Thus, by writing (7.9) in the form

$$J^* = \int_\xi \{Q_{\nu\nu}^* \vec{\nu} + Q_{\nu t}^* \vec{t}\} \vec{U}^* ds_t. \tag{7.11}$$

and replacing $\vec{\nu}$ with its expression which follows from (Chapter VII, 7.17_1)

$$\vec{\nu} = -\frac{1}{\varkappa_t} \frac{d\vec{t}}{ds_t} - \frac{1}{\varkappa_t R_t} \vec{n}, \tag{7.12}$$

we obtain, integrating in parts, /213

$$J^* = \int_g \left\{ \left[\dot{Q}^*_{vt} + \frac{d}{ds_t}\left(\frac{Q^*_{vv}}{\varkappa_t}\right)\right] u^*_t + \dot{Q}^*_{vv}\left[-\frac{\dot{w}^*}{\varkappa_t R_t} + \frac{1}{\varkappa_t}\vec{t}\,\frac{d\vec{U}^*}{ds_t}\right]\right\} ds_t.$$

By using the equality (Chapter VIII, 3.14 and 3.15)

$$\vec{t}\cdot\frac{d\vec{U}}{ds_t} = \dot{\varepsilon}_{tt} = \frac{d\dot{u}_t}{ds_t} + \varkappa_t \dot{u}_v + \frac{\dot{w}}{R_t} \tag{7.13}$$

we may impart the following final form to J^*

$$J^* = \int_g \left\{ \left[\dot{Q}^*_{vt} + \frac{d}{ds_t}\left(\frac{Q^*_{vv}}{\varkappa_t}\right)\right] u^*_t + \dot{Q}^*_{vv}\left[\dot{u}^*_v + \frac{1}{\varkappa_t}\frac{du^*_t}{ds_t}\right]\right\} ds_t, \tag{7.14}$$

from which we can see that the quantities

$$\begin{array}{ll} Q^{*\prime}_{vt} = Q^*_{vt} + \dfrac{d}{ds_t}\left(\dfrac{Q^*_{vv}}{\varkappa_t}\right), & Q^*_{vv} \\ u^*_t & u^{*\prime}_v = u^*_v + \dfrac{1}{\varkappa_t}\dfrac{du^*_t}{ds_t} \end{array} \tag{7.15}$$

may be looked upon as generalized forces and displacements which correspond to each other.

In exactly the same way, by replacing $\vec{t}$ in (7.11) with its expression

$$\vec{t} = \frac{1}{\varkappa_t}\frac{d\vec{v}}{ds_t} - \frac{1}{\varkappa_t R_{vt}}\vec{n}, \tag{7.16}$$

we obtain one other version for the generalized forces and displacements

$$\begin{array}{ll} Q^{*\prime}_{vv} = Q^*_{vv} - \dfrac{d}{ds_t}\left(\dfrac{Q^*_{vt}}{\varkappa_t}\right), & Q^*_{vt} \\ u^*_v & u^{*\prime}_t = u^*_t - \dfrac{1}{\varkappa_t}\dfrac{du^*_v}{ds_t} \end{array} \tag{7.17}$$

From these expressions we can see that they are applicable only in the case when $\varkappa_t \neq 0$, i.e., when the boundary contour does not coincide with the geodetic

line. For cases when the contour coincides with the coordinate β-line, according to (Chapter VII, 8.58 and 7.69) ($\gamma = 0$)

$$\varkappa_t = \frac{1}{AB}\frac{\partial B}{\partial \alpha}; \quad R_t = R_\beta; \quad \frac{d}{ds_t} = \frac{1}{B}\frac{\partial}{\partial \beta}. \tag{7.18}$$

In particular, for shells of revolution which are bounded by parallel circles (section 1, Chapter V)

$$\begin{aligned} &A = R_1;\ B = R_2 \sin\theta;\ \frac{\partial B}{\partial \alpha} = R_1 \cos\theta;\ \varkappa_t = \frac{\cos\theta}{R_2 \sin\theta};\\ &u_\nu^* = u^*;\ u_t = v^*;\ Q_{\nu\nu}^* = T_1^*;\ Q_{\nu t}^* = S^*;\ R_t = R_2;\ \frac{d}{ds_t} = \frac{1}{R_2 \sin\theta}\frac{\partial}{\partial \varphi} \end{aligned} \tag{7.19}$$

and

$$Q_{\nu t}^{*'} = S^* + \frac{1}{\cos\theta}\frac{\partial T_1^*}{\partial \varphi}; \quad u_\nu^{*'} = u^* + \frac{1}{\cos\theta}\frac{\partial v^*}{\partial \varphi}; \tag{7.20}$$

$$Q_{\nu\nu}^{*'} = T_1^* - \frac{1}{\cos\theta}\frac{\partial S^*}{\partial \varphi}; \quad u_t^{*'} = v^* - \frac{1}{\cos\theta}\frac{\partial u^*}{\partial \varphi}. \tag{7.21}$$

For a shallow spherical shell ($\cos\theta \approx 1$) the quantities /214

$$Q_{\nu t}^{*'} = S^* + \frac{\partial T_1^*}{\partial \varphi}, \quad Q_{\nu\nu}^{*'} = T_1^* - \frac{\partial S^*}{\partial \varphi} \tag{7.22}$$

were obtained in the interesting work of E. Reissner (ref. 282; see also ref. 268).

5. Between the membrane (it would be more accurate to use the term "without flexure") and purely flexural states,[1] there is close association. By using the static-geometric analogy it is possible to show that for every infinitely small bending of the surface there is a certain membrane state of stress, and vice versa. This is discussed in more detail in references 38, 251 and 256. Below we shall limit ourselves and consider only the manifestation of this association as it pertains to the understanding of the present section.

Let us assume that the quantities with the sign (2) in relationships of section 1, Chapter IX correspond to purely flexural displacements, while those with a sign (1) correspond to a state of stress without flexure, i.e.,

[1] Geometrists call it the infinitely small bending of a surface.

$$\varepsilon_1^{(2)} = \varepsilon_2^{(2)} = \omega^{(2)} = 0; \tag{7.23}$$

$$\varkappa_1^{(1)} = \varkappa_2^{(1)} = \tau^{(1)} = 0; \quad M_{\nu}^{(1)} = 0; \quad Q_{\nu n}^{(1)} = Q_n^{(1)} = 0. \tag{7.24}$$

Equating expression (Chapter IX, 1.4) to (Chapter IX, 1.5), changing the signs and taking into account equalities (7.23)-(7.24), we obtain

$$\int\int_S \vec{q} \cdot \vec{U}^{\varkappa} AB d\alpha d\beta + \int_{\varepsilon} \vec{Q}_{\nu}^{*} \cdot \vec{U}^{\varkappa} ds_t = 0, \tag{7.25}$$

where we now have

$$\vec{Q}_{\nu}^{*} = Q_{\nu n}^{*} \vec{\nu} + Q_{\nu t}^{*} \vec{t}. \tag{7.26}$$

An important necessary condition for the solution of membrane theory equations follows from equality (7.25): the external load acting on the shell must not perform any work during any purely flexural displacements of the shell (of course, the displacements must be geometrically permissible).

In particular, if we take the vector of rigid displacement for $\vec{U}^{\varkappa}$ (the so-called trivial flexure)

$$\vec{U}^{\varkappa} = \vec{U}^{\circ} + \left[\vec{\Omega}^{\circ}, \vec{r} - \vec{r}^{\circ}\right], \tag{7.27}$$

then, since the (constant) vectors $\vec{U}^{\circ}$ and $\vec{\Omega}^{\circ}$ are arbitrary, the well-known conditions for the equilibrium of the middle surface as a rigid whole follow from (7.25)

$$\int\int_S \vec{q} AB d\alpha d\beta + \int_{\varepsilon} \vec{Q}_{\nu}^{*} ds_t = 0; \tag{7.28}$$

$$\int\int_S \left[\vec{r} - \vec{r}_0; \vec{q}\right] AB d\alpha d\beta + \int_{\varepsilon} \left[\vec{r} - \vec{r}_0; \vec{Q}_{\nu}^{*}\right] ds_t = 0. \tag{7.29}$$

Let us now return to condition (7.25). As we have already stated, /215 for every specific shell it isolates a class of external loads which perform no work for purely flexural displacements which are geometrically permissible. If the external load satisfies this condition for some form of pure flexure, it would seem that it is not necessary to impose additional constraints on the edge of the shell which eliminate this form of pure flexure. However, this is not the case, because such a "membrane equilibrium" is not stable. It is

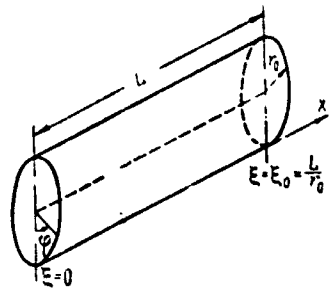

Figure 25

sufficient to have a random load of insignificant value which performs work for the specific form of pure flexure, and the shell loses the possibility of balancing this load in a membrane fashion. This is particularly important because our knowledge of the load which actually acts on the shell is approximate.

Let us clarify this thought by means of a simple example. For this purpose we consider a closed circular cylindrical shell (fig. 25) which is free of a surface load. Solving the system of equations

$$r_0\varepsilon_1 \equiv \frac{\partial u}{\partial \xi} = 0, \quad r_0\varepsilon_2 \equiv \frac{\partial v}{\partial \varphi} + w = 0, \quad r_0\omega \equiv \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \xi} = 0, \tag{7.30}$$

we obtain the following expressions for the purely flexural displacements

$$u = u_0(\varphi); \quad v = -\xi u_0'(\varphi) + v_0(\varphi), \quad w = \xi u_0''(\varphi) - v_0'(\varphi). \tag{7.31}$$

where, if we do not set geometric boundary conditions, $u_0(\varphi)$ and $v_0(\varphi)$ are arbitrary periodic functions. For the shell under consideration, condition (7.25) takes the form

$$\int_0^{2\pi} \{T_1^+(\varphi)u_0(\varphi) + S^+(\varphi)[-\xi_0 u_0'(\varphi) + v_0(\varphi)]\}\, d\varphi - \int_0^{2\pi} \{T_1^-(\varphi)u_0(\varphi) + S^-(\varphi)v_0(\varphi)\}\, d\varphi = 0 \tag{7.32}$$

$$\begin{pmatrix} T_1^-(\varphi) = T_1^*(0, \varphi) & T_1^+(\varphi) = T_1^*(\xi_0, \varphi) \\ S^-(\varphi) = S^*(0, \varphi) & S^+(\varphi) = S^*(\xi_0, \varphi) \end{pmatrix}.$$

Let us assume the static boundary conditions /216

$$S^+ = S^- = 0. \tag{7.33}$$

Then equality (7.32) is simplifi. .

$$\int_0^{2\pi} [T_1^*(\varphi) - T_1^-(\varphi)]\, u_0(\varphi)\, d\varphi = 0. \tag{7.34}$$

From this equality and from the fact that $u_0(\varphi)$ is arbitrary, it follows that

$$T_0^+(\varphi) = T_1^-(\varphi). \tag{7.35}$$

(In the rather simple case considered, this equality may be obtained directly by integrating the system

$$\frac{\partial T_1^*}{\partial \xi} + \frac{\partial S^*}{\partial \varphi} = 0, \quad \frac{\partial S^*}{\partial \xi} + \frac{\partial T_2^*}{\partial \varphi} = 0, \quad T_2^* = 0, \tag{7.36}$$

whose solution is given by the expressions

$$T_2^* = 0, \quad S^* = S_0(\varphi), \quad T_1^* = -\xi S_0'(\varphi) + T_0(\varphi).) \tag{7.37}$$

(Indeed, by subjecting the latter to conditions (7.33) we obtain

$$T_1^+(\varphi) = T_1^-(\varphi) = T_0(\varphi).) \tag{7.38}_1$$

Let us consider in more detail the case when

$$T_1^+(\varphi) \neq T_1^-(\varphi), \quad S^+(\varphi) = S^-(\varphi) = 0. \tag{7.38}_2$$

It is easy to see that for this type of loading, condition (7.25) or, what is the same, (7.34) is violated, for example, for the following form of pure flexure

$$u(\xi, \varphi) = K(T_1^+(\varphi) - T_1^-(\varphi)) \quad (K = \text{const}). \tag{7.39}$$

In connection with these statements, when the considered integral is equal to zero, the external edge load $(7.38)_2$ performs work for the purely flexural displacement (7.39). In other words, the shell cannot resist this flexure in a membrane fashion. The formulated necessary condition for the solution of the

membrane theory is disrupted, and the equilibrium of the shell cannot be achieved without the introduction of moments and cutting forces. The same conclusion is reached by considering the solution of form $(7.38)_1$, which signals the impossibility of obtaining a membrane solution for the considered problem. (It is recommended that the reader examine this instructive problem in its moment formulation, for example, using the following boundary conditions

$$T_1^+(\varphi)=A\cos 2\varphi,\quad T_1^-(\varphi)=-A\cos 2\varphi,\quad S^+(\varphi)=S^-(\varphi)=0,$$
$$w(0,\ \varphi)=w(\xi_0,\ \varphi)=0,\quad \vartheta_1(0,\ \varphi)=\vartheta_1(\xi_0,\ \varphi)=0, \tag{7.40}$$

which satisfy the equilibrium conditions of the shell as a whole.)

The picture will be entirely different if the boundary conditions /217 eliminate the displacement due to pure flexure. As we have already stated, in this case condition (7.25) is reduced to the sole requirement that the external load be self-balanced. If, on the other hand, the boundary conditions also remove the displacements of the shell as a rigid whole, (trivial flexure), then no conditions whatsoever are imposed on the external load. It is automatically balanced by the support reactions which realize the fixing. In practice, an effort is always made to fix the edges in such a way as to remove displacements due to pure flexure.

In conclusion, we emphasize once again that condition (7.25) is necessary, but not sufficient for the formal solution of the equilibrium equations for assigned boundary conditions. In particular, it is easy to see this by means of the given example. Indeed, it is easy to see directly that the external load

$$T_1^+\neq T_1^-,\quad S^+=S^-=0,\quad q_1=q_2=q_n=0 \tag{7.41}$$

in the case of geometric boundary conditions

$$u|_{\xi=0}=u|_{\xi=\xi_0}=0 \tag{7.42}$$

satisfies equation (7.32). However, this membrane state, according to (7.37), cannot be realized, regardless of how the arbitrary functions $S_0(\varphi)$ and $T_0(\varphi)$ are selected.

6. Let us consider shells whose boundary contours of the middle surfaces coincide with the asymptotic lines. As we clarified in section 4, all of the arbitrary functions in the membrane solution for shells of zero Gaussian curvature depend on the coordinate α_2. Thus, when the boundaries coincide with the asymptotic line (along which the coordinate α_1 varies), it is not possible to satisfy a single boundary condition.

For shells with a negative Gaussian curvature and with rectilinear asymptotic lines only two of the arbitrary functions depend on α^1 (section 4). Therefore, only two boundary conditions can be satisfied on the asymptotic lines. This property is also common to shells with a negative Gaussian curvature of the general form. Indeed, if we seek, for example, the solution of equation (4.11) (in the same manner as in section 3) in the form of expansions

$$\sqrt{\frac{\rho}{a}}\,u_1 = U_0 + \frac{\alpha^1}{1!}U_1 + \frac{(\alpha^1)^2}{2!}U_2 + \ldots,$$
$$\sqrt{\frac{\rho}{a}}\,u_2 = V_0 + \frac{\alpha^1}{1!}V_1 + \frac{(\alpha^1)^2}{2!}V_2 + \ldots,$$

it is easy to see that only one function ($U_0(\alpha^2)$) may be assigned in an /218

arbitrary manner. Furthermore, only one arbitrary function exists in the analogous investigation of system (4.9).

The transient asymptotic line has interesting properties. For this line, as we have established in section 3, the arbitrary functions S_0^*, v_0^* are associated with the membrane condition which we can obtain from (3.18)-(3.21) by setting $q_{\nu i} = q_{ti} = q_{ni} = 0$ in these. Therefore, in this case we can satisfy only one boundary condition.

With regard to very long shells, the direct solution of homogeneous membrane equations is not applicable to them. The fact is that these solutions do not possess the important membrane solution attribute of being damped when we move away from the boundary. However, as we shall show in section 10 of the next chapter, the membrane solutions may be utilized to construct approximate solutions in these cases.

7. The shortcoming associated with the arbitrary nature of the solution and the numerous qualifications (part of which are considered in this section) force us to give a negative answer to the second question raised at the beginning of the chapter: as a rule, the membrane solution does not satisfy all of the boundary conditions and therefore cannot be considered as the total solution of a moment problem.

This situation only partially reduces the value of the membrane theory. Indeed, it is usually possible to break a shell up into several parts such that the external load and geometry are smooth over each part. As we established in section 1, the membrane solution may be adopted as the particular solution for each of such parts. Thus, the membrane theory gives us a particular solution of merit.

Furthermore, when designing a shell, operating under assigned conditions, the designer is usually able (within certain limits) to assign at his own discretion the form of the middle surface, the law for the variation of thickness

and the boundary elements which support the edge. This makes it possible in a whole series of cases of practical interest to produce shells which operate in a state close to the membrane state of stress: domes (refs. 33 and 165), vessels (ref. 33). For such shells the membrane solution completely solves the problem of stress analysis.

Usually, however, it is not possible to design a shell which operates entirely in the membrane state. In this case the problem before the designer is complicated by a whole series of auxiliary requirements (from the point of view of strength), such as the technology of fabrication, economic factors, dimensions and so forth. Therefore, the problem of the optimum design of a given specific shell may be considered solved if a large part of the shell op- /219
erates in a state close to a membrane state, while the mixed state of stress is localized in narrow zones adjoining the points where there is a sharp change in the shell geometry and in the external loading or if it is localized to the edges of the shell. In this case the design consists of combining the membrane solution with a solution of the edge effect type (section 6, Chapter XIII).

The uniform (across the thickness of the shell) operation of the material in the direction of maximum structural rigidity makes it possible to make the membrane shells lighter and more economical, compared with shells with a mixed state (particularly when the moments are high). This establishes the practical value of membrane shells and the practical importance of the membrane theory as the basic design approach in the theory of shells.

However, we should state that there are whole classes of shells which, in view of their function, operate in a state of stress which is far from a membrane state: thermal expansion tanks, curved tubes, elastic elements of certain instruments, musical instruments. For such shells it is necessary to devise special design methods which take into account the special features associated with their operation.

8. Membrane Criteria. Brief Historic Survey

The basic work on the membrane theory may be considered to be that of Lamé and Clapeyron (ref. 57) who considered the symmetric deformation of shells of revolution in 1928. In the general formulation the relationships of membrane theory were considered by Beltrami (ref. 277) and Lecornu (ref. 279), who were apparently the first to associate the membrane theory with the question of infinitely small flexures of the surface. The clarification of the basic relationships was achieved much later in the works of V. V. Sokolovskiy (ref. 169) and Yu. N. Rabotnov (ref. 168).

The wide application of thin-walled structures in the building industry at the beginning of this century revived the interest in the membrane theory. In designing specific shells by means of the membrane theory, the question naturally always arose concerning the validity of neglecting moments and cutting forces (and making other simplifications). Furthermore, engineers frequently encountered cases when the application of the membrane theory produced results which were completely incorrect. It became necessary to

formulate the conditions which, if satisfied, would guarantee the valid application of the membrane theory to a particular shell under consideration.

The works of Lagally (ref. 278), Löbell (ref. 280), Dinnger (ref. 165), Pöschl (ref. 281), the discussion of A. A. Gvozdev with V. E. Novodvorskiy (refs. 254, 266 and 255) and many other works helped to gain a broader /220 understanding of the operating nature of thin shells and helped to formulate the classical criteria for the membrane state.

These are presented in the books of V. V. Novozhilov in clear distinct form (refs. 31 and 33):

(a) the boundaries of the shell are free from cutting forces and moments, while the normal displacements and the rotations of shell edges are not restricted;

(b) the geometric parameters of the shell, the components of the surface and edge loading are sufficiently smooth and do not increase substantially when the function is differentiated;

(c) the tangential geometric boundary conditions limit the displacements due to pure flexure.

In this case a clear understanding of the fact that these criteria are insufficient is significant. It is possible to have exceptions (singular cases).

A great deal of attention is devoted in the works of A. L. Gol'denveyzer to the criteria for the membrane theory. In his monograph (ref. 11) he formulated the following five conditions for the membrane state:

(1) the distortion lines must not form on excessively dense mesh;

(2) none of the distortion lines must touch the asymptotic lines of the surface (at any point);

(3) the external loads and the reaction forces must not have an excessively large variability index (i.e., must be smooth functions which do not increase appreciably during differentiation);

(4) the middle surface must not be singular;

(5) the tangential geometric boundary conditions must provide for the rigidity of the middle surface (A. L. Gol'denveyzer calls this the Novodvorskiy hypothesis).

Conditions 3 and 5 were mentioned above. Let us clarify the other three. The term "distortion lines" used by A. L. Gol'denveyzer means:

(a) edge of the shell;

(b) the discontinuity line for the components of the external load or for some of its derivatives;

(c) lines where there is a break or a discontinuous variation in curvature;

(d) lines where there is a discontinuous variation in the rigidity or the thickness of a shell. (It would be more correct to include the zones of sufficiently rapid variation in the geometry of the shell, even though this variation is not discontinuous.)

Thus condition 1 shows that the criteria of A. L. Gol'denveyzer (unlike the "classical" criteria) do not strive to provide for the membrane state of the shell as a whole. A provision is made for the edge effect. In this connection condition 2 requires that the edge effect be simple, so that a mixed state of stress can exist in the narrow zones adjoining the lines of distortion /221 (section 3, Chapter XIII). It is interesting to note that this approach was characteristic of V. E. Novodvorskiy, who used this rule to break down shells into statically variable and statically nonvariable (ref. 30).

Condition 4 was explained by A. L. Gol'denveyzer in the following manner: "It is difficult for us to specify general criteria which could be used to predict the occurrence of a special basic state of stress, and we limit ourselves to the enumeration of those cases which have been encountered in practice. These include: (a) cylindrical shells which extend infinitely in the direction of the generatrix; (b) shells whose middle surface has sharp points (for example, a nontruncated cone); (c) shells whose middle surface touches the plane along a closed curve (for example, a torus)." (Ref. 11, p. 358.)

A. L. Gol'denveyzer calls his conditions sufficient. It is hardly possible to agree with this. Indeed, in order to affirm that a membrane state of stress exists in a shell, we must be convinced (criterion 4) that the middle surface is not singular. Since a clear determination of this concept is not given, there is always the danger of augmenting the list of singular middle surfaces evolving from experience with yet one more example. It is not difficult to augment the above list by including, for example, bent tubes, very shallow shells, very long shells, which differ from cylindrical shells (section 5). In a later work (ref. 257) A.L. Gol'denveyzer proposes an integral criterion for the membrane state. We shall discuss it briefly. Let

$$\{\varepsilon_\alpha,\ \varepsilon_\beta,\ \omega\} = O\left\{\left(\frac{h}{\lambda}\right)^k \lambda x_\beta,\ \left(\frac{h}{\lambda}\right)^k \lambda x_\beta,\ \left(\frac{h}{\lambda}\right)^k \lambda \tau\right\}. \qquad (8.1)$$

(Here the symbol $O\{\ldots\}$ represents a quantity of the same order, while λ is some characteristic dimension of the middle surface.) For the stresses

$$\sigma_{T_\alpha} = \frac{T_\alpha}{h} = \frac{E}{1-\nu^2}(\varepsilon_\alpha + \nu\varepsilon_\beta),\ \sigma_{M_\alpha} = \frac{6M_\alpha}{h^2} = \frac{Eh}{2(1-\nu_2)}(x_\alpha + \nu x_\beta)$$

. . . . . .

the following evaluations are valid

$$\left\{\sigma_{T_\alpha},\ \sigma_{T_\beta},\ \sigma_S\right\}=O\left\{E\lambda\varkappa\left(\frac{h}{\lambda}\right)^k\right\};$$
$$\left\{\sigma_{M_\alpha},\ \sigma_{M_\beta},\ \sigma_H\right\}=O\left\{E\lambda\varkappa\left(\frac{h}{\lambda}\right)\right\} \qquad (8.2)$$
$$(\varkappa=O\{\varkappa_\alpha,\ \varkappa_\beta,\ \tau\}).$$

From this we can see that when $k > 1$ flexural stresses prevail in the entire shell, and when $k < 1$ the tangential stresses prevail. By using equalities (8.1) the energy of deformation (Chapter VIII, 8.42) may be written in the form /222

$$V=\left[1+O\left\{\left(\frac{h}{\lambda}\right)^{2(1-k)}\right\}\right]V_{\text{tangential}}. \qquad (8.3)$$

The first term corresponds to the energy of tangential deformation and the second to the energy of flexural deformation. Therefore, if $k > 1$ everywhere in the shell, the flexural state of stress prevails as a whole in the shell, while for $k < 1$ the membrane stress prevails.

A. L. Gol'denveyzer proceeded from the integral expressions (8.3) and called the quantity k the characteristic deformation. With this approach it does not follow from the inequality $k > 1$ that flexural stresses prevail at all points of the shell. We can only maintain that the zones where flexural stresses prevail occupy a large part of the shell or that there is an intense concentration of the moment state in the narrow zones.

The integral criterion is formulated in the following manner: the state of stress of thin elastic shells will be a membrane state if, among the deformations which permit geometric boundary conditions and whose displacements cause the external forces to conduct appreciable work, there are no deformations with a characteristic deformation greater or equal to unity.

As we can see from this determination, the proposed criterion has a negative character and is designed to detect cases when the membrane state of stress is absent. We can trace the basic idea of the author in the work. A formal removal of purely flexural displacements by fixing a small region of the boundary is insufficient for the shell as a whole to operate in a state of stress which is close to the membrane state. In addition to this the author utilized his criterion for changing the meaning of five membrane conditions.

The monograph by I. N. Vekua (ref. 251) considers a "purely mathematical question concerning the establishment of conditions which provide for the existence and uniqueness of static forces," which satisfy the system of equations from the membrane theory.[1] A wide use is made of the relationship noted

[1] Reference here is made to equations (1.1).

above between the membrane theory and the infinitely small flexure of the surface. Generalized analytic functions are used to obtain a series of curious results for convex shells with one or more holes.

In the works of Metskhovrishvili (refs. 262 and 264) an effort is /223
made to obtain the necessary and sufficient conditions for the existence of a membrane state of stress in toroidal shells. Unfortunately, these conditions are interpreted to mean only the conditions for the existence of a finite solution of the membrane equilibrium equations (1.1). As we have established in Example 3 (section 2), this is quite insufficient for the existence of a membrane solution.

A key place in both articles is occupied by a solution obtained by the author (fig. 26)

$$T_1^* = T_2^* = 0;\ S^* = \frac{C}{(1+\alpha\sin\theta)^3}. \tag{8.4}$$

For a closed shell it represents an example of a state of stress corresponding to zero external loading. Indeed, if we represent it in terms of the membrane equations (1.1), we become convinced that this solution corresponds to $q_1 = q_2 = q_n = 0$. The edge forces, on the other hand, are absent because there are no edges. In addition, solution (8.4) is periodic and is finite everywhere. At first glance the example is paradoxical.

Without associating the membrane problem with the determination of displacements, the author deprives himself of the possibility to explain this example correctly and erroneously associates the paradox with the presence of a surface moment (section 2, Chapter III). At the same time the imaginary paradox has a rather simple explanation. Indeed, solution (8.4) corresponds to the symmetric torsion of a toroidal shell (the second symmetric case). In this case (section 2, Chapter XV), constant C is associated with the twisting moment by means of the following relationship

$$B_z^0 = 2\pi a^2 C.$$

The displacement which corresponds to this moment is given by the equation

$$\left(U_z^1 = 0,\ \overline{U}_z^1 = -\frac{B_z^0}{Ehc}\right) \qquad \text{(Chapter XV, 3.5')}$$

$$v = \frac{2(1+\nu)}{Eha} B_z^0 (1+\alpha\sin\theta)\frac{\alpha}{2\pi}\int_0^{\theta}\frac{d\tau}{(1+\alpha\sin\tau)^3}.$$

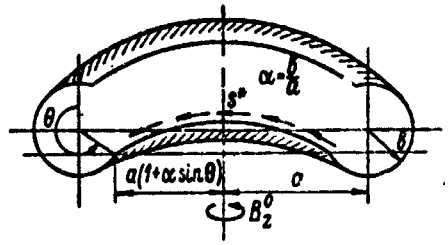

Figure 26

In the case when the torus is closed with respect to θ, the following discontinuity in the displacement takes place at the cross section $\theta = \theta_0$ according to (Chapter XV, 6.13 and 6.21) /224

$$[v] = v(\theta_0 + 2\pi) - v(\theta_0) = \frac{2(1+\nu)}{Eha}(1+\alpha\sin\theta_0)\alpha J_3(\alpha) = \\ = \frac{2(1+\nu)}{Eha}(1+\alpha\sin\theta_0)\frac{\alpha\left(1+\frac{\alpha^2}{2}\right)}{(1-\alpha^2)^{3/2}}. \tag{8.5}$$

It follows from this that the existence of the state of stress (8.4) in the shell is associated with the fact that in the shell cut along the parallel circle $\theta = \theta_0$ the cut edges were joined after their relative twisting by an angle $\Omega = \frac{[v]}{a(1+\alpha\sin\theta_0)}$. In a shell of this type, which has undergone preliminary deformation, stresses will exist even when external loads are absent.

In concluding the short survey we note that in references 33, 11, 163, 269 and 263 there is a detailed examination of conditions which provide for the rigidity of the middle surface in shells of revolution.

CHAPTER XIII. THE ASYMPTOTIC ANALYSIS OF THE EQUATIONS OF THE THEORY OF SHELLS. BASIC STATE OF STRESS TYPES. EDGE EFFECT

In section 6, Chapter VI we carried out an asymptotic investigation /225
of the characteristic equation for the circular cylindrical shell. It appears enticing to transfer the methods presented in that section to a shell of general form. However, substantial difficulties are associated with this. First of all, we have to deal with a system of equations rather than with one equation. In the second place, and this is particularly important, the coefficients of the equations are variable for a shell of general form. The variability of the coefficients deprives us of the possibility of investigating, in the general form, the state of stresses characterized by functions which do not increase substantially during differentiation (states of stress with positive index of variability). Bearing this in mind, in the present chapter we shall limit ourselves to the consideration of states of stress with substantially positive indexes of variability.

A large number of works are devoted to the construction of asymptotic solutions in the theory of shells, and some of these are presented in the list of references for this chapter. In a whole series of cases the asymptotic methods make it possible for us to achieve relatively simple solutions. Thus it is frequently possible to break up the solution of the general moment problem into two parts: membrane solution and a solution of the edge effect type. Each of these is constructed in a simpler manner than the general solution of the moment theory.

However, like any other method, the method of asymptotic integration is not universal and is limited in its scope of application. Thus, for example, in cases when the edge of the shell coincides with the asymptotic line, the edge effect substantially loses its basic property--its rapid damping away from the line. Furthermore, its application meets substantial mathematical difficulties. In view of these situations, the method of breaking up a /226
solution is of low effectiveness in this case.

Furthermore, when the asymptotic method is discussed in the theory of shells, it is customary to have in mind an asymptotic property based on the utilization of exponential functions. For a whole series of problems in the theory of shells (for example, Chapter XV), this restriction in the class of asymptotic functions is not acceptable.

Of course, what has been stated above does not belittle the role of asymptotic solutions in the theory of shells. However, this fact should not be forgotten when reading the present chapter or when selecting a method for designing specific shells.

The application of the asymptotic method to specific classes of shells has been carried out for a long time (refs. 114, 9, 25 and others). In the work of S. M. Feynberg (ref. 272), which was ahead of its time, an effort was made to apply this method to shells of the general form. A substantial development and details of the method were given in the works of A. L. Gol'denveyzer (refs. 11 and 256).

In the present chapter we present a short asymptotic analysis of the relationships appearing in the complex version of the theory of shells. The purpose of this analysis is to classify the states of stress and to construct a complete theory for the simple edge effect. The utilization of complex relationships has made it easier to apply the asymptotic method and has made it possible to obtain several new results.

1. Asymptotic Investigation of the System of Equations Utilizing Complex Displacements

As a basis we take the system of equations with complex displacements in the form proposed by V. V. Novozhilov (Chapter VIII, 8.39)

$$\begin{aligned}
&\tilde{\varepsilon}_\alpha + ia R_0(\tilde{\varkappa}_\beta - \nu\tilde{\varkappa}_\alpha) = 0;\\
&\tilde{\varepsilon}_\beta + ia R_0(\tilde{\varkappa}_\alpha - \nu\tilde{\varkappa}_\beta) = 0;\\
&\frac{\tilde{\omega}}{2} - ia R_0(1+\nu)\tilde{\tau} = 0.
\end{aligned} \tag{1.1}$$

Here $a = \frac{c}{R_0}$ and R_0 is some characteristic linear dimension of the middle surface.

Following (ref. 11) we introduce a large parameter

$$k = a^{-t} \qquad (t > 0) \tag{1.2}$$

and represent the complex displacements in the form of the following asymptotic expansions

$$\begin{aligned}
\tilde{u} &= k^\lambda \left\{U_0(\alpha, \beta) + \frac{1}{k} U_1(\alpha, \beta) + \ldots\right\} e^{kf(\alpha, \beta)};\\
\tilde{v} &= k^\mu \left\{V_0(\alpha, \beta) + \frac{1}{k} V_1(\alpha, \beta) + \ldots\right\} e^{kf(\alpha, \beta)};\\
\tilde{w} &= \left\{W_0(\alpha, \beta) + \frac{1}{k} W_1(\alpha, \beta) + \ldots\right\} e^{kf(\alpha, \beta)}.
\end{aligned} \tag{1.3}$$

If we use these displacements to compute the components of the complex deformation and substitute them into (1.1), we obtain a system of equations for which, first of all, we must find consistent triplets of parameters t; μ, λ. After this has been done by grouping terms with the same exponents k, we obtain a series of recurrent systems from which we can successively determine f (α, β), U_0 (α, β). We note that the factual realization of this approach is associated with very great difficulties (ref. 11). For our purposes it is sufficient to consider only the problem concerning the consistency of the values for parameters t; λ, μ. /227

We use equations (Chapter VIII, 8.9) to compute the components of the complex deformation

$$\begin{aligned}
\tilde{\varepsilon}_\alpha &= \left\{k^{\lambda+1}\frac{1}{A}\frac{\partial f}{\partial\alpha}U_0+\frac{W_0}{R_\alpha}+\ldots\right\}e^{kf(\alpha,\beta)};\\
\tilde{\varepsilon}_\beta &= \left\{k^{\mu+1}\frac{1}{B}\frac{\partial f}{\partial\beta}V_0+\frac{W_0}{R_\beta}+\ldots\right\}e^{kf(\alpha,\beta)};\\
\tilde{\omega} &= \left\{k^{\mu+1}\frac{1}{A}\frac{\partial f}{\partial\alpha}V_0+k^{\lambda+1}\frac{1}{B}\frac{\partial f}{\partial\beta}U_0-\frac{2W_0}{R_{\alpha\beta}}+\ldots\right\}e^{kf(\alpha,\beta)};\\
\tilde{\varkappa}_\alpha &= \left\{-\frac{k^2}{A^2}\left(\frac{\partial f}{\partial\alpha}\right)^2W_0+\ldots\right\}e^{kf(\alpha,\beta)};\\
\tilde{\varkappa}_\beta &= \left\{-\frac{k^2}{B^2}\left(\frac{\partial f}{\partial\beta}\right)^2W_0+\ldots\right\}e^{kf(\alpha,\beta)};\\
\tilde{\tau} &= \left\{-\frac{k^2}{AB}\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\beta}W_0+\ldots\right\}e^{kf(\alpha,\beta)}.
\end{aligned}\tag{1.4}$$

Here we have written out only terms with senior derivatives which play a role in the determination of parameters t; λ, μ.

Substituting the expansions which have been obtained into (1.1) and retaining terms with senior powers of k, we arrive at a so-called leading (characteristic) system of equations

$$\begin{aligned}
&k^{\lambda+1}\frac{1}{A}\frac{\partial f}{\partial\alpha}U_0+\frac{W_0}{R_\alpha}+k^{2-\frac{1}{t}}\left[iR_0\left(-\frac{1}{B^2}\left(\frac{\partial f}{\partial\beta}\right)^2+\frac{\nu}{A^2}\left(\frac{\partial f}{\partial\alpha}\right)^2\right)\right]W_0=0;\\
&k^{\mu+1}\frac{1}{B}\frac{\partial f}{\partial\beta}V_0+\frac{W_0}{R_\beta}+k^{2-\frac{1}{t}}\left[iR_0\left(-\frac{1}{A^2}\left(\frac{\partial f}{\partial\alpha}\right)^2+\frac{\nu}{B^2}\left(\frac{\partial f}{\partial\beta}\right)^2\right)\right]W_0=0;\\
&k^{\mu+1}\frac{1}{A}\frac{\partial f}{\partial\alpha}V_0+k^{\lambda+1}\frac{1}{B}\frac{\partial f}{\partial\beta}U_0-\frac{2W_0}{R_{\alpha\beta}}+\\
&\qquad+k^{2-\frac{1}{t}}\left[iR_0\,2(1+\nu)\frac{1}{AB}\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\beta}\right]W_0=0.
\end{aligned}\tag{1.5}$$

Let us consider three basic cases. /228

I. Let

$$t<\frac{1}{2}\left(k<\frac{1}{\sqrt{a}}\right). \tag{1.6}$$

Since in this case $2-\frac{1}{t}<0$, we can drop terms in equations (1.5) which are contained in the square brackets, thereby reducing the system to the form

$$\begin{aligned}
&k^{\lambda+1}\frac{1}{A}\frac{\partial f}{\partial\alpha}U_0+\frac{W_0}{R_\alpha}=0;\\
&k^{\mu+1}\frac{1}{B}\frac{\partial f}{\partial\beta}V_0+\frac{W_0}{R_\beta}=0;\\
&k^{\mu+1}\frac{1}{A}\frac{\partial f}{\partial\alpha}V_0+k^{\lambda+1}\frac{1}{B}\frac{\partial f}{\partial\beta}U_0-\frac{2W_0}{R_{\alpha\beta}}=0.
\end{aligned} \tag{1.7}$$

(We recall (section 6, Chapter VI) that when an asymptotic analysis is carried out, the relative thickness of the shell h/R_0 is assumed to be infinitely small. Therefore in the course of the analysis only terms with the maximum exponent of k remain.)

Assuming that all of the terms in (1.7) have factors with the same exponent of k, we obtain

$$\lambda=\mu=-1. \tag{1.8}$$

Then the conditions for the existence of a nontrivial solution for this system are written in the following form

$$\begin{vmatrix}
\frac{1}{A}\frac{\partial f}{\partial\alpha} & 0 & \frac{1}{R_\alpha}\\
0 & \frac{1}{B}\frac{\partial f}{\partial\beta} & \frac{1}{R_\beta}\\
\frac{1}{B}\frac{\partial f}{\partial\beta} & \frac{1}{A}\frac{\partial f}{\partial\alpha} & -\frac{2}{R_{\alpha\beta}}
\end{vmatrix}=0$$

or

$$\frac{1}{R_\beta A^2}\left(\frac{\partial f}{\partial\alpha}\right)^2+\frac{2}{R_{\alpha\beta}AB}\frac{\partial f}{\partial\alpha}\frac{\partial f}{\partial\beta}+\frac{1}{R_\alpha B^2}\left(\frac{\partial f}{\partial\beta}\right)^2=0. \tag{1.9}$$

II. Let

$$t=\frac{1}{2}\left(k=\frac{1}{\sqrt{a}}\right). \tag{1.10}$$

Assuming again

$$\lambda = \mu = -1, \tag{1.11}$$

we obtain the characteristic system

$$\begin{aligned}
&\frac{1}{A}\frac{\partial f}{\partial \alpha}U_0+\left\{\frac{1}{R_\alpha}+iR_0\left[-\frac{1}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2+\frac{\nu}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2\right]\right\}W_0=0,\\
&\frac{1}{B}\frac{\partial f}{\partial \beta}V_0+\left\{\frac{1}{R_\beta}+iR_0\left[-\frac{1}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2+\frac{\nu}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2\right]\right\}W_0=0,\\
&\frac{1}{B}\frac{\partial f}{\partial \beta}U_0+\frac{1}{A}\frac{\partial f}{\partial \alpha}V_0+\left\{-\frac{2}{R_{\alpha\beta}}+iR_0 2(1+\nu)\frac{1}{AB}\frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}\right\}W_0=0,
\end{aligned} \tag{1.12}$$

and the condition of solvability for this system has the form /229

$$\begin{aligned}
&\frac{1}{A^4}\left(\frac{\partial f}{\partial \alpha}\right)^4+\frac{2}{A^2B^2}\left(\frac{\partial f}{\partial \alpha}\right)^2\left(\frac{\partial f}{\partial \beta}\right)^2+\frac{1}{B^4}\left(\frac{\partial f}{\partial \beta}\right)^4+\\
&+\frac{i}{R_0}\left(\frac{1}{R_\beta A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2+\frac{2}{R_{\alpha\beta}AB}\frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}+\frac{1}{R_\alpha B^2}\left(\frac{\partial f}{\partial \beta}\right)^2\right)=0.
\end{aligned} \tag{1.13}$$

III. Let, finally,

$$t>\frac{1}{2}\left(k>\frac{1}{\sqrt{a}}\right).$$

Since in this case $2-\frac{1}{t}>0$, we can drop the terms $\frac{W_0}{R_\alpha}$, $\frac{W_0}{R_\beta}$, $\frac{W_0}{R_{\alpha\beta}}$ in system (1.5). Now, assuming that

$$\lambda = \mu = 1-\frac{1}{t}, \tag{1.14}$$

we arrive at the system

$$\begin{aligned}
&\frac{1}{A}\frac{\partial f}{\partial \alpha}U_0+iR_0\left[-\frac{1}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2+\frac{\nu}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2\right]W_0=0,\\
&\frac{1}{B}\frac{\partial f}{\partial \beta}V_0+iR_0\left[-\frac{1}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2+\frac{\nu}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2\right]W_0=0,\\
&\frac{1}{B}\frac{\partial f}{\partial \beta}U_0+\frac{1}{A}\frac{\partial f}{\partial \alpha}V_0+iR_0\left[2(1+\nu)\frac{1}{AB}\frac{\partial f}{\partial \alpha}\frac{\partial f}{\partial \beta}\right]W_0=0,
\end{aligned}\tag{1.15}$$

for which the condition for the existence of a nontrivial solution is the equation

$$\frac{1}{A^4}\left(\frac{\partial f}{\partial \alpha}\right)^4+\frac{2}{A^2B^2}\left(\frac{\partial f}{\partial \alpha}\right)^2\left(\frac{\partial f}{\partial \beta}\right)^2+\frac{1}{B^4}\left(\frac{\partial f}{\partial \beta}\right)^4=0.\tag{1.16}$$

By using the latter it is easy to establish from expressions (1.3) that the characteristic equations which have been obtained are the leading equations for the following differential equations

$$\text{I}\qquad D\widetilde{w}=0;\tag{1.17}$$

$$\text{II}\qquad \Delta\Delta\widetilde{w}+\frac{i}{c}D\widetilde{w}=0;\tag{1.18}$$

$$\text{III}\qquad \Delta\Delta\widetilde{w}=0,\tag{1.19}$$

where

$$\begin{aligned}
\Delta&=\frac{1}{AB}\left\{\frac{\partial}{\partial \alpha}\frac{B}{A}\frac{\partial}{\partial \alpha}+\frac{\partial}{\partial \beta}\frac{A}{B}\frac{\partial}{\partial \beta}\right\};\\
D&=\frac{1}{AB}\left\{\frac{\partial}{\partial \alpha}\frac{B}{R_\beta A}\frac{\partial}{\partial \alpha}+\frac{\partial}{\partial \beta}\frac{1}{R_{\alpha\beta}}\frac{\partial}{\partial \alpha}+\frac{\partial}{\partial \alpha}\frac{1}{R_{\alpha\beta}}\frac{\partial}{\partial \beta}+\frac{\partial}{\partial \beta}\frac{A}{R_\alpha B}\frac{\partial}{\partial \beta}\right\}.
\end{aligned}\tag{1.20}$$

From this and from (Chapter XI, 7.14) we can see that the differential equation which corresponds to the second case is the V. Z. Vlasov equation for hollow shells. The equation for the first case is obtained by retaining the membrane part in the latter; the equation for the third case is obtained by retaining the "moment" part. /230

We note the following curious situation: although we proceeded from a system of equations in terms of complex displacements, containing terms multiplied by the Poisson ratio ν, the characteristic equations (1.17)-(1.19) do not contain such terms. This, once again, emphasizes the situation noted above: the terms in the complex equations containing the multiplier ν are small.

Let us carry out the asymptotic evaluation of the accuracy associated with the V. Z. Vlasov equations for shallow shells. For this purpose we consider the homogeneous equation (Chapter XI, 7.12_1) which, if we follow its derivation, is asymptotically exact, i.e., its asymptotic error a.e. $\leq a^1$ (see below). In all three cases the quantities $\tilde{u}$ and $\tilde{v}$ are substantially less than $\tilde{w}$. More precisely we assume that

$$\tilde{w} \sim W_0 \tag{1.21}$$

(the $\sim$ sign indicates that the quantity associated with it is of the same order of magnitude), then, according to (1.2), (1.3), (1.8), (1.11) and (1.14)

$$\{\tilde{u},\ \tilde{v}\} \sim a^t W_0 \quad \text{when } t < \tfrac{1}{2}; \tag{1.22}$$

$$\{\tilde{u},\ \tilde{v}\} \sim a^{1-t} W_0 \quad \text{when } t > \tfrac{1}{2}. \tag{1.23}$$

Noting that the functions which determine the state of stress increase by $k = a^{-t}$ during each differentiation, i.e., that

$$\left\{\frac{\partial\Phi}{\partial\alpha},\ \frac{\partial\Phi}{\partial\beta}\right\} \sim a^{-t}\Phi;\quad \left\{\frac{\partial^2\Phi}{\partial\alpha^2},\ \frac{\partial^2\Phi}{\partial\alpha\,\partial\beta},\ \frac{\partial^2\Phi}{\partial\beta^2}\right\} \sim a^{-2t}\cdot\Phi,\ldots, \tag{1.24}$$

we obtain, by analyzing equations (Chapter XI, 7.5, 7.9 and 7.10)

$$\left\{\tilde{x}_\alpha^0,\ \tilde{x}_\beta^0,\ \tilde{\tau}^0\right\} \sim a^{-2t}\frac{W_0}{R_0^2};\quad \Delta\Delta\tilde{w} \sim a^{-4t}\frac{W_0}{R_0^4};\quad D\tilde{w} \sim a^{-2t}\frac{W_0}{R_0^3}; \tag{1.25}$$

when $0 < t < \frac{1}{2}$:

$$\left\{\tilde{\gamma}_\alpha,\ \tilde{\gamma}_\beta,\ \tilde{\gamma}_{\alpha\beta}\right\} \sim \frac{W_0}{R_0^2};\quad \tilde{\Gamma}\left(\tilde{u},\ \tilde{v}\right) \sim \frac{W_0}{R_0^3};\quad \Delta\left(\tilde{\gamma}_\alpha + \tilde{\gamma}_\beta\right) \sim a^{-2t}\frac{W_0}{R_0^4}; \tag{1.26}$$

when $t \geq \frac{1}{2}$:

$$\left\{\tilde{\gamma}_\alpha,\ \tilde{\gamma}_\beta,\ \tilde{\gamma}_{\alpha\beta}\right\} \sim a^{1-2t}\frac{W_0}{R_0^2};\quad \tilde{\Gamma}\left(\tilde{u},\ \tilde{v}\right) \sim a^{1-2t}\frac{W_0}{R_0^3};\quad \Delta\left(\tilde{\gamma}_\alpha + \tilde{\gamma}_\beta\right) \sim a^{1-4t}\frac{W_0}{R_0^4}. \tag{1.27}$$

From this it follows that the asymptotic error associated with the dropping of terms $\tilde{u}$, $\tilde{v}$ and their derivatives in equation (Chapter XI, 7.12_1) /231 is equal to

when $0 < t < \frac{1}{2}$

$$\text{a.e.} = \frac{a^0}{a^{-2t}} = a^{2t}; \tag{1.28}$$

when $t \geq \frac{1}{2}$

$$\text{a.e.} = \frac{a^{1-2t}}{a^{-2t}} = a^1. \tag{1.29}$$

(We recall that the term "asymptotic error" of a simplfied equation was used by us in section 6, Chapter VI following (ref. 11) as the order of the ratio of the maximum neglected term to the maximum remaining term.)

The last of the relationships shows that when $t \geq \frac{1}{2}$, the V. Z. Vlasov equation for shallow shells may be considered asymptotically exact. We can also see from relationships (1.24)-(1.26) that the evaluations which have been obtained are also valid when $\tilde{u}$ and $\tilde{v}$ are dropped in expressions (Chapter XI, 7.13) for $\tilde{\varkappa}_\alpha$, $\tilde{\varkappa}_\beta$, $\tilde{\tau}$.

2. Classification of the States of Stress

1. Let us consider in more detail the first of the cases isolated in the preceding section. Its basic characteristic is the inequality $t < 1/2$ ($k < \sqrt{a}$). The characteristic equation (1.17) may be obtained from the more general one (1.18) by neglecting in the latter the "moment" part $\Delta\Delta\tilde{w}$. It is easy to evaluate the asymptotic error when this part is dropped. Indeed, according to (1.25)

$$\Delta\Delta\tilde{w} \sim a^{-4t}\frac{W_0}{R_0^4}; \quad \frac{i}{c}D\tilde{w} \sim a^{-1-2t}\frac{W_0}{R_0^4}. \tag{2.1}$$

From this it follows that the asymptotic error of the neglected moment part is equal to

$$\text{a.e.} = \frac{a^{-4t}}{a^{-1-2t}} = a^{1-2t}. \tag{2.2}$$

If we recall that the error associated with the dropping of terms depending on $\tilde{u}$ and $\tilde{v}$ in the membrane parts was determined by equality (1.28), the value of the asymptotic error in equation (1.17) was determined as the maximum of the values /232

$$\text{a.e.} = \max\{a^{2t},\ a^{1-2t}\}. \tag{2.3}$$

Equality (2.3) places rather rigid limitations on the value of the variability index t. In order to be able to consider a state of stress with a small variability index (the basic states of stress according to the terminology of A. L. Gol'denveyzer), we must not drop the terms contained in $\Gamma(\tilde{u}, \tilde{v})$. In this case the characteristic system may be taken in the following form (Chapter XI, 7.12)

$$\begin{gathered} D\tilde{w} + \tilde{\Gamma}(\tilde{u},\ \tilde{v}) = ia\frac{1}{R_0}\Delta\Delta\tilde{w}, \\ \frac{1}{B}\frac{\partial\tilde{v}}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{u} - \frac{1}{A}\frac{\partial\tilde{u}}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{v} = \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\tilde{w}, \\ \frac{1}{A}\frac{\partial\tilde{v}}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{v} + \frac{1}{B}\frac{\partial\tilde{u}}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{u} = \frac{2}{R_{\alpha\beta}}\tilde{w}. \end{gathered} \tag{2.4}$$

Now the asymptotic error is determined only by the equality (2.2). The system, which has been written out, may be used to construct approximate methods for finding the basic state of stress.

2. Let us consider the rapidly changing state of stress characterized by the inequality $t > \frac{1}{2}$. Its characteristic equation

$$\Delta\Delta\tilde{w} = 0 \tag{1.19}$$

is obtained from a more general equation (1.18) by dropping the "membrane terms" $\frac{i}{c}D\tilde{w}$, in the latter. By using equalities (2.1) we determine the asymptotic error when such terms are dropped

$$\text{a.e.} = \frac{a^{-1-2t}}{a^{-4t}} = a^{2t-1}. \tag{2.5}$$

When $t \geq 1$ characteristic equation (1.19) becomes asymptotically exact. We should point out that the curvatures of the surface do not enter at all in

equation (1.19), so that the nature of the solution does not depend on the degree to which the middle surface is distorted. This is explained by the fact that the quantities which determine the state of stress vary so rapidly that the distortion of the surface has no noticeable effect on them. Furthermore, the equation breaks down into two independent equations

$$\Delta\Delta w = 0 \text{ and } \Delta\Delta\overline{w} = 0, \tag{2.6}$$

of which the first determines the flexural deformation, while the second /233 determines the tangential deformation ($\overline{w}$ is the stress function). Thus, when $t \geq 1$, the state of stress breaks down into two independent states: nonflexural and purely flexural.

The following interesting fact (ref. 201) is associated with the high rate of change of the state of stress in the neighborhood of the point where the concentrated forces are applied: the principal singularity of the solution corresponding to the concentrated force does not depend on the curvature of the shell's middle surface.

We should like to point out one other characteristic property of the state of stress with a large variability index. For this purpose we introduce $ds_\alpha = Ad\alpha$ and $ds_\beta = Bd\beta$, which permits us to write the characteristic equation (1.16) in the following form

$$\left[\left(\frac{\partial f}{\partial s_\alpha}\right)^2 + \left(\frac{\partial f}{\partial s_\beta}\right)^2\right]^2 = 0. \tag{2.7}$$

As we can see from expressions (1.3), the nature of the solution is determined by the cofactor $e^{kf(\alpha, \beta)}$, and consequently by the properties of the so-called phase function $f(\alpha, \beta)$. From (2.7) we obtain the equality

$$\left|\frac{\partial f}{\partial s_\alpha}\right| = \left|\frac{\partial f}{\partial s_\beta}\right|, \tag{2.8}$$

which shows that the rate of variability for the phase function is the same in all directions. We have already encountered this property during the asymptotic analysis of the equation for the circular cylindrical shell (region 3 in figure 66, Part I).

From this it follows that for a state of stress with a large variability index it is impossible, in principle, to have solutions of the edge effect type, in which the characteristic functions change rapidly in one direction and slowly in the direction which is perpendicular to it.

3. The third basic case ($t = \frac{1}{2}$) includes a solution of the edge effect type. Let us assume, for example, that the edge coincides with the line $\alpha = \alpha^0 = \text{const}$. Let us assume further that the phase function does not vary at all along the edge, i.e., that $\frac{\partial f}{\partial s_\beta} = 0$. Then from (1.13) we obtain the equation

$$\left\{iR_0\left(\frac{\partial f}{\partial s_\alpha}\right)^2 - \frac{1}{R_\beta}\right\}\left(\frac{\partial f}{\partial s_\alpha}\right)^2 = 0, \tag{2.9}$$

which gives a nontrivial solution for $\frac{\partial f}{\partial s_\alpha}$.

The asymptotic analysis for real shells, carried out under the assumption that the quantity $a = \frac{c}{R_0}$ may be arbitrarily small, gives us, of course, only a /234

qualitative representation of the state of stress. However, this leading information makes it possible for us to construct relatively simple relationships for states of stress of most interest in practice. The next section is devoted to the formulation of such relationships.

3. The Simple Edge Effect

Let us assume that the considered shell edge coincides with the coordinate line $\alpha = \alpha^0 = \text{const}$. As we have already established in the two preceding sections, the simple (using the terminology of A. L. Gol'denveyzer) edge effect is characterized by the following properties

$$\frac{\partial^{k+1}\Phi}{\partial\alpha^{k+1}} \sim q\frac{\partial^k\Phi}{\partial\alpha^k}; \quad \frac{\partial^{k+1}\Phi}{\partial\beta^{k+1}} \sim \frac{\partial^k\Phi}{\partial\beta^k}, \tag{3.1}$$

where

$$q = a^{-\frac{1}{2}} = \left(\frac{R_0}{c}\right)^{\frac{1}{2}} \tag{3.2}$$

and

$$\{\tilde{u}, \tilde{v}\} \sim \frac{1}{q}\tilde{w}. \tag{3.3}$$

If we introduce the new independent variable

$$x = q\,(\alpha-\alpha^0), \tag{3.4}$$

equation (1.18) after reduction by q^4 may be written in the form

$$\begin{aligned}&\frac{1}{AB}\frac{\partial}{\partial x}\frac{B}{A}\frac{\partial}{\partial x}\frac{1}{AB}\frac{\partial}{\partial x}\ldots\frac{\partial\tilde{w}}{\partial x}+\frac{i}{R_0AB}\frac{\partial}{\partial x}\frac{B}{k_\beta A}\frac{\partial\tilde{w}}{\partial x}+\\&+\frac{1}{q}\left\{\frac{i}{R_0AB}\left(\frac{\partial}{\partial x}\frac{1}{R_{\alpha\beta}}\frac{\partial\tilde{w}}{\partial\beta}+\frac{\partial}{\partial\beta}\frac{1}{R_{\alpha\beta}}\frac{\partial\tilde{w}}{\partial x}\right)\right\}+\\&+\frac{1}{q^2}\{\ldots\}+\ldots=0.\end{aligned} \tag{3.5}$$

Limiting ourselves below to the first and second approximation, we shall not write out the terms here and below, which cannot affect the unknown approximation.

Following this (refs. 252 and 256), we replace the coefficients of this equation by the terms which have been written out for the expansions

$$\begin{aligned}&A=A^0(\beta)+\frac{x}{q}A^1(\beta)+\ldots;\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&\frac{1}{R_\alpha}=k_\alpha^0(\beta)+\frac{x}{q}k_\alpha^1(\beta)+\ldots;\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&\frac{B}{R_\beta A}=\frac{B^0k_\beta^0}{A^0}\left[1+\left(\frac{B^1}{B^0}+\frac{k_\beta^1}{k_\beta^0}-\frac{A^1}{A^0}\right)\frac{x}{q}+\ldots\right],\end{aligned} \tag{3.6}$$

and shall represent the unknown function as an asymptotic expansion /235

$$\tilde{w}=\tilde{w}^0+\frac{1}{q}\tilde{w}^1+\ldots \tag{3.7}$$

By substituting the expressions which have been written out into equation (3.5), which has been multiplied by A^4, and collecting terms with the same powers of q, we arrive at a recurrent system of equations

$$\begin{aligned}&\frac{\partial^4\tilde{w}^0}{\partial x^4}+i\frac{(A^0)^2k_\beta^0}{R_0}\frac{\partial^2\tilde{w}^0}{\partial x^2}=0;\\&\frac{\partial^4\tilde{w}^1}{\partial x^4}+i\frac{(A^0)^2k_\beta^0}{R_0}\frac{\partial^2\tilde{w}^1}{\partial x^2}=-\left(2\frac{B^1}{B^0}-6\frac{A^1}{A^0}\right)\frac{\partial^3\tilde{w}^0}{\partial x^3}+\end{aligned} \tag{3.8}$$

$$+ i\frac{(A^0)^2 k_\beta^0}{R_0}\left\{-x\left(2\frac{A^1}{A^0}+\frac{k_\beta^1}{k_\beta^0}\right)\frac{\partial^2\tilde{w}^0}{\partial x^2}-\left(\frac{B^1}{B^0}+\frac{k_\beta^1}{k_\beta^0}-\frac{A^1}{A^0}\right)\frac{\partial\tilde{w}^0}{\partial x}-\right.$$
$$\left.-2\frac{A^0}{B^0}\frac{k_{\alpha\beta}^0}{k_\beta^0}\frac{\partial^2\tilde{w}^0}{\partial x\partial\beta}-\frac{A^0}{B^0}\frac{k_{\alpha\beta}^{0'}}{k_\beta^0}\frac{\partial\tilde{w}^0}{\partial x}\right\} \tag{3.9}$$

(' means that the derivative is taken with respect to β).

Seeking the solution of equation (3.8) in the form

$$\tilde{w}^0=\tilde{\psi}_0(\beta)\,e^{f(\beta)x}, \tag{3.10}$$

we obtain the characteristic equation

$$f^2+i\frac{(A^0)^2 k_\beta^0}{R_0}=0, \tag{3.11}$$

from which we find

$$f=\pm\frac{1-i\,\mathrm{sign}\,k_\beta^0}{\sqrt{2}}\sqrt{\frac{(A^0)^2|k_\beta^0|}{R_0}}. \tag{3.12}$$

Here

$$\mathrm{sign}\,k_\beta^0=\begin{cases}+1, & \text{if} \quad k_\beta^0>0,\\ -1, & \text{if} \quad k_\beta^0<0.\end{cases} \tag{3.13}$$

Since we are interested in the solution which is damped as we move away from the edge, we should take the upper sign in solution (3.10) when $\alpha < \alpha^0$ and the lower sign when $\alpha > \alpha^0$.

Furthermore, by substituting the established value $\tilde{w}^0$ into equation (3.9) and integrating this equation, we find

$$\tilde{w}^1=\left\{\chi_1(\beta)\tilde{\psi}_0 f x^2+\left(\chi_2(\beta)\tilde{\psi}_0+\chi_3\tilde{\psi}_0'\right)x+\tilde{\psi}_1(\beta)\right\}e^{f(\beta)x}, \tag{3.14}$$

where

$$\chi_1(\beta)=\frac{1}{2}\frac{A^1}{A^0}+\frac{1}{4}\frac{k_\beta^1}{k_\beta^0}+\frac{1}{2}\frac{A^0}{B^0}\frac{k_{\alpha\beta}^0}{k_\beta^0}\chi_4(\beta);$$

$$\chi_2(\beta)=-\left[\frac{1}{2}\frac{B^1}{B^0}+\frac{3}{4}\frac{k_\beta^1}{k_\beta^0}+\frac{3}{2}\frac{A^0}{B^0}\frac{k_{\alpha\beta}^0}{k_\beta^0}\chi_4(\beta)-\frac{A^0}{B^0}\frac{k_{\alpha\beta}^{0'}}{k_\beta^0}\right]; \qquad (3.15)$$

$$\chi_3(\beta)=\frac{A^0}{B^0}\frac{k_{\alpha\beta}^0}{k_\beta^0};$$

$$\chi_4(\beta)=\frac{f'}{f}=\frac{A^{0'}}{A^0}+\frac{1}{2}\frac{k_\beta^{0'}}{k_\beta^0}.$$

Thus the basic complex function has the following form in the second approximation /236

$$\tilde{w}=\left\{\left[1+\frac{1}{q}(\chi_1 f x^2+\chi_2 x)\right]\tilde{\psi}_0+\frac{1}{q}\chi_3 x\tilde{\psi}_0'+\frac{1}{q}\tilde{\psi}_1\right\}e^{fx}. \qquad (3.16)$$

Without sacrificing generality, we can drop the "superfluous" arbitrary function $\tilde{\psi}_1(\beta)$. Its retention, however, may sometimes somewhat simplify the procedure of getting the solution to satisfy the boundary conditions.

Let us express in terms of $\tilde{w}$ the basic static and gecmetric quantities. First of all we transform the expressions for the complex forces in terms of the new variable (Chapter VIII, 8.35)

$$\tilde{T}_\alpha=iEhcq^2\left\{\frac{1}{A^2B}\frac{\partial B}{\partial x}\frac{\partial\tilde{w}}{\partial x}+\ldots\right\};$$

$$\tilde{S}=iEhcq^2\left\{\frac{1}{q}\left(-\frac{1}{AB}\frac{\partial^2\tilde{w}}{\partial x\partial\beta}+\frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial\tilde{w}}{\partial x}\right)+\ldots\right\}; \qquad (3.17)$$

$$\tilde{T}_\beta=iEhcq^2\left\{\frac{1}{A^2}\frac{\partial^2\tilde{w}}{\partial x^2}+\frac{1}{A}\frac{\partial}{\partial x}\left(\frac{1}{A}\right)\frac{\partial\tilde{w}}{\partial x}+\ldots\right\}.$$

Substituting expression (3.16) into the above equations and again making use of expansions of type (3.6), we arrive at the following expressions for the complex forces

$$\tilde{T}_\alpha=iEhcq^2\left\{\frac{1}{q}\left[\frac{B^1}{(A^0)^2B^0}f\tilde{\psi}_0\right]\right\}e^{fx};$$

$$\tilde{S}=iEhcq^2\left\{-\frac{1}{qA^0B^0}\left[\left(\chi_4 f^2x+\frac{1}{2}\frac{k_\beta^{0'}}{k_\beta^0}\right)\tilde{\psi}_0+f\tilde{\psi}_0'\right]\right\}e^{fx}; \qquad (3.18)$$

$$\widetilde{T}_\beta = iEhcq^2\left\{\frac{1}{(A^0)^2}f^2\tilde{\psi}_0 + \frac{1}{q(A^0)^2}\left[(\chi_1 f^3 x^2 + \chi_5 f^2 x + \right.\right.$$
$$\left.\left. + \chi_6 f)\tilde{\psi}_0 + \chi_3(f^2 x + 2f)\tilde{\psi}_0' + f^2\tilde{\psi}_1\right]\right\}e^{fx},$$

where

$$\chi_5 = \frac{1}{4}\frac{k_\beta^1}{k_\beta^0} - \frac{1}{2}\frac{B^1}{B^0} + \frac{1}{2}\frac{A^0}{B^0}\frac{k_{\alpha\beta}^0}{k_\beta^0}\chi_4 + \frac{A^0}{B^0}\frac{k_{\alpha\beta}^{0'}}{k_\beta^0};$$
$$\chi_6 = -\frac{k_\beta^1}{k_\beta^0} - \frac{B^1}{B^0} - 2\frac{A^0}{B^0}\frac{k_{\alpha\beta}^0}{k_\beta^0}\chi_4 + 2\frac{A^0}{B^0}\frac{k_{\alpha\beta}^{0'}}{k_\beta^0}. \qquad (3.19)$$

Separating the real and imaginary parts in the resulting relationships we obtain the forces and the moments /237

$$T_\alpha = \operatorname{Re}\widetilde{T}_\alpha;\quad T_\beta = \operatorname{Re}\widetilde{T}_\beta;\quad S = \operatorname{Re}\widetilde{S};$$
$$M_\alpha = -c\operatorname{Im}\left\{\widetilde{T}_\beta + \nu\widetilde{T}_\alpha\right\};\quad M_\beta = -c\operatorname{Im}\left\{\widetilde{T}_\alpha + \nu\widetilde{T}_\beta\right\};$$
$$H = c(1-\nu)\operatorname{Im}\tilde{S}.$$

On the other hand, to obtain the complex displacements we can use the system of equations (Chapter IV, 9.4 and 9.5)

$$\frac{1}{A}\frac{\partial\tilde{u}}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{v} = \frac{1}{Eh}\left(\widetilde{T}_\alpha - \nu\widetilde{T}_\beta\right) - \frac{\tilde{w}}{R_\alpha};$$
$$\frac{1}{B}\frac{\partial\tilde{v}}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{u} = \frac{1}{Eh}\left(\widetilde{T}_\beta - \nu\widetilde{T}_\alpha\right) - \frac{\tilde{w}}{R_\beta}; \qquad (3.20)$$
$$\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{\tilde{v}}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{\tilde{u}}{A}\right) = \frac{2(1+\nu)}{Eh}\tilde{S} + \frac{2\tilde{\omega}}{R_{\alpha\beta}}.$$

In accordance with equalities (3.3) we seek the tangential complex displacements in the form

$$\tilde{u} = \frac{1}{q}U(\beta)e^{fx};\quad \tilde{v} = \frac{1}{q}V(\beta)e^{fx}. \qquad (3.21)$$

Substituting these expressions as well as (3.18) and (3.16) into the system of equations (3.20), we obtain, by reducing the common factor e^{fx},

$$\frac{1}{A^0}\,Uf+\ldots=i\,\frac{Ehc}{Eh}\,q^2(-\nu)\,\frac{1}{(A^0)^2}\,f^3\tilde{\psi}_0-k_1^0\tilde{\psi}_0+\ldots;$$

$$\ldots=i\,\frac{Ehc}{Eh}\,q^2\,\frac{1}{(A^0)^2}\,f^2\tilde{\psi}_0-k_3^0\tilde{\psi}_0+\ldots; \qquad (3.22)$$

$$\frac{1}{A}\,Vf+\ldots=2k_{\alpha 3}^0\tilde{\psi}_0+\ldots$$

(here the dots designate small terms of the order of $1/q$).

Now, by substituting into the right side of this system the value of f^2 given by (3.11) we become convinced that the second equation is satisfied identically and obtain from the first and third equation

$$U=-\frac{A^0}{f}\left(k_\alpha^0+\nu k_3^0\right)\tilde{\psi}_0;\quad V=\frac{A^0}{f}\,2k_{\alpha 3}^0\tilde{\psi}_0. \qquad (3.23)$$

Thus the complex displacements may be taken in the following form

$$\tilde{u}=-\frac{1}{q}\,\frac{A^0}{f}\left(k_\alpha^0+\nu k_3^0\right)\tilde{\psi}_0 e^{fx};\quad \tilde{v}=\frac{1}{q}\,\frac{A^0}{f}\,2k_{\alpha 3}^0\tilde{\psi}_0 e^{fx}. \qquad (3.24)$$

(We recall that $u = \mathrm{Re}\,\tilde{u}$, $v = \mathrm{Re}\,\tilde{v}$.) (3.25)

We note that the tangential complex displacements were established by us /238 essentially in the first approximation. It is not necessary to determine them more accurately, because they are, according to (3.3), small quantities of the order $\frac{1}{q}\tilde{w}$. The following useful relationship follows from $(3.24)_2$ with the same degree of accuracy

$$\frac{1}{A}\,\frac{\partial\tilde{v}}{\partial\alpha}=\frac{2}{R_{\alpha\beta}}\,\tilde{w}. \qquad (3.26)$$

Let us obtain the expressions for the complex boundary quantities at the boundary $\alpha = \alpha^0$. For this purpose we assume that in equalities (Chapter VIII, 8.40, 8.19, 8.9 and 8.10), $\gamma = 0$, $Q_{\nu\nu}^* = Q_{\nu t}^* = Q_{\nu n}^* = 0$, $M_{\nu\nu}^* = 0$ and taking into account (3.26) and the Codazzi-Gauss relationships (Chapter VII, 7.78) we obtain

$$\widetilde{M}_{\nu\nu} = iEhc\left\{\frac{1}{R_\beta}\widetilde{w}\right\};$$
$$\widetilde{Q}_{\nu\nu} = iEhc\left\{\frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{\partial\widetilde{w}}{A\partial\alpha} + \frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial\widetilde{w}}{\partial\beta} - \frac{1}{R_{\alpha\beta}^2}\widetilde{w}\right\};$$
$$\widetilde{Q}_{\nu t} = iEhc\left\{-\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{A}\frac{\partial\widetilde{w}}{\partial\alpha} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\frac{\partial\widetilde{w}}{B\partial\beta} + \frac{\widetilde{w}}{R_\beta R_{\alpha\beta}}\right\}; \qquad (3.27)$$
$$\widetilde{Q}_{\nu n} = iEhc\left\{\frac{1}{R_\beta}\frac{\partial\widetilde{w}}{A\partial\alpha} + \frac{2}{R_{\alpha\beta}}\frac{\partial\widetilde{w}}{B\partial\beta} + \frac{\partial}{B\partial\beta}\left(\frac{1}{R_{\alpha\beta}}\right)\widetilde{w}\right\}.$$

4. Simplified Version (First Approximation)

In the preceding section we obtained a solution of a type of simple edge effect which was "exact" in the sense that when the corresponding relationships were derived, we dropped quantities of the order $\frac{1}{q^2} = \frac{c}{R_0}$ compared with 1. In practical calculations it is usually sufficient to utilize a cruder approximation by dropping terms of the order $\frac{1}{q} = \sqrt{\frac{c}{R_0}}$. These simplified relationships are obtained directly from those presented in the preceding section by dropping quantities of the order of unity $\sqrt{\frac{c}{R_0}}$.

First of all, by returning to the variable α we have

$$\widetilde{w} = (\varphi(\beta) + i\psi(\beta))\, e^{\pm\left(1 - i\,\mathrm{sign}\,R_\beta^0\right)\beta}, \qquad (4.1)$$

where

$$\beta = \frac{A^0}{\sqrt{2c\,|R_\beta^0|}}(\alpha - \alpha^0). \qquad (4.2)$$

We recall that here the upper sign pertains to the case when $\alpha \leq \alpha^0$ and /239

the lower sign to the case when $\alpha \geq \alpha^0$. The zero sign means that the quantity which contains it was computed at the edge of the shell. It is easy to see that with the assumed degree of accuracy all relationships may be obtained, if we assume that $\widetilde{w}$ is the solution of equation

$$\frac{\partial^2 \tilde{w}}{\partial \alpha^2} + i \frac{A^2}{cR_\beta} \tilde{w} = 0. \tag{4.3}$$

Using this equation and dropping small terms in relationships (3.17) and remembering that $x = q(\alpha - \alpha^0)$ we obtain

$$\begin{aligned} &\tilde{T}_\alpha = iEhc \left\{ \frac{1}{AB} \frac{\partial B}{\partial \alpha} \frac{\partial \tilde{w}}{A \partial \alpha} \right\}, \quad \tilde{T}_\beta = Eh \frac{1}{R_\beta} \tilde{w}, \\ &\tilde{S} = iEhc \left\{ -\frac{1}{B} \frac{\partial}{\partial \beta} \frac{\partial \tilde{w}}{A \partial \alpha} \right\}. \end{aligned} \tag{4.4}$$

It follows from this that the basic quantities in the considered approximation will be as follows

$$T_\beta = \operatorname{Re} \tilde{T}_\beta; \quad M_\alpha = -c \operatorname{Im} \tilde{T}_\beta; \quad M_\beta = \nu M_\alpha, \tag{4.5}$$

and in accordance with (4.1)

$$\begin{aligned} T_\beta &= \frac{Eh}{R_\beta} (\varphi \cos \beta \pm \operatorname{sign} R_\beta^0 \psi \sin \beta) e^{\pm \beta}; \\ M_\alpha &= -\frac{Ehc}{R_\beta} (\mp \operatorname{sign} R_\beta^0 \varphi \sin \beta + \psi \cos \beta) e^{\pm \beta}. \end{aligned} \tag{4.6}$$

At the same time, by comparing expressions (3.24)-(3.25) with (3.16), we see that for the assumed degree of accuracy

$$\begin{aligned} u &= v = 0; \\ w &= (\varphi \cos \beta \pm \operatorname{sign} R_\beta^0 \psi \sin \beta) e^{-\beta}. \end{aligned} \tag{4.7}$$

Finally, by simplifying relationships (3.27) we obtain the following simplified expressions for the complex boundary conditions

$$\begin{aligned} &\tilde{Q}_{vv} = iEhc \left\{ \frac{1}{AB} \frac{\partial B}{\partial \alpha} \frac{\partial \tilde{w}}{A \partial \alpha} \right\}; \quad \tilde{Q}_{vn} = iEhc \left\{ \frac{1}{R_\beta} \frac{\partial \tilde{w}}{A \partial \alpha} \right\}; \\ &\tilde{Q}_{vt} = iEhc \left\{ -\frac{1}{B} \frac{\partial}{\partial \beta} \frac{1}{A} \frac{\partial \tilde{w}}{\partial \beta} \right\}; \quad \tilde{M}_{vv} = iEhc \left\{ \frac{1}{R_\beta} \tilde{w} \right\}. \end{aligned} \tag{4.8}$$

We note that within the accuracy adopted in this section the radii of curvature and the Lamé parameters in all of the presented relationships may be replaced by their boundary values.

5. The Generalized Edge Effect for Shells With a Zero Gaussian Curvature

Let us consider the case when the coordinate line is the asymptotic line of a shell with a zero Gaussian curvature. Let us assume that this line is $\beta = \beta_0 = \text{const}$. For shells of general form with a zero Gaussian curvature (section 4, Chapter XII) /240

$$\begin{gathered}\frac{1}{R_\alpha}=\frac{1}{R_{\alpha\beta}}=0;\quad A=1;\\ B=\alpha f_2(\beta)+f_3(\beta);\quad R_\beta=\frac{B}{f_1(\beta)}.\end{gathered} \tag{5.1}$$

In the adopted system of coordinates, equation (1.18) is written in the following form

$$\begin{gathered}\frac{1}{B}\left(\frac{\partial}{\partial\alpha}B\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial}{\partial\beta}\right)\frac{1}{B}\left(\frac{\partial}{\partial\alpha}B\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial}{\partial\beta}\right)\tilde{w}+\\ +i\frac{a^{-1}}{R_0R_\beta}\frac{\partial^2\tilde{w}}{\partial\alpha^2}=0.\end{gathered} \tag{5.2}$$

(The last term is transformed by means of the equality which follows from (5.1)

$$\frac{\partial}{\partial\alpha}\left(\frac{B}{R_\beta}\right)=0.) \tag{5.3}$$

Let us replace the independent variables by introducing

$$x=q\alpha,\quad y=p(\beta-\beta_0), \tag{5.4}$$

where

$$q=a^{-t}=\left(\frac{R_0}{c}\right)^t;\quad p=a^{-s}. \tag{5.5}$$

In terms of the new variables equation (5.2) is written in the form

$$q^4\left\{\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial}{\partial x}\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial\tilde{w}}{\partial x}\right\}+q^2p^2\left\{\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial}{\partial x}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial\tilde{w}}{\partial y}+\right.$$
$$\left.+\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial\tilde{w}}{\partial x}\right\}+p^4\underline{\left\{\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial\tilde{w}}{\partial y}\right\}}+ \tag{5.6}$$
$$\underline{+i\frac{a^{-1}q^2}{R_0R_3}\frac{\partial^2\tilde{w}}{\partial x^2}}=0.$$

We seek a solution which varies at a greater rate with the coordinate β than the coordinate α. Therefore we should assume that $p \gg q$. More precisely, assuming that the underlined terms have the same order of magnitude, we obtain

$$p^4=a^{-1}q^2, \text{ т. е. } -4s=-1-2t. \tag{5.7}$$

From this it follows that

$$s=\frac{1+2t}{4};\quad q=a^{-t};\quad p=a^{-\frac{1+2t}{4}}. \tag{5.8}$$

Now, by introducing a new large parameter

$$\lambda=\frac{p}{q}=a^{-\frac{1-2t}{4}}, \tag{5.9}$$

we write equation (5.6), (first dividing all of its terms by p^4) in the form /241

$$\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial\tilde{w}}{\partial y}+\frac{i}{R_0R_3}\frac{\partial^2\tilde{w}}{\partial x^2}+$$
$$+\frac{1}{\lambda^2}\left\{\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial}{\partial x}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial\tilde{w}}{\partial y}+\right.$$
$$\left.+\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial y}\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial\tilde{w}}{\partial x}\right\}+ \tag{5.10}$$
$$+\frac{1}{\lambda^4}\left\{\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial}{\partial x}\frac{1}{B}\frac{\partial}{\partial x}B\frac{\partial\tilde{w}}{\partial x}\right\}=0.$$

Utilizing equalities (5.4) and (5.9) we represent the coefficients and the equations in the form of the following expansions

$$B(x, y)=B_0\left(\frac{x}{q}\right)+\frac{y}{\lambda}\frac{B_1(\ldots)}{q}+\ldots;$$
$$\frac{1}{R_\beta}=k_\beta^0\left(\frac{x}{q}\right)+\frac{y}{\lambda}\frac{k_\beta^1(\ldots)}{q}+\ldots \tag{5.11}$$

Now we shall seek $\tilde{w}$ in the form

$$\tilde{w}=\tilde{w}^{(0)}+\frac{1}{\lambda}\tilde{w}^{(1)}+\frac{1}{\lambda^2}\tilde{w}^{(2)}+\ldots \tag{5.12}$$

Substituting these expansions into equation (5.10) and collecting the multipliers in front of the same powers of λ, we obtain the following recurrent equations

$$\frac{\partial^4\tilde{w}^{(0)}}{\partial y^4}+i\frac{B_0^4k_\beta^0}{R_0}\frac{\partial^2\tilde{w}^{(0)}}{\partial x^2}=0;$$
$$\frac{\partial^4\tilde{w}^{(1)}}{\partial y^4}+i\frac{B_0^4k_\beta^0}{R_0}\frac{\partial^2\tilde{w}^{(1)}}{\partial x^2}=\frac{6B_1}{qB_0}\frac{\partial^3\tilde{w}^{(0)}}{\partial y^3}-i\frac{B_0^4}{R_0q}\left(\frac{4B_1^0k_\beta^0}{B_0}+k_\beta^1\right)\frac{\partial^2\tilde{w}^{(0)}}{\partial x^2}; \tag{5.13}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus the problem is reduced to a successive search of solutions for equations (5.13). In this case, since the characteristic equation in y is of the fourth order, we may require that the unknown solution satisfy boundary conditions

$$\tilde{w}^{(0)}=\tilde{\varphi}(x);\quad\frac{\partial\tilde{w}^{(0)}}{\partial y}=\tilde{\psi}(x);$$
$$\tilde{w}^{(k)}=\frac{\partial\tilde{w}^{(k)}}{\partial y}=0\quad(k\geqslant 1). \tag{5.14}$$

In accordance with (5.12) this provides for the fulfillment of two boundary /242 conditions

$$\tilde{w}=\tilde{\varphi}(x);\quad\frac{\partial\tilde{w}}{\partial y}=\tilde{\psi}(x), \tag{5.15}$$

where $\tilde{\varphi}(x)$ and $\tilde{\psi}(x)$ are arbitrary complex functions which make it possible to satisfy four real boundary conditions on the considered asymptotic edge of the shell. We recall that in the case of a simple edge effect (section 3) the solution gave us only one arbitrary complex function. This is far from being accidental. Indeed, as we have seen in section 3, Chapter XII, in the case when the boundary is not an asymptotic line, the membrane theory places at our command two arbitrary real functions on this line. If to these we add the arbitrary complex function which has been mentioned (which is contained in the

solution of the simple edge effect type), then together they permit us to satisfy four boundary conditions. On the other hand, in the case of a surface with zero Gaussian curvature (section 3, Chapter XII), the membrane solution gives us no measure of arbitrariness at the boundary which coincides with the asymptotic line. In this case a solution of the generalized edge effect type gives us all of the necessary (4) arbitrary real functions (Re $\widetilde{\Phi}(x)$, Im $\widetilde{\Phi}(x)$, Re $\widetilde{\psi}(x)$, Im $\widetilde{\psi}(x)$).

Let us now return to equations (5.13). They are parabolic equations with partial derivatives of the fourth order (ref. 238) having the form

$$\frac{\partial^4 X}{\partial y^4} + \Phi(x)\frac{\partial^2 X}{\partial x^2} = f(x, y). \tag{5.16}$$

The author knows nothing about methods for effectively constructing solutions of the edge effect type for such equations which would satisfy conditions (5.14).

We shall say a few words concerning the nature of the considered generalized effect. First of all we can see from (5.8) that it is much less pronounced than the simple edge effect (section 3). Indeed, if we assume that the order of the characteristic functions does not change during differentiation with respect to α, i.e., that t = 0, then $p = a^{-\frac{1}{4}} = \sqrt[4]{\frac{R_0}{h}}$. We recall that in the case of the simple edge effect the rate of variability was determined by the quantity $a^{-\frac{1}{2}} = \sqrt{\frac{R_0}{c}}$. A corollary of this is the situation when the zone of the generalized edge effect is not localized along the edge, but covers a substantial part of the shell and sometimes the entire shell. This, in turn, does not make it possible for us to apply the method of separate fulfillment of boundary conditions.

Furthermore, in the case of the generalized edge effect, unlike that /243
of the simple edge effect, the variability rate of the characteristic functions into the depth of the region depends substantially, (as we can see from $(5.8)_3$) on the rate of their variability along the edge. Thus, for example, if the functions decrease substantially (t < 0) during differentiation with respect to α, then, as we can see from (5.8), the variability rate with respect to β also decreases. A. L. Gol'denveyzer proposed that such a state of stress be called the degenerate edge effect. In regard to this effect, the method proposed above in general becomes inapplicable, because we obtain powers of a large factor in expansions (5.11)

$$\frac{1}{q} = a^t \quad (t < 0).$$

In essence, the latter means that in the case of a degenerate edge effect a large role is played by the variability of the coefficients. The generalized edge effect is discussed in detail in reference 11.

It is known that the application of variational methods and various numerical methods for the solution of problems in the theory of shells is usually complicated by the fact that the unknown solution contains parts which vary rapidly. In the utilization of variational methods, this situation makes it difficult to select the competitive (coordinate) functions (Chapter IX). On the other hand, in the utilization of numerical methods it becomes necessary to use extremely dense meshes (intervals). The role of the edge effect theory is reduced in essence to the construction of a simple expression for the rapidly varying part of the unknown solution. This, in turn, makes it easier to apply the variational and the numerical methods and also improves their convergence.

If we take these statements into account it becomes clear that the difficulties associated with the construction of the generalized edge effect do not pay for themselves, because in those isolated cases when it can be constructed we can be sure beforehand that it will not be simple. In addition, since the generalized edge effect (we have already mentioned this) is weakly defined, it does not hinder the application of variational and numerical methods. In other words, it is not necessary to break down the solution in this case.

These statements also pertain to the generalized edge effect associated with the asymptotic lines of the shell with negative Gaussian curvature. This case may be considered in a similar manner. In some sense it is an intermediate case, because the variability rate for it is determined by the quantity $a^{-\frac{1}{3}} = \sqrt[3]{\frac{R_0}{c}}$.

It is possible that the point of view expressed here is somewhat categorical and subjective. We cannot deny the fact that for very thin (refined) shells with simple geometry the introduction of the generalized edge effect/244 may simplify the calculation.

We recommend that the reader who is interested in the mathematical aspects of the problems considered in this chapter and who has the necessary mathematical background become familiar with references 252 and 256. In these references, in particular, the existence of a generalized edge effect (which is also called the parabolic boundary layer) is associated with the fact that the asymptotic lines of the middle surface coincide with the characteristics of the "membrane" part of the characteristic differential equation.

6. Separation of the State of Stress and of the Boundary Conditions

In cases when the boundary contour does not coincide with the asymptotic lines of the middle surface the unknown stress-strain state is usually separated

into the basic state (which varies slowly) and the edge effect. Usually the membrane state is taken as the basic state. Sometimes the basic state is different: a state of high moments (section 11) or a state of semimoments (section 10).

In separating the state of stress the most complicated problem concerns the separation of the boundary conditions. It is to this problem that we shall devote our prime attention.

As in section 3 of the preceding chapter, we introduce a parallel system of coordinates taking the boundary curve (g) as the reference curve. Taking into account equations (3.27) and the fact that in a parallel system of coordinates

$$\alpha=s_\nu,\quad \beta=s_t,\quad A=1,\quad B|_g=1,\quad \varkappa_t=\frac{1}{B}\frac{\partial B}{\partial s_\nu}, \tag{6.1}$$

we obtain the following expressions for the complex boundary quantities

$$\begin{aligned}
\widetilde{Q}_{\nu\nu} &\equiv Q_{\nu\nu}-iEhc\varkappa_{tt}=Q^0_{\nu\nu}-iEhc\varkappa^0_{tt}+iEhc\times\\
&\quad\times\left\{\varkappa_t\frac{\partial\widetilde{w}^{\kappa}}{\partial s_\nu}+\frac{\partial^2\widetilde{w}^{\kappa}}{\partial s_t^2}-\frac{1}{R^2_{\nu t}}\widetilde{w}^{\kappa}\right\};\\
\widetilde{Q}_{\nu t} &\equiv Q_{\nu t}+iEhc\varkappa_{t\nu}=Q^0_{\nu t}+iEhc\varkappa^0_{t\nu}+iEhc\times\\
&\quad\times\left\{-\frac{\partial^2\widetilde{w}^{\kappa}}{\partial s_t\partial s_\nu}+\varkappa_t\frac{\partial\widetilde{w}^{\kappa}}{\partial s_t}+\frac{1}{R_tR_{\nu t}}\widetilde{w}^{\kappa}\right\};\\
\widetilde{Q}_{\nu n} &\equiv Q_{\nu n}-iEhc\varkappa_{tn}=Q^0_{\nu n}-iEhc\varkappa^0_{tn}+iEhc\times\\
&\quad\times\left\{\frac{1}{R_t}\frac{\partial\widetilde{w}^{\kappa}}{\partial s_\nu}+\frac{2}{R_{\nu t}}\frac{\partial\widetilde{w}^{\kappa}}{\partial s_t}+\frac{\partial}{\partial s_t}\left(\frac{1}{R_{\nu t}}\right)\widetilde{w}^{\kappa}\right\};\\
\widetilde{M}_{\nu\nu} &=M_{\nu\nu}+iEhc\varepsilon_{tt}=M^0_{\nu\nu}+iEhc\varepsilon^0_{tt}+iEhc\left\{\frac{1}{R_t}\widetilde{w}^{\kappa}\right\}.
\end{aligned} \tag{6.2}$$

(K = edge)

Here and below the sign (^{0}) means that the quantity marked with it refers to the basic state of stress. The sign ($^{\kappa}$), on the other hand, means that the quantity refers to the edge effect. /245

By using the last two equations we can transform this system to the form

$$\begin{aligned}
\widetilde{Q}'_{\nu\nu} &\equiv Q'_{\nu\nu}-iEhc\varkappa'_{tt}=Q'^0_{\nu\nu}-iEhc\varkappa'^0_{tt};\\
\widetilde{Q}'_{\nu t} &\equiv Q'_{\nu t}+iEhc\varkappa'_{t\nu}=Q'^0_{\nu t}+iEhc\varkappa'^0_{t\nu};\\
\widetilde{Q}'_{\nu n} &\equiv Q'_{\nu n}-iEhc\varkappa'_{tn}=Q'^0_{\nu n}-iEhc\varkappa'^0_{tn}+iEhc\left\{\frac{1}{R_t}\frac{\partial\widetilde{w}^{\kappa}}{\partial s_\nu}\right\};
\end{aligned} \tag{6.3}$$

$$\widetilde{M}_{\nu\nu} \equiv M_{\nu\nu} + iEhc\varepsilon_{tt} = M^{\circ}_{\nu\nu} + iEhc\varepsilon^{\circ}_{tt} + iEhc\left\{\frac{1}{R_t}\widetilde{w}^{\kappa}\right\}, \tag{6.4}$$

where

$$\begin{aligned}
\widetilde{Q}'_{\nu\nu} &= \widetilde{Q}_{\nu\nu} - \varkappa_t R_t \widetilde{Q}_{\nu n} + \left\{-\frac{\partial^2 R_t \widetilde{M}_{\nu\nu}}{\partial s_t^2} + \frac{R_t}{R_{\nu t}^2}\widetilde{M}_{\nu\nu} + \right.\\
&\left. + \varkappa_t R_t\left[\frac{\partial}{\partial s_t}\left(\frac{R_t}{R_{\nu t}}\widetilde{M}_{\nu\nu}\right) + \frac{1}{R_{\nu t}}\frac{\partial R_t \widetilde{M}_{\nu\nu}}{\partial s_t}\right]\right\};\\
\widetilde{Q}'_{\nu t} &= \widetilde{Q}_{\nu t} + \frac{\partial R_t \widetilde{Q}_{\nu n}}{\partial s_t} + \left\{-\varkappa_t \frac{\partial R_t \widetilde{M}_{\nu\nu}}{\partial s_t} - \frac{1}{R_{\nu t}}\widetilde{M}_{\nu\nu} - \right.\\
&\left. - \frac{\partial}{\partial s_t} R_t\left[\frac{\partial}{\partial s_t}\left(\frac{R_t}{R_{\nu t}}\widetilde{M}_{\nu\nu}\right) + \frac{1}{R_{\nu t}}\frac{\partial R_t \widetilde{M}_{\nu\nu}}{\partial s_t}\right]\right\};\\
\widetilde{Q}'_{\nu n} &= \widetilde{Q}_{\nu n} + \left\{-\frac{\partial}{\partial s_t}\left(\frac{R_t}{R_{\nu t}}\widetilde{M}_{\nu\nu}\right) - \frac{1}{R_{\nu t}}\frac{\partial R_t \widetilde{M}_{\nu\nu}}{\partial s_t}\right\}.
\end{aligned} \tag{6.5}$$

In accordance with (6.3) the complex combinations $\widetilde{Q}'_{\nu\nu}$ and $\widetilde{Q}'_{\nu t}$ are entirely independent of the edge effect. Therefore, it is pertinent to call them the basic combinations. Such will also be the edge forces

$$\begin{aligned}
Q'_{\nu\nu} &= Q_{\nu\nu} - \varkappa_t R_t Q_{\nu n} + \left\{-\frac{\partial^2 R_t M_{\nu\nu}}{\partial s_t^2} + \frac{R_t}{R_{\nu t}^2} M_{\nu\nu} + \right.\\
&\left. + \varkappa_t R_t\left[\frac{\partial}{\partial s_t}\left(\frac{R_t}{R_{\nu t}} M_{\nu\nu}\right) + \frac{1}{R_{\nu t}}\frac{\partial R_t M_{\nu\nu}}{\partial s_t}\right]\right\};\\
Q'_{\nu t} &= Q_{\nu t} + \frac{\partial R_t Q_{\nu n}}{\partial s_t} + \left\{-\varkappa_t \frac{\partial R_t M_{\nu\nu}}{\partial s_t} - \frac{1}{R_{\nu t}} M_{\nu\nu} - \right.\\
&\left. - \frac{\partial}{\partial s_t} R_t\left[\frac{\partial}{\partial s_t}\left(\frac{R_t}{R_{\nu t}} M_{\nu\nu}\right) + \frac{1}{R_{\nu t}}\frac{\partial R_t M_{\nu\nu}}{\partial s_t}\right]\right\},
\end{aligned} \tag{6.6}$$

obtained by isolating the real parts from (6.5).

On the other hand, by isolating the imaginary parts we find that the deformation components are also basic /246

$$\begin{aligned}
\varkappa'_{tt} &= \varkappa_{tt} - \varkappa_t R_t \varkappa'_{tn} + \frac{\partial^2 R_t \varepsilon_{tt}}{\partial s_t^2} - \frac{R_t}{R_{\nu t}^2}\varepsilon_{tt};\\
\varkappa'_{t\nu} &= \varkappa_{t\nu} - \frac{\partial R_t \varkappa'_{tn}}{\partial s_t} - \varkappa_t \frac{\partial R_t \varepsilon_{tt}}{\partial s_t} - \frac{1}{R_{\nu t}}\varepsilon_{tt},
\end{aligned} \tag{6.7}$$

where

$$\varkappa'_{tn} = \varkappa_{tn} + \frac{\partial}{\partial s_t}\left(\frac{R_t \varepsilon_{tt}}{R_{vt}}\right) + \frac{1}{R_{vt}} \frac{\partial R_t \varepsilon_{tt}}{\partial s_t} \tag{6.8}$$

is the imaginary part of the complex force $\widetilde{Q}'_{\nu n}$.

Let us consider them in more detail. By applying relationships (Chapter VIII, 3.15 and 3.18) the first may be written out in the following form

$$\begin{aligned}\varkappa'_{tt} = &\left\{\frac{\partial}{\partial s_t}\left(-\frac{\partial w}{\partial s_t} + \frac{u_t}{R_t} - \frac{u_\nu}{R_{vt}}\right) + \varkappa_t \vartheta_\nu + \frac{1}{R_{vt}}\left(-\frac{\partial u_\nu}{\partial s_t} + \varkappa_t u_t + \right.\right. \\ &\left.\left. + \frac{w}{R_{vt}}\right)\right\} - \varkappa_t R_t\left[-\frac{\partial}{\partial s_t}\left(-\frac{\partial u_\nu}{\partial s_t} + \varkappa_t u_t + \frac{w}{R_{vt}}\right) + \frac{\vartheta_\nu}{R_t} + \frac{1}{R_{vt}}\left(-\frac{\partial w}{\partial s_t} + \right.\right. \\ &\left.\left. + \frac{u_t}{R_t} - \frac{u_\nu}{R_{vt}}\right)\right] + \left[\frac{\partial}{\partial s_t^2} - \frac{1}{R_{vt}^2} - \varkappa_t R_t\left(\frac{\partial}{\partial s_t}\left(\frac{1}{R_{vt}}\right) + \right.\right. \\ &\left.\left. + \frac{1}{R_{vt}} \frac{\partial}{\partial s_t}\right)\right] R_t\left(\frac{\partial u_t}{\partial s_t} + \varkappa_t u_\nu + \frac{w}{R_t}\right).\end{aligned}$$

We can see from this that the quantity $\varkappa'_{tt}$ may be looked upon as the deformation component $\varkappa_{tt}$ (it is contained in the figure brackets) from which w and ϑ_ν are eliminated by means of equalities

$$\begin{aligned}&\varepsilon_{tt} \equiv \frac{\partial u_t}{\partial s_t} + \varkappa_t u_\nu + \frac{w}{R_t} = 0; \\ &\varkappa'_{tn} \equiv \left(-\frac{\partial \omega_{tt}}{\partial s_t} + \frac{\vartheta_\nu}{R_t} + \frac{\vartheta_t}{R_{vt}}\right) + \frac{\partial}{\partial s_t}\left(\frac{R_t}{R_{vt}} \varepsilon_{tt}\right) + \frac{1}{R_{vt}} \frac{\partial R_t \varepsilon_{tt}}{\partial s_t} = 0,\end{aligned} \tag{6.9}$$

i.e., they are replaced by the quantities

$$\begin{aligned}&w^{\infty} = -R_t\left(\frac{\partial u_t}{\partial s_t} + \varkappa_t u_\nu\right); \\ &\vartheta_\nu^{\infty} = R_t\left(\frac{\partial \omega_{tt}^{\infty}}{\partial s_t} - \frac{\vartheta_t^{\infty}}{R_{vt}}\right) = R_t\left\{\frac{\partial}{\partial s_t}\left[-\frac{\partial u_\nu}{\partial s_t} + \varkappa_t u_t - \right.\right. \\ &\left.\left. - \frac{R_t}{R_{vt}}\left(\frac{\partial u_t}{\partial s_t} + \varkappa_t u_\nu\right)\right] - \frac{1}{R_{vt}}\left[\frac{\partial}{\partial s_t} R_t\left(\frac{\partial u_t}{\partial s_t} + \varkappa_t u_\nu\right) + \frac{u_t}{R_t} - \frac{u_\nu}{R_{vt}}\right]\right\},\end{aligned} \tag{6.10}$$

where

$$\omega_{tt}^{00} = -\frac{\partial u_\nu}{\partial s_t} + \varkappa_t u_t + \frac{w^{00}}{R_{\nu t}}; \quad \vartheta_\nu^{00} = -\frac{\partial w^{00}}{\partial s_t} + \frac{u_t}{R_t} - \frac{u_\nu}{R_{\nu t}}. \tag{6.11}$$

The examination of $\varkappa'_{t\nu}$ leads to the same conclusion.

As in the case when we utilize the solution of the edge effect type, /247 we refer the tangential displacements to the basic state. Then[1]

$$u_\nu = u_\nu^0, \quad u_t = u_t^0 \tag{6.12}$$

and from (6.10), (6.11), and (Chapter VIII, 3.15) we find

$$w^\nu = w^{00} + R_t \varepsilon_{tt}^0; \quad \vartheta_\nu^0 = \vartheta_\nu^{00} + R_t \varkappa_{tn}^{\prime 0}; \tag{6.13}$$

$$\omega_{tt}^0 = \omega_{tt}^{00} + \frac{R_t}{R_{\nu t}} \varepsilon_{tt}^0; \quad \vartheta_t^0 = \vartheta_t^{00} - \frac{\partial R_t \varepsilon_{tt}^0}{\partial s_t}. \tag{6.14}$$

Furthermore, since the edge effect is characterized only by one component of the displacement vector--the normal flexure w^K

$$\vartheta_\nu^K = -\frac{\partial w^K}{\partial s_\nu}; \quad \omega_{tt}^K = \frac{w^K}{R_{\nu t}}; \quad \vartheta_t^K = -\frac{\partial w^K}{\partial s_t};$$
$$\varepsilon_{tt}^K = \frac{w^K}{R_t}; \quad \varkappa_{tn}^{\prime K} = -\frac{1}{R_t}\frac{\partial w^K}{\partial s_\nu}. \tag{6.15}$$

By collecting all of the equations we now obtain

$$u_\nu = u_\nu^0; \quad u_t = u_t^0; \quad w = w^{00} + R_t \varepsilon_{tt}^0 + w^K;$$
$$\vartheta_\nu = \vartheta_\nu^{00} + R_t \varkappa_{tn}^{\prime 0} - \frac{\partial w^K}{\partial s_\nu}; \tag{6.16}$$

$$\vartheta_t = \vartheta_t^{00} - \frac{\partial R_t \varepsilon_{tt}^0}{\partial s_t} - \frac{\partial w^K}{\partial s_t}; \quad \omega_{tt} = \omega_{tt}^{00} + \frac{R_t}{R_{\nu t}} \varepsilon_{tt}^0 + \frac{w^K}{R_{\nu t}}; \tag{6.17}$$

$$\varkappa'_{tt} = \varkappa_{tt}^{00}; \quad \varkappa'_{t\nu} = \varkappa_{tt}^{00}; \quad \varkappa'_{tn} = \varkappa_{tn}^{\prime 0} - \frac{1}{R_t}\frac{\partial w^K}{\partial s_\nu};$$
$$\varepsilon_{tt} = \varepsilon_{tt}^0 + \frac{1}{R_t} w^K. \tag{6.18}$$

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[1] There are versions of boundary conditions for which the solution of the edge effect produces substantial corrections for the tangential displacements (ref. 256). We do not consider them, because such boundary conditions are rather artificial and are of little practical interest.

We have carried out the necessary separation of the boundary geometric quantities.

7. Continuation. Conditions for Elastic Junction of Shells

1. Let us replace the displacements u_ν, u_t, w in the relationships of the preceding section by the corresponding stress functions $\bar{u}_\nu$, $\bar{u}_t$, $\bar{w}$. From the static-geometric analogy which relieves us from the necessity of repeating the discussions of the preceding section, we obtain /248

$$\bar{u}_\nu = \bar{u}_\nu^0; \quad \bar{u}_t = \bar{u}_t^0; \quad \bar{w} = \bar{w}^{00} + R_t \bar{\varepsilon}_{tt}^0 + \bar{w}^\kappa;$$
$$\bar{\vartheta}_\nu = \bar{\vartheta}_\nu^{00} + R_t \bar{\varkappa}_{tn}^0 - \frac{\partial \bar{w}^\kappa}{\partial s_\nu}; \tag{7.1}$$

$$\bar{\vartheta}_t = \bar{\vartheta}_t^{00} - \frac{\partial R_t \bar{\varepsilon}_{tt}^0}{\partial s_t} - \frac{\partial \bar{w}^\kappa}{\partial s_t}; \quad \bar{\omega}_{tt} = \bar{\omega}_{tt}^{00} + \frac{R_t}{R_{\nu t}} \bar{\varepsilon}_{tt}^0 + \frac{\bar{w}^\kappa}{R_{\nu t}}; \tag{7.2}$$

$$\bar{\varkappa}'_{tt} = \bar{\varkappa}_{tt}^{00}, \quad \bar{\varkappa}'_{t\nu} = \bar{\varkappa}_{t\nu}^{00}; \quad \bar{\varkappa}'_{tn} = \bar{\varkappa}_{tn}^{0} - \frac{1}{R_t} \frac{\partial \bar{w}^\kappa}{\partial s_\nu}; \quad \bar{\varepsilon}_{tt} = \bar{\varepsilon}_{tt}^0 + \frac{\bar{w}^\kappa}{R_t}, \tag{7.3}$$

where

$$\bar{w}^{00} = -R_t \left(\frac{\partial \bar{u}_t}{\partial s_t} + \varkappa_t \bar{u}_\nu \right);$$

$$\bar{\vartheta}_\nu^{00} = R_t \left\{ \frac{\partial}{\partial s_t} \left[-\frac{\partial \bar{u}_\nu}{\partial s_t} + \varkappa_t \bar{u}_t - \frac{R_t}{R_{\nu t}} \left(\frac{\partial \bar{u}_t}{\partial s_t} + \varkappa_t \bar{u}_\nu \right) \right] - \right. \tag{7.4}$$

$$\left. - \frac{1}{R_{\nu t}} \left[\frac{\partial}{\partial s_t} R_t \left(\frac{\partial \bar{u}_t}{\partial s_t} + \varkappa_t \bar{u}_\nu \right) + \frac{\bar{u}_t}{R_t} - \frac{\bar{u}_\nu}{R_{\nu t}} \right] \right\}; \tag{7.5}$$

$$\bar{\omega}_{tt}^{00} = -\frac{\partial \bar{u}_\nu}{\partial s_t} + \varkappa_t \bar{u}_t + \frac{\bar{w}^{00}}{R_{\nu t}}, \quad \bar{\vartheta}_t^{00} = -\frac{\partial \bar{w}^{00}}{\partial s_t} + \frac{\bar{u}_t}{R_t} - \frac{\bar{u}_\nu}{R_{\nu t}}.$$

We recall that the vectors

$$\vec{\bar{U}} = \bar{u}_\nu \vec{\nu} + \bar{u}_t \vec{t} + \bar{w} \vec{n}, \quad \vec{\bar{\Omega}}_t = -\bar{\vartheta}_t \vec{\nu} + \bar{\vartheta}_\nu \vec{t} + \bar{\omega}_{tt} \vec{n} \quad \text{(Chapter VIII, 8.31, 8.30)}$$

are associated with the principal vector

$$\vec{F} = \int_{s_0}^{s_t} \vec{Q}_\nu ds_t' \qquad \text{(Chapter VIII, 8.32)}$$

and with the principal moment

$$\vec{B} = \int_{s_0}^{s_1} M_{\nu\nu}\vec{t}'\,ds_t' + \int_{s_0}^{s_1} \left[\vec{r}' - \vec{r},\ \vec{Q}_\nu\right] ds_t' \qquad \text{(Chapter VIII, 8.33)}$$

by the following relationships

$$\vec{\Omega}_t = -\frac{1}{Ehc}\vec{F};\quad \vec{U} = -\frac{1}{Ehc}\vec{B}. \qquad \text{(Chapter VIII, 8.34)}$$

Furthermore, by using the well-known relationships

$$\begin{aligned} Q_{\nu\nu} &= Ehc\bar{\varkappa}_{tt};\quad Q_{\nu t} = -Ehc\bar{\varkappa}_{t\nu};\\ Q_{\nu n} &= Ehc\bar{\varkappa}_{tn};\quad M_{\nu\nu} = -Ehc\bar{\varepsilon}_{tt} \end{aligned} \qquad \text{(Chapter VIII, 8.29)}$$

we can obtain new relationships

$$\begin{aligned} Q'_{\nu\nu} &= Ehc\bar{\varkappa}'_{tt};\quad Q'_{\nu t} = -Ehc\bar{\varkappa}'_{t\nu};\\ Q'_{\nu n} &= Ehc\bar{\varkappa}'_{tn};\quad M'_{\nu\nu} = -Ehc\bar{\varepsilon}_{tt}. \end{aligned} \qquad (7.6)$$

From this and from the first two relationships of (7.3) we draw a conclusion which has already been made in the preceding section: $Q'_{\nu\nu}$, $Q'_{\nu t}$ are quantities which do not depend on the edge effect. In addition, from (Chapter VIII, 8.34) and $(7.1)_{1,2}$ a new interesting corollary follows: the tangential components of the principal moment B_ν and B_t are determined entirely by the basic solution.

Finally, by using the symbol $[\ ]_g$ to designate the increment of the quantity when tracing the contour g, we obtain from (Chapter VIII, 8.34) /249

$$\left[\vec{\Omega}_t\right]_g = -\frac{1}{Ehc}\vec{F}_g;\quad \left[\vec{U}\right]_g = -\frac{1}{Ehc}\vec{B}_g. \qquad (7.7)$$

Here $\vec{F}_g$ and $\vec{B}_g$ are the principal vector and moment of forces and moments applied to the contour g. It follows from this that in cases when a load which is not self-balanced is applied to g the stress functions are not single-valued.

Since $\mathfrak{W}^{K}$ (4.1) is single-valued, we can maintain that the edge effect corresponds to a self-balanced load (along each closed contour).

It follows from this that the basic state of stress "carries the responsibility" for the lack of self-balancing in the edge load.

2. Let us consider briefly the boundary conditions corresponding to the separation which has been carried out. In this case we shall limit ourselves to the consideration of the basic version which follows from the variational equation of Lagrange

$$\delta V - \int_g \{Q_{\nu\nu 0}\delta u_\nu + Q_{\nu t 0}\delta u_t + Q_{\nu n 0}\delta w + M_{\nu\nu 0}\delta\vartheta_\nu\}\, ds_t - \\ - \iint_S \{q_1\delta u_1 + q_2\delta u_2 + q_n\delta w\}\, A_1 A_2 d\alpha_1 d\alpha_2 = 0. \tag{7.8}$$

Here δV is the variation of the deformation energy, S is the region of the middle surface bounded by the boundary contour g.

Transforming as usual (section 4, Chapter IX) the variation of the deformation energy we arrive at the integral relationship

$$\int_g \{(Q_{\nu\nu} - Q_{\nu\nu 0})\delta u_\nu + (Q_{\nu t} - Q_{\nu t 0})\delta u_t + \\ + (Q_{\nu n} - Q_{\nu n 0})\delta w + (M_{\nu\nu} - M_{\nu\nu 0})\delta\vartheta_\nu\}\, ds_t = 0. \tag{7.9}$$

If we assume that the geometrically permissible displacements are permitted to compete, then by the usual method we obtain from (7.9) the (natural) static boundary conditions

$$Q_{\nu\nu} = Q_{\nu\nu 0};\quad Q_{\nu t} = Q_{\nu t 0};\quad Q_{\nu n} = Q_{\nu n 0};\quad M_{\nu\nu} = M_{\nu\nu 0}. \tag{7.10}$$

It now remains for us to add to these the geometric boundary conditions

$$u_\nu = u_{\nu 0};\quad u_t = u_{t 0};\quad w = w_0;\quad \vartheta_\nu = \vartheta_{\nu 0} \tag{7.11}$$

(the zero in the subscript shows that the particular quantity is assigned on the contour).

By using relationships of the preceding section we can transform /250
equation (7.9) to the form

$$\int_g \{(Q'_{\nu\nu} - Q'_{\nu\nu 0})\,\delta u_\nu + (Q'_{\nu t} - Q'_{\nu t 0})\,\delta u_t + (Q'_{\nu n} - Q'_{\nu n 0})\,\delta(R_t \varepsilon_{tt}) + (M_{\nu\nu} - M_{\nu\nu 0})\,\delta(R_t \varkappa'_{tn})\}\,ds_t = 0. \tag{7.12}$$

From this the "natural" static boundary conditions follow

$$Q'_{\nu\nu} = Q'_{\nu\nu 0}, \quad Q'_{\nu t} = Q'_{\nu t 0}, \quad Q'_{\nu n} = Q'_{\nu n 0}, \quad M_{\nu\nu} = M_{\nu\nu 0}. \tag{7.13}$$

To these we add the corresponding geometric conditions

$$u_\nu = u_{\nu 0}, \quad u_t = u_{t 0}, \quad R_t \varepsilon_{tt} = R_t \varepsilon_{tt 0}, \quad R_t \varkappa'_{tn} = R_t \varkappa'_{tn 0}. \tag{7.14}$$

Substituting the expressions (6.16)-(6.18), (7.3) and (7.6) we obtain

$$\begin{aligned} &u_\nu^0 = u_{\nu 0}; \quad w^\varkappa = R_t(\varepsilon_{tt 0} - \varepsilon_{tt}^0); \\ &u_t^0 = u_{t 0}; \quad -\frac{\partial w^\varkappa}{\partial s_\nu} = R_t(\varkappa'_{tn 0} - \varkappa'^0_{tn}); \end{aligned} \tag{7.15}$$

$$\begin{aligned} &Q'^0_{\nu\nu} = Q'_{\nu\nu 0}; \quad Q'^\varkappa_{\nu n} = Q'_{\nu n 0} - Q'^0_{\nu n}; \\ &Q'^0_{\nu t} = Q'_{\nu t 0}; \quad M^\varkappa_{\nu\nu} = M_{\nu\nu 0} - M^0_{\nu\nu}. \end{aligned} \tag{7.16}$$

The boundary conditions which have been obtained are equivalent to conditions (7.10)-(7.11), but differ from them in an advantageous manner. Indeed, the expressions for $Q'_{\nu\nu}$ and $Q'_{\nu t}$ do not contain a solution of the boundary effect type. Therefore, the problem breaks down completely into two: the determination of the basic state of stress with its boundary conditions

$$\begin{aligned} &Q'^0_{\nu\nu} = Q'_{\nu\nu 0}, \quad Q'^0_{\nu t} = Q'_{\nu t 0}, \\ &u_\nu^0 = u_{\nu 0}, \quad u_t^0 = u_{t 0} \end{aligned}$$

and a subsequent determination of arbitrary elements in the edge effect solution. In this case the geometric part of the conditions for the latter is formulated in terms of the components for the tangential deformation which produce (section 9) the edge effect. In particular, in these boundary conditions there is absolutely no role played by the purely flexural displacements and the displacements of the shell as a rigid whole, which are insignificant.

Considering the integral

$$J=\int_{g}\{Q'_{\nu\nu}u_{\nu}+Q'_{\nu t}u_{t}+Q'_{\nu n}(R_{t}\varepsilon_{tt})+M_{\nu\nu}(R_{t}\varkappa'_{tn})\,ds_{t}, \tag{7.17}$$

we can arrive at a conclusion in the usual manner (section 2, Chapter IX), namely, that the solution of the equations of the theory of shells is single-valued for the boundary conditions (7.15)-(7.16).

The relationships presented above make it possible to carry out the separation for other versions of boundary conditions: deformation conditions, elastic constraints, etc.

3. Let us assume that two shells are joined elastically and that the /251 first of the following conditions is satisfied

(a) the shells are joined smoothly without a break;

$$\text{(b)}\ \{\mathrm{Ehc}\}^{\mathrm{I}}=\{\mathrm{Ehc}\}^{\mathrm{II}}. \tag{7.18}$$

(The symbols $^{\mathrm{I}}$ and $^{\mathrm{II}}$ indicate that the quantities pertain respectively to the first and to the second shell.)

From the relationships presented above we can see that along the common junction the static quantities: $Q'_{\nu\nu}$, $Q'_{\nu t}$, $Q'_{\nu n}$, $M_{\nu\nu}$ as well as the deformation quantities: $\varkappa'_{tt}$, $\varkappa'_{t\nu}$, $\varkappa'_{tn}$, ε_{tt} must be continuous. If at the same time condition "b" is satisfied, the complex boundary quantities $\widetilde{Q}'_{\nu\nu}$, $\widetilde{Q}'_{\nu t}$, $\widetilde{Q}'_{\nu n}$, $\widetilde{M}_{\nu\nu}$ are also continuous. The conditions of continuity or, as we call them, the conditions of complex junction, are written in accordance with (6.3)-(6.4) in the following manner

$$\begin{gathered}
\{Q'^{0}_{\nu\nu}-iEhc\varkappa^{00}_{tt}\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}};\quad \{Q'^{0}_{\nu t}+iEhc\varkappa^{00}_{t\nu}\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}};\\
\left\{Q'^{0}_{\nu n}-iEhc\varkappa'^{0}_{tn}+iEhc\frac{1}{R_{t}}\frac{\partial\widetilde{w}^{\kappa}}{\partial s_{\nu}}\right\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}};\\
\left\{M^{0}_{\nu\nu}+iEhc\varepsilon^{0}_{tt}+iEhc\frac{1}{R_{t}}\widetilde{w}^{\kappa}\right\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}}.
\end{gathered} \tag{7.19}$$

If, on the other hand, condition "b" is not satisfied, the real and imaginary parts of these expressions are equated separately (after reducing the imaginary parts by Ehc).

8. Simplified Versions. Compliance Coefficients

1. The relationships obtained in the two preceding sections are "exact" in the sense that when we derived them we neglected only quantities of the order

c/R_0 compared with 1. For practical calculations it is usually sufficient to consider the edge effect in a cruder (first) approximation based on the neglect of terms of order $\sqrt{\frac{c}{R_0}}$ compared with 1 (section 4).

In this case expressions (6.3) retain their form, but the quantities contained in them are substantially simplified. Thus

$$\widetilde{Q}'_{\nu\nu} = \widetilde{Q}_{\nu\nu} - \varkappa_t R_t \widetilde{Q}_{\nu n};\ \widetilde{Q}'_{\nu t} = \widetilde{Q}_{\nu t} + \frac{\partial R_t \widetilde{Q}_{\nu n}}{\partial s_t};\ \widetilde{Q}'_{\nu n} = \widetilde{Q}_{\nu n}; \tag{8.1}$$

$$Q'_{\nu\nu} = Q_{\nu\nu} - \varkappa_t R_t Q_{\nu n};\ Q'_{\nu t} = Q_{\nu t} + \frac{\partial R_t Q_{\nu n}}{\partial s_t};\ Q'_{\nu n} = Q_{\nu n}; \tag{8.2}$$

$$\varkappa'_{tt} = \varkappa_{tt} - \varkappa_t R_t \varkappa_{tn};\ \varkappa'_{t\nu} = \varkappa_{t\nu} - \frac{\partial R_t \varkappa_{tn}}{\partial s_t};\ \varkappa'_{tn} = \varkappa_{tn}. \tag{8.3}$$

With regard to the simplified expression for $\widetilde{w}^K$ (4.1) in the parallel system of coordinates, it is written in the following manner /252

$$\widetilde{w}^K = [\varphi(s_t) + i\psi(s_t)]\, e^{(1-t\,\mathrm{sign}\,R_{t0})\beta}, \tag{8.4}$$

where

$$\mathrm{sign}\,R_{t0} = \begin{cases} +1 & \text{for } R_{t0} > 0 \\ -1 & \text{for } R_{t0} < 0 \end{cases};\quad \beta = \frac{s_\nu}{\sqrt{2c\,|R_{t0}|}}. \tag{8.5}$$

We recall that the quantity s_ν represents the distance along the geodetic normal from the considered point in the region to the boundary contour. It is significant that s_ν which becomes equal to zero at the contour is a negative quantity, so that solution (8.4) is rapidly damped as we move away from the contour. The arbitrary functions $\varphi(s_t)$ and $\psi(s_t)$ which are contained in the solution are determined from the boundary conditions. Anticipating, we note that in specific problems (section 13) no necessity arises for the actual construction of geodetic normals. By means of expressions (6.4) we can introduce compliance (rigidity) coefficients for the edge in the same manner as when we considered the symmetric deformation of the shells of revolution (section 8, Chapter V). Indeed, by separating the real parts and introducing the designations

$$Q_0 = Q_{\nu n}|_g;\ Q_0^0 = Q_{\nu n}^0|_g;\ M_0 = M_{\nu\nu}|_g;\ M_0^0 = M_{\nu\nu}^0|_g;\ R_{t0} = R_t|_g, \tag{8.6}$$

we obtain

$$Q_0 - Q_0^0 = \frac{Ehc}{R_{t0}} \frac{1}{\sqrt{2c|R_{t0}|}} (\varphi \operatorname{sign} R_{t0} - \psi);$$
$$M_0 - M_0^0 = -\frac{Ehc}{R_{t0}} \psi \tag{8.7}$$

or

$$\psi = -\frac{R_{t0}}{Ehc}(M_0 - M_0^0); \varphi = \operatorname{sign} R_{t0} \frac{R_{t0}}{Ehc} \times$$
$$\times |\sqrt{2c|R_{t0}|}(Q_0 - Q_0^0) - (M_0 - M_0^0)|. \tag{8.8}$$

Then, by separating the imaginary parts in the same relationships and introducing expressions (8.8) into them, we obtain

$$(R_t \varepsilon_{tt})_0 = (R_t \varepsilon_{tt}^0)_0 + \beta_{11}(Q_0 - Q_0^0) + \beta_{12}(M_0 - M_0^0);$$
$$(R_t \varkappa_{tn})_0 = (R_t \varkappa_{tn}^c)_0 + \beta_{12}(Q_0 - Q_0^0) + \beta_{22}(M_0 - M_0^0), \tag{8.9}$$

where

$$\beta_{11} = \frac{2\sqrt[4]{3(1-\nu^2)}}{E}\left(\frac{|R_{t0}|}{h}\right)^{\frac{3}{2}};$$
$$\beta_{12} = -\frac{2\sqrt{3(1-\nu^2)}}{Eh}\left(\frac{|R_{t0}|}{h}\right); \tag{8.10}$$
$$\beta_{22} = \frac{4[3(1-\nu^2)]^{\frac{3}{4}}}{Eh^2}\left(\frac{|R_{t0}|}{h}\right)^{\frac{1}{2}}.$$

By determining the quantities $(Q_0 - Q_0^0)$, $(M_0 - M_0^0)$, from here (in those cases /253 when they are not directly assigned) and substituting expressions (8.8) into (4.6), we arrive at the following expressions for the stresses

$$\sigma_{M_\nu} = \sigma_{M_\nu^0} + \sigma_{M_0}(\cos\beta - \sin\beta)e^3 + \operatorname{sign} R_{t0} \frac{3}{\sqrt{3(1-\nu^2)}} \sigma_\kappa \sin\beta e^3;$$
$$\sigma_{M_t} = \sigma_{M_t^0} + \nu\left[\sigma_{M_0}(\cos\beta - \sin\beta)e^3 + \operatorname{sign} R_{t0} \frac{3}{\sqrt{3(1-\nu^2)}} \sigma_\kappa \sin\beta e^3\right]; \tag{8.11}$$
$$\sigma_{T_t} = \sigma_{T_t^0} + \sigma_\kappa \cos\beta e^3 - \operatorname{sign} R_{t0} \frac{\sqrt{3(1-\nu^2)}}{3} \sigma_{M_0}(\cos\beta + \sin\beta)e^3;$$

$$\sigma_{T_\nu} = \sigma_{T_\nu^0};\ \sigma_S = \sigma_{S^0};\ \sigma_H = \sigma_{H^0}$$

$$\left(\beta = \frac{s_\nu}{\sqrt{2c\,|R_{t0}|}}\right).$$

Here

$$\sigma_{M_0} = \frac{6(M_0 - M_0^0)}{h^2};\ \sigma_K = \operatorname{sign} R_{t0} \cdot 2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{|R_{t0}|}{h}}\,\frac{Q_0 - Q_0^0}{h}. \tag{8.12}$$

It is easy to obtain relationships which are the converse of (8.9)

$$\begin{aligned} Q_0 &= Q_0^0 + B_{11}[R_t(\varepsilon_{tt} - \varepsilon_{tt}^0)]_0 + B_{12}[R_t(\varkappa_{tn} - \varkappa_{tn}^0)]_0; \\ M_0 &= M_0^0 + B_{12}[R_t(\varepsilon_{tt} - \varepsilon_{tt}^0)]_0 + B_{22}[R_t(\varkappa_{tn} - \varkappa_{tn}^0)]_0. \end{aligned} \tag{8.13}$$

Here

$$\begin{aligned} B_{11} &= \frac{E}{\sqrt[4]{3(1-\nu^2)}}\left(\frac{h}{|R_{t0}|}\right)^{\frac{3}{2}}; \\ B_{12} &= \frac{Eh}{2\sqrt{3(1-\nu^2)}}\left(\frac{h}{|R_{t0}|}\right); \\ B_{22} &= \frac{Eh^2}{2[3(1-\nu^2)]^{\frac{3}{4}}}\left(\frac{h}{|R_{t0}|}\right)^{\frac{1}{2}}. \end{aligned} \tag{8.14}$$

Usually the quantities $(\varepsilon_{tt} - \varepsilon_{tt}^0)$ and $R_t\,(\varkappa_{tn} - \varkappa_{tn}^0)$ are of the same order of magnitude. From this and from (8.13)-(8.14) we can see that with an accuracy achieved before quantities of order $\sqrt{\frac{c}{R_0}}$ are neglected with respect to unity, we can approximately assume that

$$Q_0 - Q_0^0 = B_{11}[R_t(\varepsilon_{tt} - \varepsilon_{tt}^0)]_0;\ M_0 - M_0^0 = B_{12}[R_t(\varepsilon_{tt} - \varepsilon_{tt}^0)]_0. \tag{8.15}$$

In this way we can assume that the occurrence of the edge effect is produced by the fact that the relative elongation of the contour corresponding to the basic state of stress does not coincide with that required by the boundary conditions of the problem.

2. In order to make it possible for us to obtain the conditions for /254
elastic joining of shells which are also valid when the shells are joined at an angle, we use the "natural" directions for the shell which are associated with the tangent, the principal normal and the binormal to the boundary contour. We recall that in this case

$$Q_{vm} = Q_{vv}\sin\varphi + Q_{vn}\cos\varphi;\quad Q_{vb} = Q_{vv}\cos\varphi - Q_{vn}\sin\varphi; \qquad \text{(Chapter VIII, 5.18)}$$
$$\varkappa_{tm} = -\varkappa_{tt}\sin\varphi - \varkappa_{tn}\cos\varphi;\quad \varkappa_{tb} = -\varkappa_{tt}\cos\varphi + \varkappa_{tn}\sin\varphi;$$
$$\cos\varphi = -\frac{1}{R_t\hat{\sigma}};\quad \sin\varphi = -\frac{\varkappa_t}{\hat{\sigma}}, \qquad \text{(Chapter VIII, 3.26)}$$

where

$$\hat{\sigma} = +\sqrt{\left(\frac{1}{R_t}\right)^2 + \varkappa_t^2} \qquad \text{(Chapter VII, 7.38-7.39)}$$

is the spatial curvature of the boundary contour.

Within the accuracy adopted in this section, relationships (6.2) may be written in the form

$$\begin{aligned}
\widetilde{Q}_{vv} &\equiv Q_{vv} - iEhc\varkappa_{tt} = Q^0_{vv} - iEhc\varkappa^0_{tt} + iEhc\varkappa_t\frac{\partial\widetilde{w}^\kappa}{\partial s_v};\\
\widetilde{Q}_{vt} &\equiv Q_{vt} + iEhc\varkappa_{tv} = Q^0_{vt} + iEhc\varkappa^0_{tv} - iEhc\frac{\partial^2\widetilde{w}^\kappa}{\partial s_t\partial s_v};\\
\widetilde{Q}_{vn} &\equiv Q_{vn} - iEhc\varkappa_{tn} = Q^0_{vn} - iEhc\varkappa^0_{tn} + iEhc\frac{\partial\widetilde{w}^\kappa}{R_t\partial s_v};\\
\widetilde{M}_{vv} &\equiv M_{vv} + iEhc\varepsilon_{tt} = M^0_{vv} + iEhc\varepsilon^0_{tt} + iEhc\frac{1}{R_t}\widetilde{w}^\kappa.
\end{aligned} \qquad (8.16)$$

From this, by using the equalities written out above, we obtain, first of all

$$\begin{aligned}
\widetilde{Q}_{vb} &\equiv Q_{vb} + iEhc\varkappa_{tb} = Q^0_{vb} + iEhc\varkappa^0_{tb};\\
\widetilde{Q}_{vm} &\equiv Q_{vm} + iEhc\varkappa_{tm} = Q^0_{vm} + iEhc\varkappa^0_{tm} - iEhc\hat{\sigma}\frac{\partial\widetilde{w}^\kappa}{\partial s_v}.
\end{aligned} \qquad (8.17)$$

Eliminating $\widetilde{w}^\kappa$ from $(8.17)_2$ and $(8.16)_2$ we obtain the complex combination

$$\widetilde{Q}_{\nu t}-\frac{\partial}{\partial s_t}\left(\frac{\widetilde{Q}_{\nu m}}{\hat{a}}\right)\equiv Q_{\nu t}-\frac{\partial}{\partial s_t}\left(\frac{Q_{\nu m}}{\hat{a}}\right)+\widehat{iEhc}\left[\varkappa_{t\nu}-\frac{\partial}{\partial s_t}\left(\frac{\varkappa_{tm}}{\hat{a}}\right)\right]=$$
$$=Q^0_{\nu t}-\frac{\partial}{\partial s_t}\left(\frac{Q^0_{\nu m}}{\hat{a}}\right)+iEhc\left[\varkappa^0_{t\nu}-\frac{\partial}{\partial s_t}\left(\frac{\varkappa^0_{tm}}{\hat{a}}\right)\right].$$

Then by adding equality $(8.16)_4$ to the three expressions which have been obtained we have

$$\begin{aligned}
&\widetilde{Q}_{\nu b}\equiv Q_{\nu b}+iEhc\varkappa_{tb}=Q^0_{\nu b}+iEhc\varkappa^0_{tb};\\
&\widetilde{Q}'_{\nu t}\equiv Q'_{\nu t}+iEhc\varkappa'_{t\nu}=Q'^0_{\nu t}+iEhc\varkappa'^0_{t\nu};\\
&\widetilde{Q}_{\nu m}\equiv Q_{\nu m}+iEhc\varkappa_{tm}=Q^0_{\nu m}+iEhc\varkappa^0_{tm}-iEhc\hat{\mathfrak{z}}\frac{\partial\widetilde{w}^{\kappa}}{\partial s_{\nu}};\\
&\widetilde{M}_{\nu\nu}\equiv M_{\nu\nu}+iEhc\varepsilon_{tt}=M^0_{\nu\nu}+iEhc\varepsilon^0_{tt}+iEhc\frac{1}{R_t}\widetilde{w}^{\kappa},
\end{aligned}\tag{8.18}$$

where, now

$$Q'_{\nu t}=Q_{\nu t}-\frac{\partial}{\partial s_t}\left(\frac{Q_{\nu m}}{\hat{a}}\right);\quad \varkappa'_{t\nu}=\varkappa_{t\nu}-\frac{\partial}{\partial s_t}\left(\frac{\varkappa_{tm}}{\hat{a}}\right).\tag{8.19}$$

As in equalities (6.3)-(6.4), of the four complex combinations two do not depend on the edge effect and are therefore basic ("membrane"). /255

If two shells of the same thickness and of the same material are joined (even at an angle), the complex conditions at the joints according to (8.18) are written in the following manner

$$\begin{aligned}
&\{Q^0_{\nu b}+iEhc\varkappa^0_{tb}\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}},\quad \{Q'^0_{\nu t}+iEhc\varkappa'^0_{t\nu}\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}};\\
&\left\{Q^0_{\nu m}-iEhc\varkappa^0_{tm}-iEhc\hat{\mathfrak{z}}\frac{\partial\widetilde{w}^{\kappa}}{\partial s_{\nu}}\right\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}};\\
&\left\{M^0_{\nu\nu}+iEhc\varepsilon^0_{tt}+iEhc\frac{1}{R_t}\widetilde{w}^{\kappa}\right\}^{\mathrm{I}}=\{\ldots\}^{\mathrm{II}}.
\end{aligned}\tag{8.20}$$

However, in the case when condition (7.18) is not satisfied, it is necessary to equate the real and imaginary parts of these combinations separately (after reducing the imaginary parts by Ehc).

By using the equalities obtained we can reduce relationships (8.9)-(8.10) to a form which is frequently more convenient. First of all, it follows from (Chapter VIII, 3.26) that

$$\varkappa_{tn} = -\varkappa_{tm}\cos\varphi + \varkappa_{tb}\sin\varphi;\quad \varkappa^{0}_{tn} = -\varkappa^{0}_{tm}\cos\varphi + \varkappa^{0}_{tb}\sin\varphi.$$

Subtracting term by term the second equality from the first and taking into account the following equality from $(8.18)_1$ $\varkappa_{tb} = \varkappa^{0}_{tb}$, we obtain

$$(\varkappa_{tn} - \varkappa^{0}_{tn}) = -\cos\varphi\,(\varkappa_{tm} - \varkappa^{0}_{tm}) = \frac{1}{R_t\hat{\sigma}}(\varkappa_{tm} - \varkappa^{0}_{tm}).$$

In a similar manner we obtain

$$Q_{vn} - Q^{0}_{vn} = -\frac{1}{R_t\hat{\sigma}}(Q_{vm} - Q^{0}_{vm}).$$

Now, by substituting the established expressions into (8.9) we reduce the latter to the form

$$\begin{aligned} u_0 &= u^{0}_{0} + a_{11}(Q_0 - Q^{0}_{0}) + a_{12}(M_0 - M^{0}_{0});\\ \vartheta_0 &= \vartheta^{0}_{0} + a_{12}(Q_0 - Q^{0}_{0}) + a_{22}(M_0 - M^{0}_{0}), \end{aligned} \tag{8.21}$$

where we now have

$$Q_0 = (-Q_{vm})_0;\quad Q^{0}_{0} = (-Q^{0}_{vm})_0;$$

$$\vartheta_0 = \left(\frac{\varkappa_{tm}}{\hat{\sigma}}\right)_0;\quad \vartheta^{0}_{0} = \left(\frac{\varkappa^{0}_{tm}}{\hat{\sigma}}\right)_0;\quad u_0 = \left(\frac{\varepsilon_{tt}}{\hat{\sigma}}\right)_0;\quad u^{0}_{0} = \left(\frac{\varepsilon^{0}_{tt}}{\hat{\sigma}}\right)_0; \tag{8.22}$$

$$\begin{aligned} a_{11} &= \frac{2\sqrt[4]{3(1-\nu^2)}}{E}\left(\frac{|R_{tn}|}{h}\right)^{\frac{3}{2}}\frac{1}{(R_t\hat{\sigma})^2_0};\\ a_{12} &= -\frac{2\sqrt{3(1-\nu^2)}}{Eh}\left(\frac{|R_{t0}|}{h}\right)\frac{1}{(R_t\hat{\sigma})_0};\\ a_{22} &= \frac{4[3(1-\nu^2)]^{\frac{3}{4}}}{Eh^2}\left(\frac{|R_{t0}|}{h}\right)^{\frac{1}{2}}. \end{aligned} \tag{8.23}$$

The relations which are the converse of (8.21) are written in the form /256

$$\begin{aligned} Q_0 &= Q^{0}_{0} + A_{11}(u_0 - u^{0}_{0}) + A_{12}(\vartheta_0 - \vartheta^{0}_{0});\\ M_0 &= M^{0}_{0} + A_{12}(u_0 - u^{0}_{0}) + A_{22}(\vartheta_0 - \vartheta^{0}_{0}), \end{aligned} \tag{8.24}$$

where

$$A_{11}=\frac{E}{\sqrt[4]{3(1-\nu^2)}}\left(\frac{h}{|R_{t0}|}\right)^{\frac{3}{2}}(R_t\hat{\sigma})_0^2;$$
$$A_{12}=-\frac{Eh}{2\sqrt[4]{3(1-\nu^2)}}\left(\frac{h}{|R_{t0}|}\right)(\hat{\sigma}R_t)_0; \qquad (8.25)$$
$$A_{22}=\frac{Eh^2}{2[3(1-\nu^2)]^{\frac{3}{4}}}\left(\frac{h}{|R_{t0}|}\right)^{\frac{1}{2}}.$$

As in the preceding item they can frequently be taken in a simplified form

$$Q_0-Q_0^0=A_{11}(u_0-u_0^0);\ M_0-M_0^0=A_{12}(u_0-u_0^0). \qquad (8.26)$$

To determine the stresses, on the other hand, we may utilize expressions (8.11) as before, with the exception that we replace σ_K by the following expression $(8.22)_{1,2}$

$$\sigma_\kappa=\operatorname{sign}R_{t0}\,2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{|R_{t0}|}{h}}\frac{1}{(R_t\hat{\sigma})_0}\frac{Q_0-Q_0^0}{h}. \qquad (8.27)$$

3. Let us illustrate the relationships obtained above by considering as an example a shell of revolution with a boundary which coincides with a parallel circle. For such an edge (section 1, Chapter V, fig. 20)

$$A_2=R_2\sin\theta=r;\ A_1=R_1;\ \varkappa_t=\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}=\frac{\cos\theta}{r};$$
$$R_t=R_2=\frac{r}{\sin\theta};\ \hat{\sigma}=\frac{1}{r};\ \frac{\partial}{\partial s_\nu}=\frac{1}{R_1}\frac{\partial}{\partial\theta};\ \frac{\partial}{\partial s_t}=\frac{1}{r}\frac{\partial}{\partial\varphi}; \qquad (8.28)$$
$$Q_{\nu\nu}=T_1;\ Q_{\nu t}=Q_{12}=S+\frac{\sin\theta}{r}2H;\ M_{\nu\nu}=M_1;$$
$$Q_{\nu n}=Q_{1n}=\frac{1}{r}\left(\frac{\partial rM_1}{R_1\partial\theta}+\frac{\partial 2H}{\partial\varphi}-\cos\theta M_2\right); \qquad (8.29)$$
$$u_\nu=u;\ u_t=v;\ w=w;\ \vartheta_\nu=\vartheta;$$
$$\varkappa_{tt}=\varkappa_2;\ \varkappa_{t\nu}=\varkappa_{21}=\tau-\frac{\sin\theta}{r}\omega;\ \varepsilon_{tt}=\varepsilon_2; \qquad (8.30)$$
$$\varkappa_{tn}=\varkappa_{2n}=\frac{1}{r}\left(-\frac{\partial r\varepsilon_2}{R_1\partial\theta}+\frac{\partial\omega}{\partial\varphi}+\cos\theta\varepsilon_1\right); \qquad (8.31)$$
$$w^{00}=-\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}-\operatorname{ctg}\theta u;\ \vartheta^{00}=\frac{1}{r}\left(-\frac{1}{\sin\theta}\frac{\partial^2u}{\partial\varphi^2}+\operatorname{ctg}\theta\frac{\partial v}{\partial\varphi}\right);$$
$$\varkappa_{tt}^{00}=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial^2v}{\partial\varphi^2}+v\right)\right];\ \varkappa_{t\nu}^{00}=-\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial^2u}{\partial\varphi^2}+u\right)\right]; \qquad (8.32)$$

$$
\begin{aligned}
Q'^{\circ}_{\nu\nu} &= T^{\circ}_1 - \operatorname{ctg}\theta Q^{\circ}_{1n} - \frac{1}{r\sin\theta}\frac{\partial^2 M^{\circ}_1}{\partial\varphi^2};\\
Q'^{\circ}_{\nu t} &= Q^{\circ}_{12} + \frac{1}{\sin\theta}\frac{\partial Q^{\circ}_{1n}}{\partial\varphi} - \frac{\operatorname{ctg}\theta}{r}\frac{\partial M^{\circ}_1}{\partial\varphi};\\
Q'^{\circ}_{1n} &= Q^{\circ}_{1n}.
\end{aligned}
\tag{8.33}
$$

The case of symmetric and antisymmetric deformation is of prime interest. For both cases /257

$$
\varkappa^{\circ\circ}_{tt} = \varkappa^{\circ\circ}_{t\nu} = 0. \tag{8.34}
$$

This is not accidental, because we can show that this condition provides for periodicity (with respect to φ) of the displacements (Chapter V). By making use of the relationships which are presented there, we establish the fact that the introduced static quantities are related in a particularly simple manner to the components of the principal vector and the principal moment ($Q'^{\circ}_{\rho} = \cos\theta Q'^{\circ}_{\nu\nu}$, $Q'^{\circ}_{z} = \sin\theta Q'^{\circ}_{\nu\nu}$):

$$
\begin{aligned}
&F_z = -2\pi r Q'^{\circ}_z \quad B_y = \pi r^2 Q'^{\circ}_{z,1} \quad F_x = \pi r\left(-Q'^{\circ}_{\nu t,1} + Q'^{\circ}_{\rho,1}\right)\\
&B_z = 2\pi r^2 Q'^{\circ}_{\nu t}; \quad B_x = -\pi r^2 Q'^{\circ}_{z,1}; \quad F_y = \pi r\left(Q'^{\circ}_{\nu t,1} + Q'^{\circ}_{\rho,1}\right).
\end{aligned}
\tag{8.35}
$$

Furthermore, by going over to simplified versions we find that (section 1, Chapter V)

$$
Q_{\nu m} = -Q_{\rho};\ Q_{\nu b} = -Q_z;\ Q'_{\nu t} = Q_{12} + \frac{\partial Q_{\rho}}{\partial\varphi}; \tag{8.36}
$$

$$
\varkappa_{tm} = \varkappa_{\rho} = \frac{1}{r}\left\{\vartheta - \frac{1}{r}\frac{\partial^2 u_z}{\partial\varphi^2}\right\};\ \varkappa_{tb} = -\varkappa_z = \frac{1}{r^2}\left\{\frac{\partial v}{\partial\varphi} - \frac{\partial^2 u_{\rho}}{\partial\varphi^2}\right\};
$$

$$
\varkappa'_{t\nu} = \frac{1}{r^2}\left\{\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u_z}{\partial\varphi^2} + u_z\right)\right\};\ \varepsilon_{tt} = \frac{1}{r}\left\{\frac{\partial v}{\partial\varphi} + u_{\rho}\right\}. \tag{8.37}
$$

Here the symmetric and the antisymmetric cases are of the greatest interest. In the symmetric case

$$
\frac{\varepsilon_{tt}}{\hat{\sigma}} = u_{\rho},\quad \frac{\varkappa_{tm}}{\hat{\sigma}} = \vartheta \tag{8.38}
$$

and relationships (8.21)-(8.23) coincide (when $Q^{\circ}_{\nu n} = 0$, $M^{\circ}_{\nu\nu} = 0$) with the /258 expressions in section 8, Chapter V for the compliance coefficients. In the antisymmetric case (section 9, Chapter V)

$$\frac{\varepsilon_{tt,1}}{\hat{a}} = r\varepsilon_{2,1} = r\gamma, \quad \frac{\varkappa_{tm,1}}{\hat{a}} = \vartheta_{,1} + \frac{u_{z,1}}{r} = \chi, \tag{8.39}$$

where χ is a so-called elastic rotation. The system of relationships (8.21)-(8.23) again coincides with the one obtained in section 10, Chapter V.

As we have already stated in section 7 of Chapter XII, the boundary quantity $(8.36)_3$ in the case of a shallow spherical shell coincides with the one proposed by E. Reissner.

It is customary to assume that $Q_{\nu\nu}$ and $Q_{\nu t}$ should be referred to the generalized boundary forces of the basic state of stress. In this case it is usually assumed that $Q_{\nu\nu} = Q^*_{\nu\nu} = T^*_\nu$, $Q_{\nu t} = Q^*_{\nu t} = S^*$, i.e., these quantities are borrowed from the membrane solution.

By using the relationships presented above we shall show that this approach is not entirely correct. First of all, in cases when the basic state of stress is semimembrane (section 10) or is associated with strong moments (section 11), the moment terms cannot be neglected in the expressions for $Q_{\nu\nu}$ and $Q_{\nu t}$. Furthermore, let us assume that the basic state of stress is a membrane state. Then, for example, in the symmetric case

$$Q'_{\nu\nu} = T_1^0 - \operatorname{ctg}\theta Q^0_{1\tau} = \frac{1}{\sin\theta} Q^0_z = \frac{1}{\sin\theta} Q^*_z,$$

i.e., within the accuracy given by the multiplier, the quantity $Q'_{\nu\nu}$ coincides with the vertical force, which is known beforehand not to depend on the edge effect (section 6, Chapter V). On the other hand the expression for $Q_{\nu\nu} = T_1$ contains the solution of the edge effect (ref. 33, Chapter IV, section 8).

These statements show that in this case, which has been investigated in detail, the introduced boundary conditions are rather convenient, because they are organically related to the deformations and to the equilibrium of the shell. Earlier they were obtained by utilizing the specific peculiarities of the stress-strain state (the presence of the predominantly spatial direction-axis of the shell, an edge load which is not self-balanced, the possibility of realizing (within the framework of the adopted relationship as a function of the angle φ) the rigid displacement of the shell).

The transformations carried out in the present chapter generalize, in a natural fashion, these quantities for the case of shells (and loads) of general

form, when the considerations employed in the examination of a particular case lose their validity.

9. Mechanical Interpretation of the Edge Effect

The association obtained above between the edge effect and the tangential deformation of a shell makes it possible to give a simple mechanical interpretation of this curious phenomenon inherent in shells. /259

Let us assume for simplicity that the boundary conditions of a particular problem are purely geometric, and that the basic state of stress consists of a membrane state and a purely flexural state. The particular solution (which removes the surface load) as a rule does not satisfy the boundary conditions of the problem. The resulting inconsistency in the boundary conditions $u_{\nu 0}$, u_{t0}, w_0, $\vartheta_{\nu 0}$ (ε_{tt0}, $\varkappa_{tn0}$) may be eliminated by applying corresponding boundary forces and moments $Q^0_{\nu\nu}$, $Q^0_{\nu t}$, $Q^0_{\nu n}$, $M^0_{\nu\nu}$ to the edge of the shell. If we do this, we can have two cases

(a) $$\varepsilon^0_{tt} = \varepsilon_{tt0};\ \varkappa^0_{tn} = \varkappa_{tn0}, \tag{9.1}$$

i.e., the tangential components of the edge deformation corresponding to the basic state of stress are equal to the desired ones. Then the edge effect does not take place and $Q^0_{\nu\nu} = Q^*_{\nu\nu}$, $Q^0_{\nu t} = Q^*_{\nu t}$, $Q^0_{\nu n} = Q^f_{\nu n}$, $M^0_{\nu\nu} = M^f_{\nu\nu}$, i.e., the total solution consists of three parts: particular solution, moment solution and the solution for pure flexure.

(b) Usually, however, condition (9.1) is not satisfied and, in particular, $\varepsilon^K_{tt} = \varepsilon_{tt0} - \varepsilon^0_{tt} \neq 0$. The extension of the contour induces the occurrence of an additional tension force

$$T^\varkappa_t = \frac{Eh}{1-\nu^2}\varepsilon^\varkappa_{tt}. \tag{9.2}$$

In view of the contour curvature ($\frac{1}{R_t} \neq 0$) this force produces a component along the normal to the middle surface $\frac{1}{R_t} T^K_t$. By setting up conditions whereby the normal component of the principal vector of forces acting on the element of the middle surface which adjoins the edge is equal to zero, we obtain

$$\frac{T_t^\kappa}{R_t}-\left(\frac{\partial Q_\nu^\kappa}{\partial s_\nu}+\frac{\partial Q_t^\kappa}{\partial s_t}\right)=0, \tag{9.3}$$

where

$$Q_\nu^\kappa=\frac{\partial M_\nu^\kappa}{\partial s_\nu}+\frac{\partial H^\kappa}{\partial s_t};\quad Q_t^\kappa=\frac{\partial M_t^\kappa}{\partial s_t}+\frac{\partial H^\kappa}{\partial s_\nu}; \tag{9.4}$$

$$M_\nu^\kappa=\frac{Eh}{1-\nu^2}\frac{h^2}{12}(\varkappa_\nu^\kappa+\nu\varkappa_t^\kappa);\qquad H^\kappa=\frac{Eh}{1-\nu^2}\frac{h^2}{12}(1-\nu)\tau^\kappa; \tag{9.5}$$

$$M_t^\kappa=\frac{Eh}{1-\nu^2}\frac{h^2}{12}(\varkappa_t^\kappa+\nu\varkappa_\nu^\kappa);$$

$$\varkappa_\nu^\kappa=-\frac{\partial^2 w^\kappa}{\partial s_\nu^2}+\ldots;\quad \varkappa_t^\kappa=-\frac{\partial^2 w^\kappa}{\partial s_t^2}+\ldots;\quad \tau^\kappa=-\frac{\partial^2 w^\kappa}{\partial s_\nu \partial s_t}+\ldots \tag{9.6}$$

Thus the normal component of the additional tangential force must be balanced by the cutting forces, or, what is the same, by the respective derivatives of the moments. However, by comparing expressions (9.2) and (9.5) it is easy to see that the last one in equation (9.3) has a small factor of the order $(\frac{h}{R_0})^2$. Therefore, equation (9.3) may be satisfied only under the condition when the variation in the curvature for part of the middle surface adjoining the contour is very large (due to deformation). Furthermore, since the variation in the curvature along the contour cannot be large (since it is assumed that R_t and the boundary conditions are smooth), the variation in the curvature along the normal to the contour ($\varkappa_\nu^\kappa$) is high. In this case it follows from (9.6) that the normal flexure is a function which changes very rapidly along the normal to the contour.

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Taking this into account, equation (9.3) may be written in a simplified form

$$\frac{T_t^\kappa}{R_t}-Ehc^2\frac{\partial^2 \varkappa_\nu^\kappa}{\partial s_\nu^2}=0. \tag{9.7}$$

Retaining the corresponding components of deformation in the third equation of continuity (section 2, Chapter II) we obtain by means of (9.2) a second equation

$$\frac{\varkappa_\nu^\kappa}{R_t}+\frac{1-\nu^2}{Eh}\frac{\partial T_t^\kappa}{\partial s_\nu^2}=0. \tag{9.8}$$

The simultaneous solution of both equations leads to the conclusion, which is already known to us, that quantities which characterize the edge effect including w^K vary rapidly along the normal to the contour and "discard" a large factor of the order of $\sqrt{\frac{R_0}{h}}$ during each differentiation with respect to s_ν.

These discussions (which do not pretend to be rigorous) make it possible to give the following short explanation for the occurrence of a simple edge effect:

(a) When the boundary conditions of the problem are satisfied, the edge of the shell is extended (deformation in the direction of the major shell rigidity), which causes the appearance of a substantial tension force.

(b) Due to the normal curvature of the edge, this force produces a component along the normal to the surface.

(c) Since the flexural rigidity of a thin shell is substantially "less" than its tensional rigidity, this normal component may be balanced only by a sharp variation in the curvature (and in flexure) in the direction of the normal to the contour.

This in essence means that a simple edge effect occurs in the shell. /261

It is interesting to note that a similar explanation may be given to the generalized edge effect (section 5) which occurs at the asymptotic edge. In this case $\frac{1}{R_t} = 0$, but neither the quantity $\frac{1}{R_\nu}$ nor $\frac{1}{R_{\nu t}}$ is equal to zero, and the normal component of the boundary forces $(-2\frac{Q_{\nu t}}{R_{\nu t}} + \frac{Q_{\nu\nu}}{R_\nu})$ may be balanced again only due to the rapid variation in the flexure along the normal to the contour. A simple analysis (section 5) shows that this variation is more uniform and depends substantially on the nature of the boundary forces $Q_{\nu t}(s_t)$, $Q_{\nu\nu}(s_t)$.

10. Semimembrane State of Stress

In section 19 of Chapter V for a semi-infinite circular cylindrical shell, subject to edge forces and moments varying according to the law

$$\Phi(\xi, \varphi) = \Phi_{,2} \begin{matrix} \cos 2\varphi \\ \sin 2\varphi \end{matrix}, \tag{10.1}$$

we obtained the following expressions for the complex forces

$$\widetilde{T}_{1,2}=-i\frac{4}{2b^2}\widetilde{C}_1e^{\gamma_1\xi}+\left(1+i\frac{4}{2b^2}\right)\widetilde{C}_2e^{\gamma_2\xi};$$
$$\widetilde{T}_{2,2}=\left(1+i\frac{4}{2b^2}\right)\widetilde{C}_1e^{\gamma_1\xi}-i\frac{4}{2b^2}\widetilde{C}_2e^{\gamma_2\xi}; \qquad (10.2)$$
$$\widetilde{S}_{,2}=i\frac{1}{2}\frac{4}{2b^2}\gamma_1\widetilde{C}_1e^{\gamma_1\xi}-\frac{1}{2}\left(1+i\frac{4}{2b^2}\right)\gamma_2\widetilde{C}_2e^{\gamma_2\xi},$$

where

$$\gamma_1=\sqrt{8-i2b^2},\quad \gamma_2=\sqrt{\frac{6}{4-ib^2}},\quad 2b^2=\frac{r_0}{c}=\sqrt{12(1-\nu^2)}\frac{r_0}{h}. \qquad (10.3)$$

Since $2b^2 \gg 1$, we obtain the following simplified expressions for roots of the characteristic equation

$$\gamma_1=\sqrt{-i2b^2\left(1+i\frac{8}{2b^2}\right)}\approx(1-i)b\left(1+i\frac{4}{2b^2}\right)\approx(1-i)b; \qquad (10.4)$$
$$\gamma_2=\sqrt{\frac{6\cdot2}{-i2b^2}\frac{1}{1+i\frac{4}{b^2}}}\approx\frac{\sqrt{12}}{(1-i)b}\left(1-i\frac{4}{2b^2}\right)\approx\frac{\sqrt{3}(1+i)}{b}. \qquad (10.5)$$

We neglect the small underlined terms in expressions (10.2). Then /262

$$\widetilde{T}_{1,2}=\ldots\ldots\widetilde{C}_2e^{\gamma_2\xi};$$
$$\widetilde{T}_{2,2}=\widetilde{C}_1e^{\gamma_1\xi}; \qquad (10.6)$$
$$\widetilde{S}_{,2}=i\frac{2}{2b^2}\gamma_1\widetilde{C}_1e^{\gamma_1\xi}-\frac{1}{2}\gamma_2\widetilde{C}_2e^{\gamma_2\xi}.$$

It follows from (10.4) and $(10.3)_3$ that the solution corresponding to the large root γ_1 is that of the simple edge effect which has been well studied in sections 3 and 4.

The second small root γ_2 corresponds to the so-called semimembrane state of stress (ref. 33, p. 301). As shown by V. V. Novozhilov, its characteristic property is the inequality (which in our case follows from (10.1) and (10.5))

$$\frac{\partial^2\Phi}{\partial\xi^2}\ll\frac{\partial^2\Phi}{\partial\varphi^2}. \qquad (10.7)$$

Let us consider the semimembrane part of the solution in more detail. To find the complex displacements it is necessary to substitute expressions ($\gamma_2 = \gamma$, $\tilde{C}_2 = \tilde{C}$)

$$\tilde{T}_{1,2} = \tilde{C}e^{\gamma\xi},\ \tilde{S}_{,2} = -\frac{\gamma}{2}\tilde{C}e^{\gamma\xi},\ \tilde{T}_{2,2} = 0 \tag{10.8}$$

into equations (Chapter V, 11.6) which in the present case assume the following form

$$\begin{aligned}
&\frac{d\tilde{u}_{,2}}{d\xi} = \frac{r_0}{Eh}\left\{\tilde{T}_{1,2} - \nu\tilde{T}_{2,2}\right\};\\
&2\tilde{v}_{,2} + \tilde{w}_{,2} = \frac{r_0}{Eh}\left\{\tilde{T}_{2,2} - \nu\tilde{T}_{1,2}\right\};\\
&-2\tilde{u}_{,2} + \frac{d\tilde{v}_{,2}}{d\xi} = \frac{2(1+\nu)r_0}{Eh}\tilde{S}_{,2};\\
&-\frac{d^2\tilde{w}_{,2}}{d\xi^2} = i\frac{2b^2r_0}{Eh}\tilde{T}_{2,2};\\
&4\tilde{w}_{,2} + 2\tilde{v}_{,2} = i\frac{2b^2r_0}{Eh}\tilde{T}_{1,2};\\
&2\frac{d\tilde{w}_{,2}}{d\xi} + \frac{d\tilde{v}_{,2}}{d\xi} = -i\frac{2b^2r_0}{Eh}\tilde{S}_{,2}.
\end{aligned} \tag{10.9}$$

It is proposed that the reader use (10.2) to convince himself that the system obtained above is approximately compatible and has the following solution

$$\begin{aligned}
\tilde{u}_{,2} &= \frac{r_0\tilde{C}}{Eh}\left\{\frac{1}{\gamma}e^{\gamma\xi}\right\};\\
\tilde{v}_{,2} &= \frac{r_0\tilde{C}}{Eh}\left\{\left[\frac{2}{\gamma^2} - (1+\nu)\right]e^{\gamma\xi}\right\};\\
\tilde{w}_{,2} &= \frac{r_0\tilde{C}}{Eh}\left\{\left[-\frac{4}{\gamma^2} + (2+\nu)\right]e^{\gamma\xi}\right\}.
\end{aligned} \tag{10.10}$$

Let us obtain the membrane solution which corresponds to this case. /263
According to (Chapter XII, 2.1 and 2.2), if we take into account (10.1), it will be the solution of system

$$\begin{aligned}
&\frac{dT^*_{1,2}}{d\xi} + 2S^*_{,2} = 0;\ \frac{dS^*_{,2}}{d\xi} - 2T^*_{2,2} = 0;\ T^*_{2,2} = 0;\\
&\frac{du^*_{,2}}{d\xi} = \frac{r_0}{Eh}\left\{T^*_{1,2} - \nu T^*_{2,2}\right\};
\end{aligned} \tag{10.11}$$

$$2v^*_{,2} + w^*_{,2} = \frac{r_0}{Eh}\left\{T^*_{2,2} - \nu T^*_{1,2}\right\};$$
$$-2u^*_{,2} + \frac{dv^*_{,2}}{d\xi} = \frac{2(1+\nu)r_0}{Eh} S^*_{,2}. \qquad (10.12)$$

It is easy to verify that the unknown solution has the form

$$S^*_{,2} = S_0, \quad T^*_{1,2} = T_0 - 2S_0\xi; \quad T^*_{2,2} = 0;$$
$$u^*_{,2} = u_0 + \frac{r_0}{Eh}\{T_0\xi - S_0\xi^2\}; \qquad (10.13)$$

$$v^*_{,2} = v_0 + 2\xi u_0 + \frac{r_0}{Eh}\left\{\xi^2 T_0 + \left[-\frac{2}{3}\xi^3 + 2(1+\nu)\xi\right] S_0\right\}; \qquad (10.14)$$

$$w^*_{,2} = -2v_0 - 4\xi u_0 + \frac{r_0}{Eh}\left\{-(2\xi^2+\nu)T_0 + \left[\frac{4}{3}\xi^3 - 2(2+\nu)\xi\right] S_0\right\},$$

where S_0, T_0, u_0, v_0 are arbitrary constants determined by the boundary conditions of a specific edge problem.

Furthermore, by computing the components of flexural deformation and the moments from the established displacements we obtain the following expressions

$$M^*_{2,2} = \frac{Eh}{(2b^2)^2}[-6(v_0 + 2\xi u_0)] +$$
$$+ \frac{r_0}{(2b^2)^2}\{(-6\xi^2 + 8\nu)T_0 + [4\xi^3 - 12(1+\nu)\xi] S_0\};$$
$$M^*_{1,2} = \frac{Eh}{(2b^2)^2}[-6\nu(v_0 + 2\xi u_0)] + \qquad (10.15)$$
$$+ \frac{r_0}{(2b^2)^2}\{[-6\nu\xi^2 + 4(1+\nu^2)]T_0 + [4\nu\xi^3 - 4(2+3\nu+\nu^2)\xi] S_0\};$$
$$H^*_{,2} = \frac{Eh(1-\nu)}{(2b^2)^2}[-6u_0] + \frac{(1-\nu)r_0}{(2b^2)^2}\{-6\xi T_0 + [6\xi^2 - 2(3+\nu)] S_0\}.$$

The construction of the membrane solution is closely associated with the moment solution established earlier. To clarify this we replace the function $e^{\gamma\xi}$ by its expansion /264

$$e^{\gamma\xi} = 1 + \gamma\xi + \frac{1}{2}\gamma^2\xi^2 + \frac{1}{6}\gamma^3\xi^3 + \ldots \qquad (10.16)$$

and introduce the following designations

$$\gamma=(1+i)\alpha,\quad \alpha=\frac{\sqrt{3}}{b},\quad \tilde{C}=A+iB. \tag{10.17}$$

Then

$$\begin{aligned}\widetilde{T}_{1,2}&=(A+iB)\left\{1+\gamma\xi+\frac{1}{2}\gamma^2\xi^2+\frac{1}{6}\gamma^3\xi^3+\ldots\right\};\\ \widetilde{S}_{,2}&=-\frac{\gamma}{2}(A+iB)\left\{1+\gamma\xi+\frac{1}{2}\gamma^2\xi^2+\ldots\right\}.\end{aligned} \tag{10.18}$$

Separating the real and imaginary parts in the above equations, we obtain

$$\begin{aligned}T_{1,2}&=[A+(A-B)\alpha\xi]-B\alpha^2\xi^2-\frac{1}{3}(A+B)\alpha^3\xi^3+\ldots;\\ S_{,2}&=\left[-\frac{1}{2}(A-B)\alpha\right]+B\alpha^2\xi+\frac{1}{2}(A+B)\alpha^3\xi^2+\ldots;\end{aligned} \tag{10.19}$$

$$\begin{aligned}M_{2,2}&=-c\{[B+(A+B)\alpha\xi]+A\alpha^2\xi^2+\ldots\};\\ H_{,2}&=(1-\nu)c\left\{\left[-\frac{\alpha}{2}(A+B)\right]-A\alpha^2\xi-\frac{1}{2}(A-B)\alpha^3\xi^2+\ldots\right\}.\end{aligned} \tag{10.20}$$

In a similar manner the separation of the real parts in expressions (10.10) gives us

$$\begin{aligned}u_{,2}&=\frac{r_0 2b^2}{12Eh}\{(A+B)\alpha\}+\frac{r_0}{Eh}\left\{A\xi+\frac{1}{2}(A-B)\alpha\xi^2+\ldots\right\};\\ v_{,2}&=\frac{r_0 2b^2}{12Eh}\{2B+2(A+B)\alpha\xi\}+\frac{r_0}{Eh}\left\{A(\xi^2-\underline{(1+\nu)})+\right.\\ &\qquad\left.+(A-B)\alpha\left(\frac{1}{3}\xi^3-(1+\nu)\xi\right)+\ldots\right\};\\ w_{,2}&=\frac{r_0 2b^2}{12Eh}\{-4B-4(A+B)\alpha\xi\}+\frac{r_0}{Eh}\left\{-A(2\xi^2+\nu)+\right.\\ &\qquad\left.+\underline{2(1+\nu)A}-(A-B)\left(\frac{2}{3}\xi^3-(2+\nu)\xi\right)\alpha+\ldots\right\}.\end{aligned} \tag{10.21}$$

Comparing the corresponding terms (contained in the square brackets) in expressions (10.13) and (10.19), we see that they coincide, if we assume that

$$S_0=-\frac{\alpha}{2}(A-B);\quad T_0=A. \tag{10.22}$$

In a similar way the comparison of relationships (10.14) with (10.21) shows that they coincide when /265

$$u_0 = \frac{2b^2 r_0}{12Eh}\{(A+B)\alpha\};$$
$$v_0 = \frac{r_0}{Eh}\left\{\frac{2b^2}{12}2B - (1+\nu)A\right\} \approx \frac{2b^2 r_0}{12Eh}\{2B\}. \tag{10.23}$$

Taking this into account it is easy to verify the coincidence of terms contained in the square brackets of expressions (10.15) and (10.20).

By using the above, rather special example we shall show that the relationships of the membrane theory may be utilized to construct the semimembrane solution. For this purpose we write the system of equilibrium equations (Chapter XII, 2.9) in the form

$$\begin{aligned}
&\frac{dT^*_{1,2}}{d\xi} + 2S^*_{,2} = 0;\\
&-2T^*_{2,2} + \frac{dS^*_{,2}}{d\xi} + \frac{\delta}{r_0}\left\{-2M^*_{2,2} + \frac{d2H^*_{,2}}{d\xi}\right\} = 0;\\
&T^*_{2,2} - \frac{\delta}{r_0}\left\{\frac{d^2M^*_{1,2}}{d\xi^2} + 4\frac{dH^*_{,2}}{d\xi} - 4M^*_{2,2}\right\} = 0,
\end{aligned} \tag{10.24}$$

and seek the forces and displacements in the form of expansions

$$T^*_{1,2} = T^{*(0)}_{1,2} + \delta T^{*(1)}_{1,2} + \delta^2 T^{*(2)}_{1,2} + \ldots \tag{10.25}$$
$$\ldots\ldots\ldots\ldots$$

Substituting expressions (10.25) into (10.24) and collecting terms with the same powers of δ, we obtain the system

$$\begin{aligned}
&\frac{dT^{*(i)}_{1,2}}{d\xi} + 2S^{*(i)}_{,2} = 0;\\
&-2T^{*(i)}_{2,2} + \frac{dS^{*(i)}_{,2}}{d\xi} = \frac{1}{r_0}\left\{-2M^{*(i-1)}_{2,2} + \underline{\frac{d2H^{*(i-1)}}{d\xi}}\right\};\\
&T^{*(i)}_{2,2} = \frac{1}{r_0}\left\{\underline{\frac{d^2M^{*(i-1)}_{1,2}}{d\xi^2}} + 4\frac{dH^{*(i-1)}_{,2}}{d\xi} - 4M^{*(i-1)}_{2,2}\right\};\\
&(i = 0,\ 1,\ 2,\ 3,\ldots)\quad (M^{*(-1)}_{1,2} = M^{*(-1)}_{2,2} = H^{*(-1)}_{,2} = 0).
\end{aligned} \tag{10.26}$$

The parameter δ is introduced above only for establishing an order in the calculation steps. After the solution has been determined we should set δ equal to one. We also note that the terms underlined in (10.26) are small and should be dropped.

Thus, in carrying out the above method, the nonhomogeneous relationships of the membrane theory must be resolved at each step. In this case the zero approximation is the approximation (10.13)-(10.14)-(10.15). The first approximation (it should be verified!), on the other hand, has the following form

$$T_{1,2}^{*(1)} = -\frac{12}{2b^2}\left\{T_0\left(\frac{1}{2}\xi^2+\frac{\alpha}{3}\xi^3\right)+S_0\left(\frac{1}{\alpha}\xi^2+\frac{1}{3}\xi^3\right)\right\};$$
$$S_{,2}^{*(1)} = \frac{6}{2b^2}\left\{T_0(\xi+\alpha\xi^2)+S_0\left(\frac{2}{\alpha}\xi+\xi^2\right)\right\}; \qquad (10.27)$$
$$T_{2,2}^{*(1)} = \frac{4}{2b^2}\left\{T_0(1+2\alpha\xi)+S_0\left(\frac{2}{\alpha}+2\xi\right)\right\}.$$

By using equalities (10.22) and $(10.17)_2$ it is easy to convince ourselves /266 that the expressions for $T_{1,2}^{*(1)}$, $S_{,2}^{*(1)}$ coincide with terms written out in (10.19), but are not contained in the square bracket. With regard to $T_{2,2}^{*(1)}$, it is easy to see that it corresponds to the term which was dropped when transferring from (10.2) to (10.6). These statements are sufficient to show that the relationships of the membrane theory are suitable for obtaining the semimembrane solution. In this case, from the mathematical point of view, we are speaking about the approximation of the exponential function $e^{\gamma\xi}$ by the segments of its Taylor's series. This serves as a basis for obtaining the estimates for the limiting length of cylindrical shells for which we can still use the membrane solution (ref. 11, p. 237).

The rather specific example which we have considered demonstrates the properties of the basic state of stress, which is different from the membrane state. According to the criterion (Chapter XII, 0.4) the semimembrane state of stress is mixed.

In references 261 and 258 the semimembrane theory is used to consider the restricted torsion of a cylindrical shell with an arbitrary cross section. A. A. Umanskiy used the methods of structural mechanics to consider this problem (refs. 270 and 271). The history of the establishment of the semimembrane theory for cylindrical shells, which first appeared in the early works of V. Z. Vlasov (ref. 253), is briefly presented in reference 33.

The problem of creating a semimembrane theory for long shells, which are different from cylindrical shells, is interesting (section 5, Chapter XII).

The mathematical aspects of the problem of constructing a basic state of stress by means of the relationships from the membrane theory are considered in references 256 and 252.

11. A State of Stress With Strong Moments

Let us assume that the geometric boundary conditions of the considered problem do not limit purely flexural displacements (section 8, Chapter XII). The latter, which are determined from the system of equations

$$\begin{aligned}
\varepsilon_\alpha &\equiv \frac{1}{A}\frac{\partial u^н}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v^н + \frac{w^н}{R_\alpha} = 0;\\
\varepsilon_\beta &\equiv \frac{1}{B}\frac{\partial v^н}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u^н + \frac{w^н}{R_\beta} = 0;\\
\omega &\equiv \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v^н}{B}\right) + \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u^н}{A}\right) - 2\frac{w^н}{R_{\alpha\beta}} = 0,
\end{aligned} \tag{11.1}$$

Note: In this Chapter the following equations in this section н = h = homogeneous.

are associated with substantial moments and cutting forces which are computed by means of equations /267

$$\begin{aligned}
\varkappa_\alpha^н &= -\frac{1}{A}\frac{\partial}{\partial \alpha}\frac{1}{A}\frac{\partial w^н}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\frac{1}{B}\frac{\partial w^н}{\partial \beta} + \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{u^н}{R_\alpha} - \frac{v^н}{R_{\alpha\beta}}\right) +\\
&+ \frac{1}{AB}\frac{\partial A}{\partial \beta}\left(\frac{v^н}{R_\beta} - \frac{u^н}{R_{\alpha\beta}}\right) + \frac{1}{2ABR_{\alpha\beta}}\left(\frac{\partial A u^н}{\partial \beta} - \frac{\partial B v^н}{\partial \alpha}\right);\\
\varkappa_\beta^н &= -\frac{1}{B}\frac{\partial}{\partial \beta}\frac{1}{B}\frac{\partial w^н}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\frac{1}{A}\frac{\partial w^н}{\partial \alpha} + \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{v^н}{R_\beta} - \frac{u^н}{R_{\alpha\beta}}\right) +\\
&+ \frac{1}{AB}\frac{\partial B}{\partial \alpha}\left(\frac{u^н}{R_\alpha} - \frac{v^н}{R_{\alpha\beta}}\right) - \frac{1}{2ABR_{\alpha\beta}}\left(\frac{\partial A u^н}{\partial \beta} - \frac{\partial B v^н}{\partial \alpha}\right);\\
\tau^н &= -\frac{1}{AB}\left(\frac{\partial^2 w^н}{\partial \alpha \partial \beta} - \frac{1}{A}\frac{\partial A}{\partial \beta}\frac{\partial w^н}{\partial \alpha} - \frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial w^н}{\partial \beta}\right) +\\
&+ \frac{1}{R_\alpha}\left(\frac{1}{B}\frac{\partial u^н}{\partial \beta} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u^н\right) + \frac{1}{R_\beta}\left(\frac{1}{A}\frac{\partial v^н}{\partial \alpha} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v^н\right) -\\
&- \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{R_{\alpha\beta}}\right)u^н - \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{R_{\alpha\beta}}\right)v^н;
\end{aligned} \tag{11.2}$$

$$\begin{aligned}
M_\alpha^н &= \frac{Eh^3}{12(1-\nu^2)}\left(\varkappa_\alpha^н + \nu\varkappa_\beta^н\right); \quad M_\beta^н = \frac{Eh^3}{12(1-\nu^2)}\left(\varkappa_\beta^н + \nu\varkappa_\alpha^н\right);\\
H^н &= \frac{Eh^3}{12(1+\nu)}\tau^н;
\end{aligned} \tag{11.3}$$

$$\begin{aligned}
Q_{\nu n}^н &= \frac{\cos\gamma}{AB}\left(\frac{\partial BM_\alpha^н}{\partial \alpha} + \frac{1}{A}\frac{\partial A^2 H^н}{\partial \beta} - \frac{\partial B}{\partial \alpha}M_\beta^н\right) +\\
&+ \frac{\sin\gamma}{AB}\left(\frac{\partial AM_\beta^н}{\partial \beta} + \frac{1}{B}\frac{\partial B^2 H^н}{\partial \alpha} - \frac{\partial A}{\partial \beta}M_\alpha^н\right) +\\
&+ \left(\frac{-\sin\gamma}{A}\frac{\partial}{\partial \alpha} + \frac{\cos\gamma}{B}\frac{\partial}{\partial \beta}\right)\times\\
&\times\left[\left(\cos^2\gamma - \sin^2\gamma\right)H^н + \sin\gamma\cos\gamma\left(M_\beta^н - M_\alpha^н\right)\right];\\
M_{\nu\nu}^н &= \cos^2\gamma M_\alpha^н + 2\sin\gamma\cos\gamma H^н + \sin^2\gamma M_\beta^н.
\end{aligned} \tag{11.4}$$

If we approach the problem formally, it must follow from (11.1) that the forces T_α^h, T_β^h, S^h are equal to zero. However, this refinement of a purely flexural state of stress is of little interest. Indeed, it is associated only with two arbitrary functions which appear when system (11.1) is integrated. Generally speaking these two functions are not sufficient for satisfying the four boundary conditions. In addition, the equations of equilibrium are not satisfied.

Therefore, forces are introduced which are determined from equilibrium equations

$$\frac{\partial BT_\alpha^*}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2S^*}{\partial\beta}-\frac{\partial B}{\partial\alpha}T_\beta^*=-ABq_\alpha-$$
$$-\frac{1}{R_\alpha}\left(\frac{\partial BM_\alpha^н}{\partial z}+\frac{1}{A}\frac{\partial A^2H^н}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta^н\right)+$$
$$+\frac{1}{R_{\alpha\beta}}\left(\frac{\partial AM_\beta^н}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H^н}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha^н\right)-$$
$$-\frac{\partial}{\partial\beta}\left(\frac{A}{R_\alpha}H^н-\frac{A}{R_{\alpha\beta}}M_\beta^н\right)-\frac{\partial A}{\partial\beta}\left(\frac{H^н}{R_\beta}-\frac{M_\alpha^н}{R_{\alpha\beta}}\right) \qquad (11.5)$$
$$(\alpha\rightleftarrows\beta,\ A\rightleftarrows B);$$
$$\frac{T_\alpha^*}{R_\alpha}-\frac{2S^*}{R_{\alpha\beta}}+\frac{T_\beta^*}{R_\beta}=q_n+\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{1}{A}\left(\frac{\partial BM_\alpha^н}{\partial\alpha}+\frac{1}{A}\frac{\partial A^2H}{\partial\beta}-\right.\right.$$
$$\left.-\frac{\partial B}{\partial z}M_\beta^н\right)+\frac{\partial}{\partial\beta}\frac{1}{B}\left(\frac{\partial AM_\beta^н}{\partial\beta}+\frac{1}{B}\frac{\partial B^2H^н}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha^н\right)\Big\}-$$
$$-\frac{1}{R_{\alpha\beta}}\left[-\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)H^н+\frac{1}{R_{\alpha\beta}}(M_\alpha^н+M_\beta^н)\right].$$

With this approach the equations of equilibrium are satisfied and two other arbitrary functions appear. Furthermore, by determining the particular solution of the system of equations /268

$$\varepsilon_\alpha^*=\frac{1}{Eh}(T_\alpha^*-\nu T_\beta^*),\quad \varepsilon_\beta^*=\frac{1}{Eh}(T_\beta^*-\nu T_\alpha^*),\quad \omega^*=\frac{2(1+\nu)}{Eh}S^*, \qquad (11.6)$$

we obtain additional displacements u*, v*, w*.

The constructed solution ($u = u^h + u^*$, ..., T_α^*, ..., M_α^h, ...) approximately satisfies all equations of the theory of shells. Indeed, the equations of continuity are satisfied identically since we have, in fact, constructed the displacements. As it is easy to see, the equilibrium equations may be disrupted

only by the moments M_α^*, M_β^*, H^* constructed from u*, v*, w*. However, the analysis carried out in section 1, Chapter XII shows that this disruption is associated with an error whose order of magnitude is $(\frac{h}{R_0})^2$.

According to criterion (Chapter XII, 0.5), the constructed solution represents an example of the basic state of stress with strong moments. In this case it is constructed by integrating (in any order) the equations of the membrane theory. In section 3, Chapter X we present calculations for a circular cylindrical shell in which the state of stress with high moments is combined with the edge effect (section 1, Chapter XVII).

12. The Intensity of the Edge Effect

The intensity of the edge effect depends on the boundary conditions /269 of the considered problem and on the form of the basic state of stress. In the preceding sections we consider three types of basic states of stress:

(a) membrane,

(b) semimembrane,

(c) with strong moments.

We shall use these for a short analysis.

First of all, from the analysis of the equilibrium equations and the relationships of Hooke's law it follows for all three states that

$$\{T_\alpha^0,\ T_\beta^0,\ S^0\} \sim pR_0 \quad (a,\ b,\ c) \tag{12.1}$$

(p is a characteristic load; R_0 is a linear dimension characteristic for the middle surface). These forces are associated with the stresses

$$\sigma_{ij} \sim p\frac{R_0}{h} \quad (a,\ b,\ c) \tag{12.2}$$

and displacements

$$\{u^0,\ v^0,\ w^0\} \sim p\frac{R_0^2}{Eh} \quad (a,\ b,\ c). \tag{12.3}$$

We recall that for a state of stress with strong moments (section 11)

$$M^0 \sim R_0 T^0 \sim p R_0^2 \quad (c). \tag{12.4}$$

Therefore, in this case it is necessary to add the purely flexural displacements to the displacements

$$\{u^н, v^н, w^н\} \sim p \frac{R_0^4}{Eh^3} \quad (c). \tag{12.5}$$

All three forms are associated with the components of the tangential edge deformation

$$R_t \varepsilon_{tt}^0 \sim p \frac{R_0^2}{Eh}, \; R_t \varkappa_{tn}^0 \sim p \frac{R_0}{Eh} \; (a, b, c) \tag{12.6}$$

(we recall that purely flexural displacements have no effect on the quantities ε_{tt} and $\varkappa_{tn}$).

Displacements (12.3) and (12.5) correspond to moments and cutting forces

$$\begin{aligned} &(a)\; M^0 \sim ph^2, \quad Q_{vn}^0 \sim p \frac{h^2}{R_0}, \\ &(b)\; M^0 \sim phR_0, \quad Q_{vn}^0 \sim ph, \\ &(c)\; M^0 \sim pR_0^2, \quad Q_{vn}^0 \sim pR_0 \end{aligned} \tag{12.7}$$

and the flexural stresses

$$(a)\; \sigma_{M^0} \sim p, \; (b)\; \sigma_{M^0} \sim p \frac{R_0}{h}, \; (c)\; \sigma_{M^0} \sim p \left(\frac{R_0}{h}\right)^2. \tag{12.8}$$

Let us consider some basic versions of boundary conditions. /270

1. "Built-in" edge ($\varepsilon_{tt} = 0$, $\varkappa_{tn} = 0$). In this case

$$[R_t(\varepsilon_{tt} - \varepsilon_{tt}^0)]_0 \sim \frac{pR_0^2}{Eh}; \; [R_t(\varkappa_{tn} - \varkappa_{tn}^0)]_0 \sim \frac{pR_0}{Eh}. \tag{12.9}$$

Then from (8.13) and (8.12) we find

$$Q_0^{\kappa} \equiv (Q_0 - Q_0^{\circ}) \sim \left(\frac{h}{R_0}\right)^{1/2} pR_0; \quad M_0^{\kappa} \equiv (M_0 - M_0^{\circ}) \sim hpR_0; \tag{12.10}$$

$$\sigma_{M_0} \sim p\frac{R_0}{h}; \quad \sigma_k \sim p\frac{R_0}{h}. \tag{12.11}$$

Thus, in the considered case the stresses due to the edge effect are quantities of the same order as the tangential basic stresses (12.2). We emphasize that here the built-in feature is interpreted only in the sense that it eliminates the tangential deformation of the edge ($\varepsilon_{tt} = 0$, $\varkappa_{tn} = 0$).

2. Free edge ($Q_0 = 0$, $M_0 = 0$). In this case, according to (12.7)

$$\begin{aligned} &\text{(a)}\ \sigma_{M_0} \sim p, \quad \sigma_k \sim p\sqrt{\frac{h}{R_0}}, \\ &\text{(b)}\ \sigma_{M_0} \sim p\frac{R_0}{h}, \quad \sigma_k \sim p\sqrt{\frac{R_0}{h}}, \\ &\text{(c)}\ \sigma_{M_0} \sim p\left(\frac{R_0}{h}\right)^2, \quad \sigma_k \sim p\left(\frac{R_0}{h}\right)^{\frac{3}{2}}. \end{aligned} \tag{12.12}$$

Examining these equations we see that for a basic membrane state we can neglect the stresses due to the edge effect. For the semimembrane state and for a state with strong moments, they are quantities of the same order as the basic quantities. Sometimes, if we are free to control the purely flexural displacements, it is possible to satisfy the conditions

$$Q_0^{\circ} = 0, \quad M_0^{\circ} = 0. \tag{12.13}$$

As we can easily see in this case, the edge effect is absent.

3. The edge is loaded with a moment and a cutting force ($M_0 \neq 0$, $Q_0 \neq 0$).

It follows from relationships (12.10) and (12.1) that the cutting force and the bending moment are usually small compared with the tangential boundary forces. This is associated with a situation which is characteristic for thin shells having a large tangential rigidity and a small flexural rigidity. As a result the insignificant forces applied to the edge, which in themselves represent a cutting force and bending moment, produce noticeable flexure. Therefore, it is clear a priori that a cutting and a bending moment of the /271 order

$$Q_0 \sim qR_0, \quad M_0 \sim R_0 pR_0 \tag{12.14}$$

applied to the edge of a shell cause the appearance of a very strong edge effect.

More precisely, from relationships (8.12) we have for all three basic states

$$\sigma_{M_0}=p\left(\frac{R_0}{h}\right)^2;\ \sigma_k=q\left(\frac{R_0}{h}\right)^{\frac{3}{2}}\ (\text{a, b, c}). \tag{12.15}$$

In addition (8.9)

$$w^{\kappa}\sim\left(\sqrt{\frac{R_0}{h}}\frac{R_0^2q}{Eh}+\frac{R_0}{h}\frac{R_0^2p}{Eh}\right),$$
$$\vartheta^{\kappa}\sim\left(\frac{R_0}{h}\frac{R_0^2q}{Eh}+\left(\frac{R_0}{h}\right)^{\frac{3}{2}}\frac{R_0^2p}{Eh}\right). \tag{12.16}$$

The relationships which have been presented in this case show that in this case the stresses and displacements due to the edge effect substantially exceed the basic quantities for the first two states, and the stresses are quantities of the same order of magnitude for the state with strong moments.

Comparing estimates (12.14) and $(12.7)_3$ we see that it is possible to have a case when, by a suitable fixing of the arbitrariness in the solution involving strong moments, this solution "lifts" the edge load (12.14). In this case the edge effect does not occur and the estimates (12.15) and (12.5) remain valid for the stresses and displacements of the highly flexural solution.

4. Let us consider the following curious example communicated to the author by A. L. Gol'denveyzer, who attributed it to L. I. Balabukha. Let us assume that the cylindrical shell (fig. 27a and b) is built in at one edge ($\xi = 0$) and is loaded at the other end by a cutting force and a bending moment distributed according to the law $\cos\varphi$. This problem pertains to a case well studied in sections 9-10, Chapter V of an antisymmetric deformation of a shell of revolution. In view of its simplicity it may be considered directly, without reference to the relationships of Chapter V.

Thus, at the edge $\xi = \xi_0$ we are given

$$T_1=S=0;\ Q_{1n}=Q_0\cos\varphi;\ M_1=M_0\cos\varphi. \tag{12.17}$$

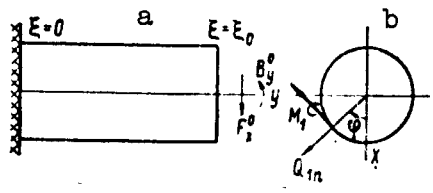

Figure 27

This edge load is not self-balanced and is reduced to the integral of the cutting force and the bending moment /272

$$F_x^0 = \int_0^{2\pi} Q_{1n} \cos\varphi \, r_0 \, d\varphi = \pi r_0 Q_0,$$
$$B_y^0 = \int_0^{2\pi} M_1 \cos\varphi \, r_0 \, d\varphi = \pi r_0 M_0. \tag{12.18}$$

The solution of the membrane system

$$\frac{\partial T_1^*}{\partial \xi} + \frac{\partial S^*}{\partial \varphi} = 0, \quad \frac{\partial S^*}{\partial \xi} + \frac{\partial T_2^*}{\partial \varphi} = 0, \quad T_2^* = 0$$

has the form

$$T_1^* = [T_0 + (\xi_0 - \xi) S_0] \cos\varphi; \quad S^* = S_0 \sin\varphi; \quad T_2^* = 0. \tag{12.19}$$

For this solution (fig. 27)

$$F_x^* = \int_0^{2\pi} S^* (-\sin\varphi) \, r_0 \, d\varphi = -\pi r_0 S_0;$$
$$B_y^* = \int_0^{2\pi} T_1^*(\xi_0) (r_0 \cos\varphi) \, r_0 \, d\varphi = \pi r_0^2 T_0. \tag{12.20}$$

Assuming that the membrane solution is "responsible" for the equilibrium of the shell as a whole (item 1, section 7), we obtain

$$F_x^0 = F_x^*; \quad B_y^0 = B_y^*. \tag{12.21}$$

From this it follows that

$$S_0 = -Q_0; \quad T_0 = \frac{1}{r_0} M_0. \tag{12.22}$$

We recall that (section 9, Chapter V) in the antisymmetric case purely flexural displacements are reduced to the rigid displacement of the shell. The two arbitrary constants corresponding to this displacement are fixed when the boundary conditions are satisfied at the built-in end. Thus, in essence, the membrane solution in the considered problem has no arbitrariness.

The following problem suggests itself: how can we satisfy four boundary conditions (12.17) by selecting two arbitrary constants of the edge effect? If the boundary conditions are not satisfied, this would mean that it is impossible to separate the solution of the considered example into a membrane solution and an edge effect solution.

A more careful examination of the obtained solution shows, however, that everything is satisfactory. Indeed, let us select $Q_0(\varphi)$ and $M_0(\varphi)$ in such a way that we satisfy the third and fourth conditions in (12.17). It is proposed that the reader use relationships (12.22), (6.2), (8.4) and (8.7) to verify that the first two conditions in (12.17) are satisfied automatically in this case. (Why?) In this case we should remember that in the considered problem /273

$$\tilde{w}_0^0=0;\quad Q_0^0=0;\frac{\partial}{\partial s_t}=\frac{1}{r_0}\frac{\partial}{\partial\tau},\ \frac{\partial}{\partial s_\nu}=-\frac{1}{r_0}\frac{\partial}{\partial\xi},\ R_t=r_0,\ \varkappa_t=0. \tag{12.23}$$

5. The brief analysis which has been carried out shows that the intensity of the edge effect depends substantially on the form of the boundary conditions and on the nature of the basic state. It was also clarified that when the edges of the shell are not fixed, this leads to a state of stress with large moments and to a rather intense edge effect. From the point of view of strength both states are extremely disadvantageous, because they are associated with very large stresses and displacements.

As we can easily see from the relationships which have been obtained and from expressions (8.11), in the general case the edge effect itself corresponds to a mixed state of stress (Chapter XII, 0.4).

13. Cylindrical Shell With a Skew Cut

Let us consider a circular cylindrical shell which is bounded on one side by a cross section normal to the axis and is bounded by a skew cross section on the other side which forms and angle δ with the axis. From figure 28 we can easily see the equation for the skew cut

$$\bar{\xi}(\bar{\varphi})=\xi_0+\frac{\cos\bar{\varphi}}{\operatorname{tg}\delta}, \tag{13.1}$$

where $\xi = \frac{x}{r_0}$, $\xi_0 = \frac{L}{r_0}$, while L is the length of the cylinder along the axis. Here and below the quantities $(\bar{\xi}, \bar{\varphi})$ will be used to designate the coordinates of the boundary contour, and the designation (ξ, φ) will be retained for the inner points of the region.

In addition to the usual Cartesian system of coordinates, we introduce a parallel system of coordinates s_ν, $s_\tau(\bar{\varphi})$ (end of section 8, Chapter VII) in the neighborhood of the boundary contour. In this case the geodetic normals on the development of the middle surface (fig. 29) are straight lines, orthogonal with respect to the contour. From this drawing we also see directly the following relationship between the parallel coordinates and the Cartesian coordinates

$$\xi = \bar{\xi}(\bar{\varphi}) + \frac{\cos\gamma}{r_0} s_\nu; \quad \varphi = \bar{\varphi} + \frac{\sin\gamma}{r_0} s_\nu. \tag{13.2}$$

It is easy to establish the variation of the quantities sin γ and cos γ as functions of $\bar{\varphi}$. Indeed, it follows from figure 28 that

$$\frac{dr_0\bar{\varphi}}{ds_\tau} = \cos\gamma; \quad \frac{dr_0\bar{\xi}}{ds_\tau} = -\sin\gamma. \tag{13.3}$$

However, (from 13.1) we find

$$\operatorname{tg}\gamma = -\frac{d\bar{\xi}}{d\bar{\varphi}} = \frac{\sin\bar{\varphi}}{\operatorname{tg}\delta}$$

and

$$\sin\gamma = \frac{\sin\bar{\varphi}}{\sqrt{\operatorname{tg}^2\delta + \sin^2\bar{\varphi}}}, \quad \cos\gamma = \frac{\operatorname{tg}\delta}{\sqrt{\operatorname{tg}^2\delta + \sin^2\bar{\varphi}}}. \tag{13.4}$$

The normal curvature and the geodetic curvature of the contour g are /274 respectively equal to (Chapter VII, 8.57 and 7.56)

$$\frac{1}{R_\tau} = \frac{\cos^2\gamma}{r_0}; \quad \varkappa_\tau = \frac{\partial\gamma}{\partial s_\tau}. \tag{13.5}$$

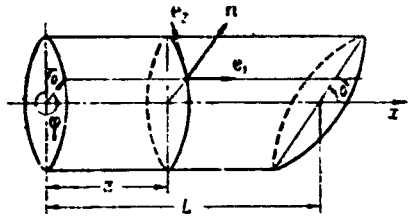

Figure 28

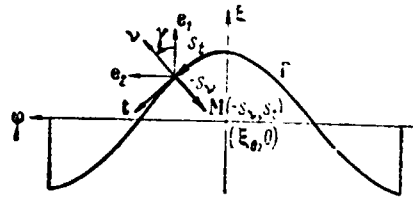

Figure 29

In this case differentiating expression $(13.4)_1$ with respect to s_t and utilizing $(13.3)_1$, we obtain

$$\frac{\partial \gamma}{\partial s_t} = \frac{\cos^3 \gamma}{r_0} \frac{\cos \bar{\varphi}}{\operatorname{tg} \delta}. \tag{13.6}$$

We shall seek the solution in the form of a sum containing the membrane solution and the edge effect solution. If we assume that the load is a uniform pressure and integrate equations (Chapter XII, 2.1 and 2.2), we obtain

$$T_1^* = -f_1'(\varphi)\xi + g_1(\varphi) + \frac{pr_0}{2}; \quad T_2^* = pr_0; \quad S^* = f_1(\varphi);$$

$$u^* = \frac{r_0}{Eh}\left[f_2(\varphi) - f_1'(\varphi)\frac{\xi^2}{2} + g_1(\varphi)\xi + \frac{1-2\nu}{2} pr_0 \xi\right]; \tag{13.7}$$

$$v^* = \frac{r_0}{Eh}\Big[(\varphi)\xi - g_2(\varphi) + f_2'(\varphi)\frac{\xi^3}{6} - g_1'(\varphi)\frac{\xi^2}{2} +$$

$$+ 2(1+\nu) f_1(\varphi)\xi\Big];$$

$$w^* = \frac{r_0}{Eh}\Big[f_2'(\varphi)\xi + g_2'(\varphi) - f_1''(\varphi)\frac{\xi^3}{6} + g_1'(\varphi)\frac{\xi^2}{2} - \tag{13.8}$$

$$-2(1+\nu) f_1'(\varphi)\xi + \nu(f_1''(\varphi)\xi - g_1(\varphi)) + \frac{(2-\nu)pr}{2}\Big].$$

With regard to the edge effect, we can determine it by utilizing relationships presented in section 8.

To be specific, we consider the case when both edges are built in. /275 According to (7.14) and (7.15) this case is associated with the boundary conditions

$$u_\nu^* = u_t^* = 0; \quad \varepsilon_{tt} = 0; \quad \varkappa_{tn} = 0 \text{ (when } \xi = \bar{\xi}, \ \varphi = \bar{\varphi}); \tag{13.9}$$

$$u^* = v^* = 0, \quad \varepsilon_2 = 0; \quad \varkappa_{2n} = 0 \text{ (when } \xi = 0), \tag{13.10}$$

where

$$u_\nu^* = u^* \cos\gamma + v^* \sin\gamma;\ u_t^* = -u^* \sin\gamma + v^* \cos\gamma;$$
$$\varepsilon_{tt} = \cos^2\gamma\,\varepsilon_2 - \sin\gamma\cos\gamma\,\omega + \sin^2\gamma\,\varepsilon_1; \tag{13.11}$$
$$\varkappa_{tn} = \frac{\partial}{\partial s_t}\left[\sin\gamma\cos\gamma(\varepsilon_2 - \varepsilon_1) + (\cos^2\gamma - \sin^2\gamma)\frac{\omega}{2}\right] -$$
$$- \frac{\cos\gamma}{r_0}\left(\frac{\partial\varepsilon_2}{\partial\xi} - \frac{1}{2}\frac{\partial\omega}{\partial\varphi}\right) - \frac{\sin\gamma}{r_0}\left(\frac{\partial\varepsilon_1}{\partial\varphi} - \frac{1}{2}\frac{\partial\omega}{\partial\xi}\right). \tag{13.12}$$

It is easy to see that the geometric boundary conditions are satisfied when

$$f_1(\varphi) = f_2(\varphi) = g_2(\varphi) = 0;\ g_1(\varphi) = -\frac{1-2\nu}{2} pr_0. \tag{13.13}$$

Then,

$$\sigma_{T_1}^* = \nu\frac{pr_0}{h};\ \sigma_{T_2}^* = \frac{pr_0}{h};\ \sigma_{S^*} = 0; \tag{13.14}$$
$$\varepsilon_1^* = \omega^* = 0;\ \varepsilon_2^* = \frac{(1-\nu^2)pr_0}{Eh}. \tag{13.15}$$

By using these relationships we compute

$$R_t\varepsilon_{tt}^* = \frac{(1-\nu^2)pr_0^2}{Eh};\ R_t\varkappa_{tn}^* = \frac{(1-\nu^2)pr_0}{Eh}(\cos^2\gamma - \tag{13.16}$$
$$- \sin^2\gamma)\cos\gamma\frac{\cos\varphi}{\operatorname{tg}\delta}.$$

By taking into account the last two boundary conditions $(13.9)_{3,4}$, we obtain from (8.11)-(8.14) $\sqrt{\frac{c}{R_0}}$.

$$\sigma_k = -2(1-\nu^2)\frac{pr_0}{h}\cos^2\gamma;$$
$$\sigma_{M_0} = -\frac{3}{\sqrt{3(1-\nu^2)}}(1-\nu^2)\frac{pr_0}{h}\cos^2\gamma. \tag{13.17}$$

Here we neglected (section 8) the effect of the quantity $R_t \varkappa^*_{tn}$. The reader should verify that if it is taken into account we shall obtain only an insignificant change in σ_k and σ_{M_0} (by amounts which are of the order of $\sqrt{\frac{c}{R_0}}$ with respect to 1).

The characteristic stresses in this problem will be

$$\begin{aligned}
\sigma_{M_\nu} &= \sigma_{M_0}(\cos\beta - \sin\beta)\, e^{\beta} + \frac{3}{\sqrt{3(1-\nu^2)}}\sigma_k \sin\beta\, e^{\beta};\\
\sigma_{T_t} &= \sigma^*_{T_t} + \sigma_k \cos\beta e^{\beta} - \frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta + \sin\beta)e^{\beta};\\
\sigma_{M_t} &= \nu\sigma_{M_\nu};\ \sigma_{T_\nu} = \sigma^*_{T_\nu},
\end{aligned} \tag{13.18}$$

where

$$\beta = \frac{s_\nu \cos\gamma}{\sqrt{2cr_0}}; \tag{13.19}$$

$$\sigma^*_{T_t} = (\nu\sin^2\gamma + \cos^2\gamma)\frac{pr_0}{h};\ \sigma^*_{T_\nu} = (\nu\cos^2\gamma + \sin^2\gamma)\frac{pr_0}{h}. \tag{13.20}$$

The expressions which have been obtained make it possible to construct stress diagrams at any point in the region. For this purpose, of course, /276 it is necessary to construct a coordinate mesh corresponding to the parallel system of coordinates. In this case its construction is not associated with any substantial difficulties since the tangential normals and the developed surface are straight lines perpendicular to the boundary contour. Also, the coordinate mesh is necessary only in the neighborhood of the boundary contour because of the rapid damping of the edge effect. Away from the contour we should utilize only the membrane solution (13.14), which is assigned in the Cartesian system of coordinates common for cylindrical shells.

Usually, however, we are interested in the maximum stresses which occur at the contour. We obtain their values by assuming that in (13.18)

$$\begin{aligned}
&\beta = 0\ (s_\nu = 0);\\
&\sigma_{M_\nu} = -\sqrt{3(1-\nu^2)}\cos^2\gamma\frac{pr_0}{h};\ \sigma_{T_t} = \nu[1-(1-\nu)\cos^2\gamma]\frac{pr_0}{h};\\
&\sigma_{M_t} = \nu\sigma_{M_\nu};\ \sigma_{T_\nu} = \sigma^*_{T_\nu} = [1-(1-\nu)\cos^2\gamma]\frac{pr_0}{h}.
\end{aligned} \tag{13.21}$$

Then, by formulating the expressions for the total stresses

$$\sigma^{\pm}_{\nu} = \sigma_{T_\nu} \pm \sigma_{M_\nu} \text{ and } \sigma^{\pm}_{t} = \sigma_{T_t} \pm \sigma_{M_t}, \tag{13.22}$$

we find, first of all, that

$$\sigma_i^{\pm} = \nu \sigma_\nu^{\pm}. \tag{13.23}$$

This equality, of course, is not accidental and is associated with the fact (should be proven!), that, in view of the boundary conditions, the elongations of the longitudinal fibers associated with the boundary element ($\varepsilon_{tt}^{0} = 0$) are absent. Figure 30 shows (for different values of δ and for $\nu = 0.3$) the ratios of the stresses at the internal edge fiber $\sigma_\nu^- = \sigma_{T_\nu} - \sigma_{M_\nu}$ to the maximum membrane stress $\sigma_0 = \sigma_{T_2}^* = \frac{pr_0}{h}$.

As we can see from this figure, the stress concentration for all δ (including the other edge of the shell which corresponds to $\delta = 90^\circ$) is determined by the following quantity

$$K = \frac{(\sigma_\nu^-)_{\varphi=0}}{\sigma_0} = 1.95. \tag{13.24}$$

With regard to the tangential stresses, as was to be expected (why?) /277 they are below the computed membrane stress $\sigma_{T_2}^* = \frac{pr_0}{h}$ $(13.21)_4$.

The resulting relationships make it possible to make two conclusions of a general nature:

(a) when the angles δ are excessively small, the maximum stresses at the point where the skew cut is built in do not depend on the angle of inclination and are equal to the corresponding value for a straight cut

$$\sigma_{max} = \left[\nu + \sqrt{3(1-\nu^2)}\right]\frac{pr_0}{h}. \tag{13.25}$$

(b) the lion's share of the stress is flexural.

The contents of this section consist of a presentation of the article in reference 261. A similar problem is considered in the work of M. I. Estrin (ref. 275). The basic difference in the formulation of the identical problem consists of the following. Estrin assumes that the edge effect dampens at a maximum rate in the direction of the cylinder generatrix. In our case, however, we assume that this direction is normal to the contour. As we can see from the material presented in the present chapter, the latter proposition follows directly from the determination of the edge effect. Since this problem is important, we shall consider it once more. For this purpose we consider a solution of the form (section 1) in the neighborhood of the boundary contour.

$$\Phi(\alpha, \beta) = e^{kf(\alpha, \beta)} \quad (k \gg 1). \tag{13.26}$$

From its determination the solution of the edge effect type is characterized by the property that it decreases rapidly as we go into the depth of the region and remains appreciably unchanged along the boundary contour. From the structure of solution (13.26) we can see that this property is realized only when we have the following condition /278

$$\frac{\partial f}{\partial s_t} = -\frac{\sin\gamma}{A}\frac{\partial f}{\partial \alpha} + \frac{\cos\gamma}{B}\frac{\partial f}{\partial \beta} = 0. \tag{13.27}_1$$

Adding the following equality to this condition

$$\frac{\partial f}{\partial s_\nu} = \frac{\cos\gamma}{A}\frac{\partial f}{\partial \alpha} + \frac{\sin\gamma}{B}\frac{\partial f}{\partial \beta} \tag{13.27}_2$$

and solving the resulting system, we obtain the following relationships

$$\frac{1}{A}\frac{\partial f}{\partial \alpha} = \cos\gamma\frac{\partial f}{\partial s_\nu}; \quad \frac{1}{B}\frac{\partial f}{\partial \beta} = \sin\gamma\frac{\partial f}{\partial s_\nu}. \tag{13.28}$$

If we let χ designate the angle between the arbitrary direction and the α-line, we obtain by means of (13.28)

$$\frac{\partial f}{\partial s_\chi} = \cos\chi\frac{1}{A}\frac{\partial f}{\partial \alpha} + \sin\chi\frac{1}{B}\frac{\partial f}{\partial \beta} = \cos(\chi - \gamma)\frac{\partial f}{\partial s_\nu}. \tag{13.29}$$

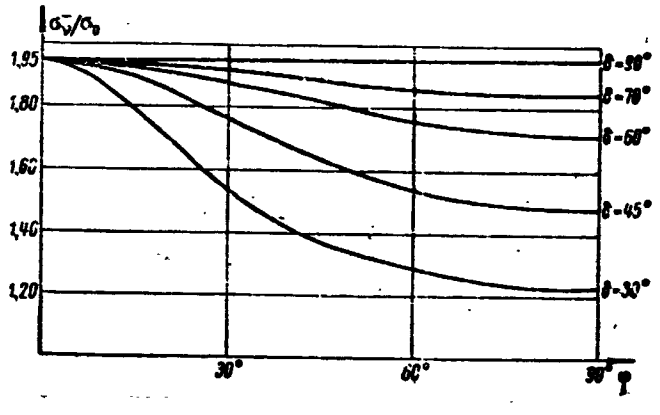

Figure 30

From this we can see that the direction of the maximum change in the solution of the edge effect type coincides with the normal to the contour. Furthermore, by expanding f (α, β) in the neighborhood of the contour point (s_t) into a double power series, we obtain

$$\Phi = e^{k\left\{f(0,0) + \left(\frac{\partial f}{A\,\partial \alpha}\right)_0 s_\alpha + \left(\frac{\partial f}{B\,\partial \beta}\right)_0 s_\beta + \ldots\right\}} \tag{13.30}$$

Limiting ourselves to the terms which have been written out for the expansion and remembering that $\cos \gamma \cdot s_\alpha + \sin \gamma \cdot s_\beta = s_\nu$, we obtain the expression

$$\Phi = \Phi_0(s_t)\, e^{k\left(\frac{\partial f}{\partial s_\nu}\right)_0 s_\nu} \qquad (\Phi_0(s_t) = e^{kf(0,0)}) \tag{13.31}$$

which has a structure quite familiar to us. The reasoning which has been presented shows that the proposition which has been utilized has an invariant (not associated with the specific selection of the coordinate system) nature and is applicable to an arbitrary (nonasymptotic) contour on a shell of general form.

Cylindrical cells with a skew cut are also considered in the works in references 276 and 259.

CHAPTER XIV. THERMOELASTIC STRESS

When shells are heated, large thermal stresses can occur in them. /279
These stresses are produced either by the nonuniform heating of the shell or by the fixed edges whose free expansion is prevented. Even when a composite shell is heated uniformly, the unmatched thermal expansion of its parts may serve as a reason for the occurrence of large temperature (contact) stresses. Therefore, along with the determination of stresses and deformations which occur when an external load is applied to the shell, it is frequently necessary to solve the corresponding thermoelastic problem.

In the present chapter we shall briefly consider the question of determining thermoelastic stresses and deformations, assuming that in this case:

(a) the law for the temperature distribution in the shell is known (the corresponding problem of thermal conductivity has been solved);

(b) the body is thermally isotropic, so that the volumetric expansion which occurs when the shell is heated may be characterized by a single quantity--the coefficient of linear expansion β;

(c) the temperatures are moderate, so that the elastic constants and the coefficient of linear expansion may be considered as quantities which are independent of the temperature.

With these assumptions the solution of thermoelastic problems does not lead to any specific difficulties. In any case, the solution of a thermoelastic problem for a specific shell is no more difficult than the calculation of the same shell loaded with external forces. Therefore, below we shall give only a brief presentation of the question concerning the design of shells for temperature effects. A more complete presentation may be obtained by becoming acquainted with the monographs in references 290, 291, 299, 300 and 301.

1. General Relationships

Let us assume that during the process of deformation the temperature /280
of the body is increased by T degrees. We shall assume that the heating produces no substantial change in the constants E and ν and that the body is thermally isotropic. Then (this constitutes the well-known Duhamel-Neumann hypothesis) a bulk thermal expansion is superposed on the deformation of the body produced by the external forces. Thus

$$e_{11}=\frac{1}{E}[\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})]+\beta T,$$

$$e_{22}=\frac{1}{E}[\sigma_{22}-\nu(\sigma_{33}+\sigma_{11})]+\beta T;$$

. . . . . .

$$e_{12}=\frac{2(1+\nu)}{E}\sigma_{12},$$

. . . . . .

where β is the coefficient of linear expansion.

From this, if we assume the static Kirchhoff hypothesis and neglect the stresses $\sigma_{\zeta\zeta}$, we obtain

$$\begin{aligned}\sigma_{11}&=\frac{E}{1-\nu^2}[e_{11}+\nu e_{22}-(1+\nu)\beta T],\\ \sigma_{22}&=\frac{E}{1-\nu^2}[e_{22}+\nu e_{11}-(1+\nu)\beta T],\\ \sigma_{12}&=\frac{E}{2(1+\nu)}e_{12}.\end{aligned}\tag{1.2}$$

Converting in the usual way to forces and moment (Chapter III) we find

$$\begin{aligned}T_1&=\frac{Eh}{1-\nu^2}\{\varepsilon_1+\nu\varepsilon_2-(1+\nu)\varepsilon_T\};\quad M_1=\frac{Eh^3}{12(1-\nu^2)}\{\varkappa_1+\nu\varkappa_2-\\ &\qquad-(1+\nu)\varkappa_T\};\\ T_2&=\frac{Eh}{1-\nu^2}\{\varepsilon_2+\nu\varepsilon_1-(1+\nu)\varepsilon_T\};\quad M_2=\frac{Eh^3}{12(1-\nu^2)}\{\varkappa_2+\nu\varkappa_1-\\ &\qquad-(1+\nu)\varkappa_T\};\\ S&=\frac{Eh}{2(1+\nu)}\omega;\quad H=\frac{Eh^3}{12(1+\nu)}\tau.\end{aligned}\tag{1.3}$$

Here

$$\varepsilon_T=\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\beta T\,d\zeta,\quad \varkappa_T=\frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}}\beta T\zeta\,d\zeta.\tag{1.4}$$

If we limit ourselves to the consideration of the (simplest) linear law governing the temperature variation across the thickness of the shell /281

$$T(\alpha_1;\ \alpha_2;\ \zeta)=t_0(\alpha_1,\ \alpha_2)+\zeta t_1(\alpha_1,\ \alpha_2)\tag{1.5}$$

and neglect the variations in the coefficient of linear expansion β along the thickness, then from (1.4) it follows that

$$\varepsilon_T = \beta t_0;\ \varkappa_T = \beta t_1. \tag{1.6}$$

It is easy to see that the quantities which have been introduced have been associated with the values of the temperature at the upper (t^+) and lower (t^-) surfaces of the shell by means of the following relationships

$$t_0 = \frac{t^+ + t^-}{2};\ t_1 = \frac{t^+ - t^-}{h}. \tag{1.7}$$

Let us return to equalities (1.3). Solving them with respect to deformation components, we obtain

$$\begin{aligned}
&\varepsilon_1 = \frac{1}{Eh}(T_1 - \nu T_2) + \varepsilon_T;\ \varkappa_1 = \frac{12}{Eh^3}(M_1 - \nu M_2) + \varkappa_T;\\
&\varepsilon_2 = \frac{1}{Eh}(T_2 - \nu T_1) + \varepsilon_T;\ \varkappa_2 = \frac{12}{Eh^3}(M_2 - \nu M_1) + \varkappa_T;\\
&\omega = \frac{2(1+\nu)}{Eh} S; \qquad \tau = \frac{12(1+\nu)}{Eh^3} H.
\end{aligned} \tag{1.8}$$

Now, by taking into account the additional temperature terms we introduce the basic relationships of the theory of shells. The complex forces and moments are introduced in a slightly different manner from that used in section 3, Chapter IV

$$\begin{aligned}
\widetilde{T}_1 &= T_1 - iEhc(\varkappa_2 - \varkappa_T),\\
\widetilde{T}_2 &= T_2 - iEhc(\varkappa_1 - \varkappa_T),\\
\widetilde{S} &= S + iEhc\tau,\\
\widetilde{M}_1 &= M_1 + iEhc(\varepsilon_2 - \varepsilon_T),\\
\widetilde{M}_2 &= M_2 + iEhc(\varepsilon_1 - \varepsilon_T),\\
\widetilde{H} &= H - iEhc\frac{\omega}{2}.
\end{aligned} \tag{1.9}$$

This makes it possible for us to use relationships (1.3) and (1.8) to obtain the same equalities as given in Chapter IV

$$\widetilde{M}_1 = ic\{\widetilde{T}_2 - \nu\widetilde{T}_1\};\ \widetilde{M}_2 = ic\{\widetilde{T}_1 - \nu\widetilde{T}_2\};\ \widetilde{H} = -ic\{\widetilde{S} + \nu\widetilde{S}\}. \tag{1.10}$$

$$
\begin{aligned}
T_1 &= \operatorname{Re}\widetilde{T}_1;\\
T_2 &= \operatorname{Re}\widetilde{T}_2;\\
S &= \operatorname{Re}\widetilde{S};\\
M_1 &= -c\operatorname{Im}\left\{\widetilde{T}_2+\nu\widetilde{T}_1\right\};\\
M_2 &= -c\operatorname{Im}\left\{\widetilde{T}_1+\nu\widetilde{T}_2\right\};\\
H &= c(1-\nu)\operatorname{Im}\widetilde{S}.
\end{aligned}
\tag{1.11}
$$

Furthermore, by using equalities (Chapter II, 2.17 and 2.18) we compute the following expression /282

$$
\begin{aligned}
&L_1\left\{\varkappa_2-\varkappa_T,\ \varkappa_1-\varkappa_T,\ -\tau,\ -\varepsilon_2+\varepsilon_T,\ -\varepsilon_1+\varepsilon_T,\ \frac{\omega}{2}\right\}=\\
&=L_1\left\{\varkappa_2,\ \varkappa_1,\ -\tau,\ -\varepsilon_2,\ -\varepsilon_1,\ \frac{\omega}{2}\right\}+L_1\left\{-\varkappa_T,\ -\varkappa_T,\ 0,\ \varepsilon_T,\ \varepsilon_T,\ 0\right\}=\\
&=A_1A_2\left(-\frac{1}{A_1}\frac{\partial\varkappa_T}{\partial\alpha_1}+\frac{1}{R_1A_1}\frac{\partial\tau_1}{\partial\alpha_1}\right).
\end{aligned}
$$

Computing L_2 and L_3 in the same manner, we obtain

$$
\begin{aligned}
&L_1\left\{\varkappa_2-\varkappa_T,\ \varkappa_1-\varkappa_T,\ -\tau,\ -\varepsilon_2+\varepsilon_T,\ -\varepsilon_1+\varepsilon_T,\ \frac{\omega}{2}\right\}=\\
&=\frac{A_1A_2}{Ehc}q_1^T,\ L_2\{\ldots\}=\frac{A_1A_2}{Ehc}q_2^T,\ L_3\{\ldots\}=-\frac{1}{Ehc}q_n^T,
\end{aligned}
\tag{1.12}
$$

where

$$
\begin{aligned}
q_1^T &= Ehc\left\{-\frac{\partial\varkappa_T}{A_1\partial\alpha_1}+\frac{1}{R_1}\frac{\partial\varepsilon_T}{A_1\partial\alpha_1}\right\};\\
q_2^T &= Ehc\left\{-\frac{\partial\varkappa_T}{A_2\partial\alpha_2}+\frac{1}{R_2}\frac{\partial\varepsilon_T}{A_2\partial\alpha_2}\right\};\\
q_n^T &= Ehc\left\{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\varkappa_T+\Delta(\varepsilon_T)\right\}.
\end{aligned}
\tag{1.13}
$$

By utilizing relationships (1.9) and (1.12) we follow the approach given in sections 5 and 6, Chapter IV to obtain the following simplified system of equations containing the complex forces

$$\frac{\partial A_2\widetilde{T}_1}{\partial\alpha_1}+\frac{1}{A_1}\frac{\partial A_1^2\widetilde{S}}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\widetilde{T}_2+ic\frac{A_2}{R_1}\frac{\partial\widetilde{T}}{\partial\alpha_1}=-A_1A_2\widetilde{q}_1;$$
$$\frac{\partial A_1\widetilde{T}_2}{\partial\alpha_2}+\frac{1}{A_2}\frac{\partial A_2^2\widetilde{S}}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\widetilde{T}_1+ic\frac{A_1}{R_2}\frac{\partial\widetilde{T}}{\partial\alpha_2}=-A_1A_2\widetilde{q}_2; \qquad (1.14)$$
$$\frac{\widetilde{T}_1}{R_1}+\frac{\widetilde{T}_2}{R_2}-ic\Delta\widetilde{T}=\widetilde{q}_n'.$$

Here

$$\widetilde{q}_1=q_1+iq_1^T,\ \widetilde{q}_2=q_2+iq_2^T,\ \widetilde{q}_n=q_n+iq_n^T;$$
$$\widetilde{q}_n'=\widetilde{q}_n+i\frac{c}{A_1A_2}\left\{\frac{\partial A_2(\widetilde{q}_1+\nu\overline{\widetilde{q}}_1)}{\partial\alpha_1}+\frac{\partial A_1(\widetilde{q}_2+\nu\overline{\widetilde{q}}_2)}{\partial\alpha_2}\right\}\approx\widetilde{q}_n. \qquad (1.15)$$

In addition, it is simple to verify that by introducing the complex boundary conditions by means of equalities /283

$$\widetilde{Q}_{\nu\nu}=Q_{\nu\nu}-iEhc(\varkappa_{tt}-\varkappa_T)=Q_{\nu\nu}^*-iEhc(\widetilde{\varkappa}_{tt}-\varkappa_T);$$
$$\widetilde{Q}_{\nu t}=Q_{\nu t}+iEhc\varkappa_{t\nu}=Q_{\nu t}^*+iEhc\widetilde{\varkappa}_{t\nu};$$
$$\widetilde{Q}_{\nu n}=Q_{\nu n}-iEhc\left(\varkappa_{tn}+\frac{d\varkappa_T}{ds_\nu}\right)=Q_{\nu n}^*-iEhc\left(\widetilde{\varkappa}_{tn}+\frac{d\varkappa_T}{ds_\nu}\right); \qquad (1.16)$$
$$\widetilde{M}_{\nu\nu}=M_{\nu\nu}+iEhc(e_{tt}-e_T)=M_{\nu\nu}^*+iEhc(\widetilde{e}_{tt}-e_T);$$
$$\left(\frac{d}{ds_\nu}=\frac{\cos\gamma}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\sin\gamma}{A_2}\frac{\partial}{\partial\alpha_2}\right).$$

we obtain for them the following simplified expressions as in the case of section 7, Chapter IV

$$\widetilde{Q}_{\nu\nu}=\cos^2\gamma\,\widetilde{T}_1+2\sin\gamma\cos\gamma\,\widetilde{S}+\sin^2\gamma\,\widetilde{T}_2;$$
$$\widetilde{Q}_{\nu t}=\sin\gamma\cos\gamma(\widetilde{T}_2-\widetilde{T}_1)+(\cos^2\gamma-\sin^2\gamma)\widetilde{S};$$
$$\widetilde{Q}_{\nu\nu}+i(1+\nu)c\frac{d\widetilde{Q}_{\nu t}}{ds_t}-ic(\widetilde{q}_\nu+\nu\overline{\widetilde{q}}_\nu)=ic\frac{d\widetilde{T}}{ds_\nu}; \qquad (1.17)$$
$$\widetilde{M}_{\nu\nu}+i(1+\nu)c\widetilde{Q}_{\nu\nu}=ic\widetilde{T}$$
$$\left(\frac{d}{ds_t}=-\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1}+\frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2},\ \widetilde{q}_\nu=\widetilde{q}_1\cos\gamma+\widetilde{q}_2\sin\gamma\right):$$

Finally, by taking the following simplified form for the complex Hooke's law (Chapter IV, 8.1)

$$
\begin{aligned}
\tilde{\varkappa}_2 &= \frac{i}{Ehc}(\tilde{T}_1 - T_1^*) + \varkappa_T;\\
\tilde{\varkappa}_1 &= \frac{i}{Ehc}(\tilde{T}_2 - T_2^*) + \varkappa_T;\\
\tilde{\tau} &= -\frac{i}{Ehc}(\tilde{S} - S^*);\\
\tilde{\varepsilon}_2 &= \frac{1}{Eh}\left(\tilde{T} + \frac{i}{c}M_1^*\right) + \varepsilon_T;\\
\tilde{\varepsilon}_1 &= \frac{1}{Eh}\left(\tilde{T} + \frac{i}{c}M_2^*\right) + \varepsilon_T;\\
\frac{\tilde{\omega}}{2} &= -\frac{i}{Ehc}H^*,
\end{aligned}
\tag{1.18}
$$

we arrive at the following system of equations containing the complex displacements (Chapter IV, 8.3)

$$
\begin{aligned}
\tilde{\varepsilon}_1 + ic(\tilde{\varkappa}_1 + \tilde{\varkappa}_2) &= \frac{1}{Eh}\left(T_1^* + T_2^* + \frac{i}{c}M_2^*\right) + \varepsilon_T + i2c\varkappa_T;\\
\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 &= \frac{i}{Ehc}(M_2^* - M_1^*);\\
\tilde{\omega} &= -\frac{i}{Ehc}2H^*.
\end{aligned}
\tag{1.19}
$$

In solving thermoelastic problems we may utilize all of the above approaches discussed in section 9, Chapter IV. /284

Thus, by comparing the relationships obtained above with the analogous ones from Chapter XI (in particular, by comparing equations (1.19) with (Chapter XI, 1.2), we can see that when we utilize the method of complex displacements, we can utilize the equations of Chapter XI by replacing in these the functions $(\tilde{\varepsilon}_1^c, \tilde{\varepsilon}_2^c, \ldots, \tilde{\tau}^c)$ by $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \ldots, \tilde{\tau})$ and the functions $(\varepsilon_1^g, \varepsilon_2^g, \omega^g, \varkappa_1^g, \varkappa_2^g, \tau^g)$ by $(-\varepsilon_1^T, -\varepsilon_2^T, 0, -\varkappa_1^T, -\varkappa_2^T, 0)$.

Incidentally, we have established a relationship, which is well known in the general theory of elasticity, between the dislocation displacements and temperature deformations.

2. Temperature Field Which Does Not Produce Stresses

It is of interest to clarify the conditions for which no stresses occur when a shell free of external loads is heated. In considering this problem we shall limit ourselves to the case when (1.5)-(1.6)

$$
\beta = \text{const}; \quad \varepsilon_T = \beta t_0; \quad \varkappa_T = \beta t_1. \tag{2.1}
$$

Since, according to the proposition, all of the forces and moments are equal to zero, it follows from (1.8) that

$$\varepsilon_1=\varepsilon_2=\varepsilon_T;\quad \varkappa_1=\varkappa_2=\varkappa_T;\quad \omega=0;\quad \tau=0. \tag{2.2}$$

Furthermore, according to (1.9) the complex forces and moments are equal to zero. Therefore it follows from (1.14) that the following equalities must be valid

$$q_1^T=q_2^T=q_n^T=0. \tag{2.3}$$

By taking into account (2.2), they can be expanded into the following system of differential equations

$$\begin{gathered}-\frac{\partial t_1}{A_1\partial\alpha_1}+\frac{1}{R_1}\frac{\partial t_0}{A_1\partial\alpha_1}=0;\quad -\frac{\partial t_1}{A_2\partial\alpha_2}+\frac{1}{R_2}\frac{\partial t_0}{A_2\partial\alpha_2}=0;\\ \left(\frac{1}{R_1}+\frac{1}{R_2}\right)t_1+\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{A_2}{A_1}\frac{\partial t_0}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\frac{A_1}{A_2}\frac{\partial t_0}{\partial\alpha_2}\right\}=0.\end{gathered} \tag{2.4}$$

It is easy to see that conditions (2.3) establish the compatibility of system (2.2), i.e., make it possible to determine the thermal displacements corresponding to the given temperature field. Finally, the fact that the forces and moments are equal to zero means that the edges of the shell are free.

In the general theory of elasticity (ref. 299, p. 21) the following proposition is proven: stresses are absent in an elastic body with a free boundary when the temperature distribution obeys the following law

$$T=a_0+a_1x+a_2y+a_3z \tag{2.5}$$

(x, y, z are Cartesian coordinates).

Let us show that this property is also retained for a shell which is considered as a body in the spatial Cartesian system of coordinates. By introducing this system we shall consider the temperature simultaneously as a function of the Cartesian coordinates x, y, z and as a function of the coordinates α_1, α_2, ζ, fixed with respect to the middle surface of the shell. Then /285

$$T=t_0(\alpha_1,\ \alpha_2)+\zeta t_1(\alpha_1,\ \alpha_2)=T(x,\ y,\ z). \tag{2.6}$$

From this we obtain

$$\begin{aligned}
&\frac{\partial T}{A_1\partial\alpha_1}=\frac{\partial T}{\partial x}\frac{\partial x}{A_1\partial\alpha_1}+\frac{\partial T}{\partial y}\frac{\partial y}{A_1\partial\alpha_1}+\frac{\partial T}{\partial z}\frac{\partial z}{A_1\partial\alpha_1}=\frac{\partial t_0}{A_1\partial\alpha_1}+\zeta\frac{\partial t_1}{A_1\partial\alpha_1};\\
&\frac{\partial T}{A_2\partial\alpha_2}=\frac{\partial T}{\partial x}\frac{\partial x}{A_2\partial\alpha_2}+\frac{\partial T}{\partial y}\frac{\partial y}{A_2\partial\alpha_2}+\frac{\partial T}{\partial z}\frac{\partial z}{A_2\partial\alpha_2}=\frac{\partial t_0}{A_2\partial\alpha_2}+\zeta\frac{\partial t_1}{A_2\partial\alpha_2};\\
&\frac{\partial T}{\partial\zeta}=\frac{\partial T}{\partial x}\frac{\partial x}{\partial\zeta}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial\zeta}+\frac{\partial T}{\partial z}\frac{\partial z}{\partial\zeta}=t_1.
\end{aligned}\tag{2.7}$$

By using the first and second equalities obtained above, the first of equations (2.4) can be transformed to the following form

$$\begin{aligned}
&-\frac{\partial t_1}{A_1\partial\alpha_1}+\frac{1}{R_1}\frac{\partial t_0}{A_1\partial\alpha_1}=\\
&=\frac{\partial T}{\partial x}\left[\frac{1}{R_1}\frac{\partial x}{A_1\partial\alpha_1}-\left(1+\frac{\zeta}{R_1}\right)\frac{\partial^2 x}{A_1\partial\alpha_1\partial\zeta}\right]-\left(1+\frac{\zeta}{R_1}\right)\frac{\partial x}{\partial\zeta}\frac{\partial^2 T}{A_1\partial\alpha_1\partial x}+\\
&+\frac{\partial T}{\partial y}\left[\frac{1}{R_1}\frac{\partial y}{A_1\partial\alpha_1}-\left(1+\frac{\zeta}{R_1}\right)\frac{\partial^2 y}{A_1\partial\alpha_1\partial\zeta}\right]-\left(1+\frac{\zeta}{R_1}\right)\frac{\partial y}{\partial\zeta}\frac{\partial^2 T}{A_1\partial\alpha_1\partial y}+\\
&+\frac{\partial T}{\partial z}\left[\frac{1}{R_1}\frac{\partial z}{A_1\partial\alpha_1}-\left(1+\frac{\zeta}{R_1}\right)\frac{\partial^2 z}{A_1\partial\alpha_1\partial\zeta}\right]-\left(1+\frac{\zeta}{R_1}\right)\frac{\partial z}{\partial\zeta}\frac{\partial^2 T}{A_1\partial\alpha_1\partial z}=0.
\end{aligned}\tag{2.8}$$

We shall represent the radius vector of an arbitrary point on the shell in two ways

$$\vec{R}=x\vec{e}_x+y\vec{e}_y+z\vec{e}_z=\vec{r}+\zeta\vec{n}.\tag{2.9}$$

From this we find (taking into consideration equalities (Chapter I, 1.4 and 1.29)

$$\begin{aligned}
&\frac{\partial x}{A_1\partial\alpha_1}=\frac{\partial\vec{R}}{A_1\partial\alpha_1}\vec{e}_x=\vec{e}_x\left(\frac{\partial\vec{r}}{A_1\partial\alpha_1}+\zeta\frac{\partial\vec{n}}{A_1\partial\alpha_1}\right)=\left(1+\frac{\zeta}{R_1}\right)\left(\vec{e}_1\vec{e}_x\right);\\
&\frac{\partial^2 x}{A_1\partial\alpha_1\partial\zeta}=\frac{\partial^2 x}{\partial\zeta A_1\partial\alpha_1}=\frac{1}{R_1}\left(\vec{e}_1\vec{e}_x\right).
\end{aligned}\tag{2.10}$$

In the same way we establish the following equalities

$$\begin{aligned}
&\frac{\partial y}{A_1\partial\alpha_1}=\left(1+\frac{\zeta}{R_1}\right)\left(\vec{e}_1\vec{e}_y\right);\quad\frac{\partial^2 y}{A_1\partial\alpha_1\partial\zeta}=\frac{1}{R_1}\left(\vec{e}_1\vec{e}_y\right);\\
&\frac{\partial z}{A_1\partial\alpha_1}=\left(1+\frac{\zeta}{R_1}\right)\left(\vec{e}_1\vec{e}_z\right);\quad\frac{\partial^2 z}{A_1\partial\alpha_1\partial\zeta}=\frac{1}{R_1}\left(\vec{e}_1\vec{e}_z\right);\\
&\frac{\partial x}{\partial\zeta}=\left(\vec{n}\vec{e}_x\right);\quad\frac{\partial y}{\partial\zeta}=\left(\vec{n}\vec{e}_y\right);\quad\frac{\partial z}{\partial\zeta}=\left(\vec{n}\vec{e}_z\right).
\end{aligned}\tag{2.11}$$

Now, by substituting the resulting equations into condition (2.8), we obtain /286

$$\left(\vec{n}\vec{e}_x\right)\frac{\partial^2 T}{A_1\partial\alpha_1\partial x}+\left(\vec{n}\vec{e}_y\right)\frac{\partial^2 T}{A_1\partial\alpha_1\partial y}+\left(\vec{n}\vec{e}_z\right)\frac{\partial^2 T}{A_1\partial\alpha_1\partial z}=0. \qquad (2.12)_1$$

The remaining two conditions are transformed in exactly the same manner

$$\left(\vec{n}\vec{e}_x\right)\frac{\partial^2 T}{A_2\partial\alpha_2\partial x}+\left(\vec{n}\vec{e}_y\right)\frac{\partial^2 T}{A_2\partial\alpha_2\partial y}+\left(\vec{n}\vec{e}_z\right)\frac{\partial^2 T}{A_2\partial\alpha_2\partial z}=0;$$
$$\left(\vec{e}_1\vec{e}_x\right)\frac{\partial^2 T}{A_1\partial\alpha_1\partial x}+\left(\vec{e}_1\vec{e}_y\right)\frac{\partial^2 T}{A_1\partial\alpha_1\partial y}+\left(\vec{e}_1\vec{e}_z\right)\frac{\partial^2 T}{A_1\partial\alpha_1\partial z}+ \qquad (2.12)_{2,3}$$
$$+\left(\vec{e}_2\vec{e}_x\right)\frac{\partial^2 T}{A_2\partial\alpha_2\partial x}+\left(\vec{e}_2\vec{e}_y\right)\frac{\partial^2 T}{A_2\partial\alpha_2\partial y}+\left(\vec{e}_2\vec{e}_z\right)\frac{\partial^2 T}{A_2\partial\alpha_2\partial z}=0.$$

This system can be satisfied by assuming that

$$\frac{\partial T}{\partial x}=a_1;\ \frac{\partial T}{\partial y}=a_2;\ \frac{\partial T}{\partial z}=a_3,$$

i.e., by assigning the following temperature distribution

$$T=a_0+a_1x+a_2y+a_3z. \qquad (2.5)$$

Thus we have established that the temperature distribution which is linear with respect to the spatial Cartesian system of coordinates does not lead to the occurrence of stresses in a shell with free edges.

In contrast to the three dimensional body, the temperature field (2.5) in a shell does not exhaust all of the case corresponding to the absence of thermal stresses. In other words, the system of equations (2.4) has nontrivial solutions which differ from (2.5). We clarify this by means of three examples.

I. Let us consider a spherical shell for which

$$R_1=R_2=A_1=R_0;\ A_2=R_0\sin\theta;\ \alpha_1=\theta;\ \alpha_2=\varphi.$$

Assuming that the temperature distribution is symmetric (independent of φ), we obtain from (2.4)

$$t_0=R_0t_1;\ \frac{d^2t_1}{d\theta^2}+\operatorname{ctg}\theta\frac{dt_1}{d\theta}+2t_1=0.$$

The last equation has two solutions

$$t_1^{(1)}=c\cos\theta,\quad t^{(2)}=cR_0\left(\cos\theta\ln\operatorname{tg}\frac{\theta}{2}+1\right). \tag{2.13}$$

The temperature corresponding to the first solution

$$T=t_0+\zeta t_1=c(R_0+\zeta)\cos\theta \tag{2.14}$$

changes in a linear fashion along the spatial axis z, which coincides with the shell's axis of rotation (fig. 20). The temperature distribution (2.14) was found by A. I. Lur'ye (ref. 23, p. 49) by integrating directly the following system of differential equations

$$\varepsilon_1=\varepsilon_2=\beta t_0,\quad \varkappa_1=\varkappa_2=\beta t_1 \tag{2.15}$$

assuming that the displacements were symmetric. The second solution /287
corresponds to the symmetric displacements which may even be nonperiodic with respect to φ.

II. Let us consider a circular cylindrical plate. For this plate

$$A_1=A_2=R_2=r_0;\quad \frac{1}{R_1}=0;\quad \alpha_1=\xi;\quad \alpha_2=\varphi,$$

and system (2.4) is written in the form

$$-\frac{\partial t_1}{r_0\partial\xi}=0;\ -\frac{\partial t_1}{r_0\partial\varphi}+\frac{1}{r_0^2}\frac{\partial t_0}{\partial\varphi}=0;$$
$$\frac{t_1}{r_0}+\frac{1}{r_0^2}\left\{\frac{\partial^2 t_0}{\partial\xi^2}+\frac{\partial^2 t_0}{\partial\varphi^2}\right\}=0.$$

It follows from the first two equations that

$$t_1=\Phi(\varphi);\quad t_0=r_0\Phi(\varphi)+\sum(\xi).$$

Substituting these expressions into the third equation we obtain

$$\{\Phi''(\varphi)+\Phi(\varphi)\}+\frac{1}{r_0}\sum''(\xi)=0.$$

From this we find

$$\Phi = a + b_1 \sin\varphi + b_2 \cos\varphi;$$
$$\sum = -ar_0 \frac{\xi^2}{2} + b_3\xi + b_4.$$

It is easy to see that the terms with the constants b_1, b_2, b_3, b_4 lead to a linear temperature distribution. Dropping them we consider in detail the temperature distribution corresponding to

$$\Phi = a, \quad \sum = -ar_0 \frac{\xi^2}{2},$$

i.e.,

$$t_1 = a, \quad t_0 = ar_0\left(1 - \frac{\xi^2}{2}\right). \tag{2.16}$$

We shall determine the displacements corresponding to this (nonlinear) temperature distribution. To do this, it is necessary to solve the system of differential equations (2.15)

$$\begin{aligned}
r_0\varepsilon_1 &= \frac{\partial u}{\partial \xi} = \beta a r_0^2\left(1 - \frac{\xi^2}{2}\right);\\
r_0\varepsilon_2 &= \frac{\partial v}{\partial \varphi} + w = \beta a r_0^2\left(1 - \frac{\xi^2}{2}\right);\\
r_0\omega &= \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \xi} = 0;\\
r_0^2\varkappa_1 &= -\frac{\partial^2 w}{\partial \xi^2} = \beta a r_0^2;\\
r_0^2\varkappa_2 &= -\frac{\partial}{\partial \varphi}\left(\frac{\partial w}{\partial \varphi} - v\right) = \beta a r_0^2;\\
r_0^2\tau &= -\frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial \varphi} - v\right) = 0.
\end{aligned} \tag{2.17}$$

Having solved this system and dropped the rigid displacements which are not significant in the present problem (ref. 302), we obtain /288

$$u = \beta a r_0^2\left(\xi - \frac{\xi^3}{6}\right); \quad v = \beta a r_0^2 \varphi; \quad w = -\beta a r_0^2 \frac{\xi^2}{2}. \tag{2.18}$$

The problem has been solved completely: we have obtained a temperature distribution which is different from (2.5) and we have constructed the corresponding displacements. The presence of the underlined dislocation term in the solution (section 2, Chapter X) once again confirms the relationship noted above between the temperature and dislocation terms.

III. As the last case we consider a plate. We place the origin of the spatial Cartesian system of coordinates at some point in the middle surface of the plate and direct the z axis along the normal to this plate. Then, by referring the middle surface to the Cartesian system of coordinates x and y ($\alpha_1 = x$, $\alpha_2 = y$, $\zeta = z$, $A_1 = A_2 = 1$, $\frac{1}{R_1} = \frac{1}{R_2} = 0$), we write system (2.4) in the form

$$\frac{\partial t_1}{\partial x} = \frac{\partial t_1}{\partial y} = 0; \quad \frac{\partial^2 t_0}{\partial x^2} + \frac{\partial^2 t_0}{\partial y^2} = 0. \tag{2.19}$$

From this it follows that $t_1 = t_1^0$, while t_0 is an arbitrary harmonic function which can differ to any substantial degree from a linear one

$$t_0 = a_0 + a_1 x + a_2 y.$$

The examples which have been considered confirm the curious peculiarity of shells noted above--the presence in these shells of additional temperature fields, compared with the three-dimensional body, which do not produce stresses.

We note in conclusion that for cases which are more general than (2.1) conditions (2.4) are replaced by

$$\begin{gathered} -\frac{\partial \varkappa_T}{A_1 \partial \alpha_1} + \frac{1}{R_1}\frac{\partial \varepsilon_T}{A_1 \partial \alpha_1} = 0; \quad -\frac{\partial \varkappa_T}{A_2 \partial \alpha_2} + \frac{1}{R_2}\frac{\partial \varepsilon_T}{A_2 \partial \alpha_2} = 0; \\ \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varkappa_T + \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{A_2}{A_1}\frac{\partial \varepsilon_T}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\frac{A_1}{A_2}\frac{\partial \varepsilon_T}{\partial \alpha_2}\right\} = 0. \end{gathered} \tag{2.20}$$

3. Thermoelastic Problems for Shells of Rotation

If we take into account the temperature, equations (Chapter V, 2.7) /289 retain their form, with the only difference that the terms q_1, q_2, q_n are replaced by

$$\tilde{q}_1 = q_1 + iq_1^T; \quad \tilde{q}_2 = q_2 + iq_2^T; \quad \tilde{q}_n = q_n + iq_n^T,$$

where we now have

$$
\begin{aligned}
q_1^T &= Ehc\left\{-\frac{1}{R_1}\frac{\partial \varkappa_T}{\partial \theta}+\frac{1}{R_1^2}\frac{\partial \varepsilon_T}{\partial \theta}\right\};\\
q_2^T &= Ehc\left\{-\frac{1}{R_2\sin\theta}\frac{\partial \varkappa_T}{\partial \varphi}+\frac{1}{R_2^2\sin\theta}\frac{\partial \varepsilon_T}{\partial \varphi}\right\};\\
q_n^T &= Ehc\left\{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\varkappa_T+\frac{1}{R_1R_2\sin\theta}\left[\frac{\partial}{\partial \theta}\times\right.\right.\\
&\left.\left.\times\left(\frac{R_2\sin\theta}{R_1}\cdot\frac{\partial \varepsilon_T}{\partial \theta}\right)+\frac{R_1}{R_2\sin\theta}\frac{\partial^2 \varepsilon_T}{\partial \varphi^2}\right]\right\}.
\end{aligned}
\tag{3.1}
$$

On the other hand, the complex displacements are determined from system (Chapter V, 2.12). The following terms

$$R_1\varepsilon_T,\ R_2\sin\theta\varepsilon_T,\ \varkappa_T,\ \varkappa_T$$

are added to the first, second, fourth and fifth equations respectively. The expressions for the complex boundary quantities (Chapter V, 1.21) are replaced by the following

$$
\begin{aligned}
\widetilde{Q}_\rho &= Q_\rho - iEhc\left[\varkappa_\rho-\left(\cos\theta\varkappa_T-\frac{\sin\theta}{R_1}\frac{\partial \varepsilon_T}{\partial \theta}\right)\right]=\\
&= Q_\rho^*-iEhc\left[\widetilde{\varkappa}_\rho-\left(\cos\theta\varkappa_T-\frac{\sin\theta}{R_1}\frac{\partial \varepsilon_T}{\partial \theta}\right)\right];\\
\widetilde{Q}_z &= Q_z+iEhc\left[\varkappa_z+\left(\sin\theta\varkappa_T+\frac{\cos\theta}{R_1}\frac{\partial \varepsilon_T}{\partial \theta}\right)\right]=\\
&= Q_z^*+iEhc\left[\widetilde{\varkappa}_z+\left(\sin\theta\varkappa_T+\frac{\cos\theta}{R_1}\frac{\partial \varepsilon_T}{\partial \theta}\right)\right];\\
\widetilde{Q}_{12} &= Q_{12}+iEhc\varkappa_{21}=Q_{12}^*+iEhc\widetilde{\varkappa}_{21};\\
\widetilde{M}_1 &= M_1+iEhc\,(\varepsilon_2-\varepsilon_T)=M_1^*+iEhc\left(\widetilde{\varepsilon}_2-\varepsilon_T\right).
\end{aligned}
\tag{3.2}
$$

By expanding the components of the thermal deformation in Fourier series

$$\varkappa_T=\sum_{k=0}^{\infty}\varkappa_{Tk}\begin{matrix}\cos k\varphi\\ \sin k\varphi,\end{matrix}\qquad \varepsilon_T=\sum_{k=0}^{\infty}\varepsilon_{Tk}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}$$

and using the method, which we have just considered, to modify the relationships of section 3, Chapter V, we obtain the necessary qualities for the k-th harmonics of the unknown and assigned values. We shall limit ourselves to the consideration of the first symmetric case, which is of most interest in practice.

For this case (section 6, Chapter V) the complex equation takes the following form, if we take into account the remarks made above /290

$$\frac{d^2\widetilde{T}}{d\theta^2}+\left[\left(2\frac{R_1}{R_2}-1\right)\operatorname{ctg}\theta-\frac{1}{R_1}\frac{dR_1}{d\theta}\right]\frac{d\widetilde{T}}{d\theta}+i\frac{R_1^2}{cR_2}\widetilde{T}=i\frac{R_1^2}{cR_2}\widetilde{F}(\theta) \tag{3.3}$$

In this case

$$\widetilde{F}(\theta)=R_2\widetilde{q}_n-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\widetilde{U}}{\sin^2\theta}; \tag{3.4}$$

$$\widetilde{U}=\widetilde{c}_1+\int_0^\theta\left(\widetilde{q}_n\cos\theta-\widetilde{q}_1\sin\theta\right)R_1R_2\sin\theta d\theta. \tag{3.5}$$

By using the values $\widetilde{U}$ and $\widetilde{T}$ obtained from the above, we can express the complex forces by means of the following relationships

$$\begin{aligned}\widetilde{T}&=\frac{\widetilde{U}}{R_2\sin^2\theta}+ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}}{d\theta};\\ \widetilde{T}_2&=-\frac{\widetilde{U}}{R_2\sin^2\theta}+\widetilde{T}-ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{T}}{d\theta}.\end{aligned} \tag{3.6}$$

The complex boundary conditions take on the form

$$\begin{aligned}\widetilde{Q}_\rho&=Q_\rho-iEhc\left\{\frac{\theta}{R_2\sin\theta}-\left(\cos\theta\varkappa_T-\frac{\sin\theta}{R_1}\frac{d\varepsilon_T}{d\theta}\right)\right\}=\\&=Q_\rho^*-iEhc\left\{\frac{\theta}{R_2\sin\theta}-\left(\cos\theta\varkappa_T-\frac{\sin\theta}{R_1}\frac{d\varepsilon_T}{d\theta}\right)\right\};\\ \widetilde{Q}_z&=Q_z+iEhc\left\{\sin\theta\varkappa_T+\frac{\cos\theta}{R_1}\frac{d\varepsilon_T}{d\theta}\right\}=\\&=Q_z^*+iEhc\left\{\sin\theta\varkappa_T+\frac{\cos\theta}{R_1}\frac{d\varepsilon_T}{d\theta}\right\};\\ \widetilde{M}_1&=M_1+iEhc\left\{\frac{u_\rho}{R_2\sin\theta}-\varepsilon_T\right\}=M_1^*+iEhc\left\{\frac{\widetilde{u}_\rho}{R_2\sin\theta}-\varepsilon_T\right\}.\end{aligned} \tag{3.7}$$

For these the approximate expressions given in section 6 of Chapter V are valid

$$\begin{aligned}\widetilde{Q}_\rho&=\cos\theta\widetilde{T}_1+ic\frac{\sin\theta}{R_1}\frac{d\widetilde{T}}{d\theta};\\ \widetilde{Q}_z&=\sin\theta\widetilde{T}_1-ic\frac{\cos\theta}{R_1}\frac{d\widetilde{T}}{d\theta}=\frac{\widetilde{U}}{R_2\sin\theta};\\ \widetilde{M}_1&=ic\left\{\widetilde{T}_2-\nu\widetilde{T}_1\right\}.\end{aligned} \tag{3.8}$$

Furthermore, for the angle of rotation and for the displacements, equalities (Chapter V, 6.8 and 6.11) are replaced by

$$\begin{aligned}
\vartheta &= -\frac{1}{Eh}\frac{R_2}{R_1}\operatorname{Re}\frac{d\widetilde{T}}{d\theta} - \frac{R_2}{R_1}\frac{d\varepsilon_T}{d\theta};\\
u_t &= \frac{R_2\sin\theta}{Eh}\left\{\operatorname{Re}\widetilde{T} + (1+\nu)\frac{c}{R_1}\operatorname{ctg}\theta\operatorname{Im}\frac{d\widetilde{T}}{d\theta} - \frac{(1+\nu)}{R_2\sin^2\theta}\operatorname{Re}\widetilde{U}\right\} + \\
&\quad + R_2\sin\theta\varepsilon_T;\\
u_z &= a_1 + \int_{\theta_0}^{\theta}\left\{\frac{1}{Eh}\left[\operatorname{Re}\left(\frac{R_2\cos\theta}{R_1}\frac{d\widetilde{T}}{d\theta} + \nu\sin\theta\widetilde{T}\right) - \frac{1+\nu}{R_2\sin\theta}\operatorname{Re}\widetilde{U}\right] + \right.\\
&\quad \left. + \left[\frac{R_2\cos\theta}{R_1}\frac{d\varepsilon_T}{d\theta} - \sin\theta\varepsilon_T\right]\right\}R_1 d\theta.
\end{aligned} \tag{3.9}$$

The function $\widetilde{U}$ may be separated into two terms $\widetilde{U} = U + iU^T$ of which the first one has already been widely applied by us (Chapter V, 6.2"), while /291

$$U^T = C_1^T + \int_{\theta_0}^{\theta}\left(q_n^T\cos\theta - q_1^T\sin\theta\right)R_1R_2\sin\theta d\theta. \tag{3.10}$$

Replacing q_1^T and q_n^T in the above equation by their expressions (3.1), we obtain

$$U^T = C_1^T + EhcR_2\sin\theta\left(\sin\theta\varkappa_T + \frac{\cos\theta}{R_1}\frac{d\varepsilon_T}{d\theta}\right). \tag{3.11}$$

We shall show that the constant C_1^T should be set equal to zero. Indeed, it follows from relationships (3.2) and (3.8) that

$$\varkappa_z = \frac{\operatorname{Im}\widetilde{U}}{EhcR_2\sin\theta} - \left(\sin\theta\varkappa_T + \frac{\cos\theta}{R_1}\frac{d\varepsilon_T}{d\theta}\right). \tag{3.12}$$

Substituting expression (3.11) we find

$$\varkappa_z = \frac{1}{EhcR_2\sin\theta}C_1^T.$$

At the beginning of section 6, Chapter V it was shown that in the first symmetric case the equality $\varkappa_z = 0$ is the condition for the periodicity of the displacements with respect to φ. It follows from this that if we limit ourselves

to the consideration of periodic displacements, we must assume that $C_1^T = 0$ in the expression for U^T.

Now the conditions for complex junction which generalize equalities (Chapter V, 6.18 and 6.17) are written in the form (3.7) and (3.8)

$$\left\{\cos\theta\widetilde{T}_1 + ic\sin\theta\frac{d\widetilde{T}}{ds_1} - iEhc\left(\cos\theta\varkappa_T - \sin\theta\frac{d\varepsilon_T}{ds_1}\right)\right\}^{\prime} = \{\cdots\}^{\prime\prime};$$
$$\left\{\widetilde{T} - (1+\nu)\widetilde{T}_1 + iEhc\varepsilon_T\right\}^{\prime} = \{\cdots\}^{\prime\prime}; \tag{3.13}$$
$$\left\{\frac{ic}{\sin\theta}\frac{d\widetilde{T}}{ds_1} + \frac{\operatorname{ctg}\theta}{r}U + iEhc\frac{1}{\sin\theta}\frac{d\varepsilon_T}{ds_1}\right\}^{\prime} = \{\cdots\}^{\prime\prime};$$

$$\left\{\widetilde{T} - ic(1+\nu)\operatorname{ctg}\theta\frac{d\widetilde{T}}{ds_1} - (1+\nu)\frac{U}{r\sin\theta} + iEhc\varepsilon_T\right\}^{\prime} = \{\cdots\}^{\prime\prime} \tag{3.14}$$
$$(r = R_2\sin\theta, \quad ds_1 = R_1 d\theta).$$

Let us return to the basic equation (3.3). Let us assume that a condition is satisfied which makes it possible for us to use the membrane solution as the particular solution, and that we have a uniform temperature distribution. Then /292

$$\widetilde{T}^{\mathrm{p}} = \widetilde{F}(\theta) = R_2\widetilde{q}_n - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\widetilde{U}}{\sin^2\theta};$$
$$\widetilde{T}_1^{\mathrm{p}} = \frac{\widetilde{U}}{R_2\sin^2\theta} + \underline{ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{F}(\theta)}{d\theta}} \approx \frac{\widetilde{U}}{R_2\sin^2\theta}; \tag{3.15}$$
$$\widetilde{T}_2^{\mathrm{p}} = -\frac{\widetilde{U}}{R_2\sin^2\theta} + \widetilde{F}(\theta) - \underline{ic\frac{\operatorname{ctg}\theta}{R_1}\frac{d\widetilde{F}(\theta)}{d\theta}} \approx -\frac{\widetilde{U}}{R_2\sin^2\theta} + \widetilde{F}(\theta).$$

p in this case = particular

Furthermore, by utilizing equality (3.11) and introducing T_1^*, T_2^*, we obtain the following expressions for the complex forces which correspond to the particular solution

$$\widetilde{T}_1^{\mathrm{p}} = T_1^* + iEhc\varkappa_2^T, \quad \widetilde{T}_2^{\mathrm{p}} = T_2^* + iEhc\varkappa_1^T. \tag{3.16}$$

Here we have utilized the following designations

$$\varkappa_1^T = \varkappa_T + \frac{1}{R_1}\frac{d}{d\theta}\frac{R_2\varepsilon_T}{R_1 d\theta}, \quad \varkappa_2^T = \varkappa_T + \frac{\operatorname{ctg}\theta}{R_1}\frac{d\varepsilon_T}{d\theta}. \tag{3.17}$$

Separating the real and imaginary parts in (3.16), we obtain

$$T_1^p = T_1^*,\ T_2^p = T_2^*,\ M_1^p = -M_1^T,\ M_2^p = -M_2^T, \tag{3.18}$$

where

$$M_1^T = D(\varkappa_1^T + \nu\varkappa_2^T),\ M_2^T = D(\varkappa_2^T + \nu\varkappa_1^T)$$
$$\left(D = Ehc^2 = \frac{Eh^3}{12(1-\nu^2)}\right). \tag{3.19}$$

Furthermore, relationships (3.9) give us

$$\vartheta^p = \vartheta^* + \vartheta^T;\ u_\rho^p = u_\rho^* + u_\rho^T;\ u_z^p = u_z^* + u_z^T. \tag{3.20}$$

Here

$$\vartheta^* = -\frac{1}{Eh}\frac{R_2}{R_1}\frac{dT^*}{d\theta};\ \vartheta^T = -\frac{R_2}{R_1}\frac{d\varepsilon_T}{d\theta};$$
$$u_\rho^* = R_2\sin\theta\frac{T_2^* - \nu T_1^*}{Eh};\ u_\rho^T = R_2\sin\theta\varepsilon_T;$$
$$u_z^* = \frac{1}{Eh}\int_{\theta_0}^{\theta}\left\{\frac{R_2\cos\theta}{R_1}\frac{dT^*}{d\theta} - (T_1^* - \nu T_2^*)\sin\theta\right\}R_1 d\theta + a_1; \tag{3.21}$$
$$u_z^T = \int_{\theta_0}^{\theta}\left(\frac{R_2\cos\theta}{R_1}\frac{d\varepsilon_T}{d\theta} - \sin\theta\varepsilon_T\right)R_1 d\theta.$$

In the case when the established partial solution does not satisfy the boundary conditions of the problem, it is necessary to supplement it by the general solution of the corresponding homogeneous problem. It is obvious /293 that the latter will not differ in any way from the one considered in sections 7 and 8, Chapter V.

In particular, by adopting the simplifications presented in section 8, Chapter V it is only necessary to change slightly the relationships obtained there. Thus, instead of (Chapter V, 8.15) we obtain the following relationships, which associate the boundary values of the bending moments of the horizontal forces, of the horizontal displacements and the angle of rotation

$$u_0 = u_0^* + u_0^T + a_{11}(Q_0 - Q_0^*) + a_{12}(M_0 + M_0^T);$$
$$\vartheta_0 = \vartheta_0^* + \vartheta_0^T + a_{12}(Q_0 - Q_0^*) + a_{22}(M_0 + M_0^T), \tag{3.22}$$

where

$$u_0=(u_\rho)_{\theta_0};\ u_0^*=(u_\rho^*)_{\theta_0};\ \vartheta_0=(\vartheta)_{\theta_0};\ \vartheta_0^*=(\vartheta^*)_{\theta_0};$$
$$u_0^T=(u_\rho^T)_{\theta_0};\ \vartheta_0^T=(\vartheta^T)_{\theta_0};\ M_0^T=(M_1^T)_{\theta_0}; \qquad (3.23)$$
$$a_{11}=\mu 2\sqrt[4]{3(1-\nu^2)}\cdot\left(\frac{r_0}{h}\right)^{\frac{3}{2}}\sqrt{\sin\theta_0}\,\frac{1}{E};$$
$$a_{12}=a_{21}=-2\sqrt{3(1-\nu^2)}\left(\frac{r_0}{h}\right)\frac{1}{Eh}; \qquad \text{(Chapter V, 8.16)}$$
$$a_{22}=\mu 4\,[3(1-\nu^2)]^{\frac{3}{4}}\sqrt{\frac{r_0}{h}}\,\frac{1}{\sqrt{\sin\theta_0}}\,\frac{1}{Eh^2};$$

while μ = -1 at the edge $\theta_0 = \theta_1$ and μ = +1 at the edge $\theta_0 = \theta_2$ ($\theta_1 \le \theta \le \theta_2$).

The expressions for the stresses are also somewhat different[1]

$$\sigma_{M_1}=-\sigma_{M_1}^T+\sigma_{M_0}(\cos\beta+\sin\beta)\,e^{-\beta}-\mu\frac{3}{\sqrt{3(1-\nu^2)}}\sigma_k\sin\beta e^{-\beta};$$
$$\sigma_{M_2}=-\sigma_{M_2}^T+\nu\left[\sigma_{M_0}(\cos\beta+\sin\beta)\,e^{-\beta}-\mu\frac{3}{\sqrt{3(1-\nu^2)}}\sigma_k\sin\beta e^{-\beta}\right]; \qquad (3.24)$$
$$\sigma_{T_2}=\sigma_{T_2}^*+\mu\sigma_k\cos\beta e^{-\beta}-\frac{\sqrt{3(1-\nu^2)}}{3}\sigma_{M_0}(\cos\beta-\sin\beta)\,e^{-\beta};$$
$$\sigma_{T_1}=\sigma_{T_1}^*,$$

where

$$\sigma_{M_0}=\frac{6(M_0+M_0^T)}{h^2};\quad \sigma_k=2\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{r_0}{h}}\sqrt{\sin\theta_0}\,\frac{Q_0-Q_0^*}{h};$$
$$\sigma_{M_1}^T=\frac{6M_1^T}{h^2};\quad \sigma_{M_2}=\frac{6M_2^T}{h^2};\quad \sigma_{T_1}^*=\frac{T_1^*}{h};\quad \sigma_{T_2}^*=\frac{T_2^*}{h}; \qquad (3.25)$$
$$\beta=\left|\int_{\theta_0}^{\theta}\sqrt{\frac{R_1^2}{2c|R_2|}}\,d\theta\right|.$$

Finally, equality (Chapter V, 8.24) assumes the form

[1]The quantity β should not be confused with the coefficient of linear expansion.

$$(u_z)_0=(u_z^*)_0+(u_z^T)_0+\gamma_1(Q_0-Q_0^*)+\gamma_2(M_0+M_0^T);$$
$$\gamma_1=\mu 2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)^{\frac{3}{2}}\frac{\cos\theta_0}{\sqrt{\sin\theta_0}}\frac{1}{E}; \qquad (3.26)$$
$$\gamma_2=-2\sqrt[4]{3(1-\nu^2)}\left(\frac{r_0}{h}\right)\operatorname{ctg}\theta_0\frac{1}{Eh}.$$

There is no special need to write out the relationships for the circular cylindrical cell. The temperature corrections for equalities of section 11, Chapter V may be obtained by assuming that in the relationships of the present section /294

$$\sin\theta=1;\ \cos\theta=0;\ R_2=r_0=\text{const};$$
$$\frac{1}{R_1}=0;\ \frac{1}{R_1 d\theta}=\frac{1}{r_0}\frac{d}{d\xi}. \qquad (3.27)$$

4. Examples

1. As the first example we consider a closed spherical shell. As we know, for closed shells the boundary conditions are replaced by periodicity conditions and conditions of finite value. Since the particular solutions established below satisfy the last requirements, they should be considered as complete solutions.

(a) Let us assume that the sphere is uniformly heated, i.e.,

$$t_0 = t_m = \text{const}, \quad t_1 = 0. \qquad (4.1)$$

Since, in this case according to equalities (3.17)-(3.19) $M_1^T = M_2^T = 0$ and since the external load is absent $T_1^* = T_2^* = 0$, the stresses in the shell are equal to zero. This could have been determined immediately, if we had noted that the distribution of temperature (4.1) satisfies the condition stipulating the absence of stresses in the shell (2.4).

Furthermore, from equality (3.21) we find (by dropping a_1)

$$\vartheta=\vartheta^T=0;\ u_\rho=u_\rho^T=\beta t_m R_0\sin\theta;\ u_z=\beta t_m R_0\cos\theta.$$

These displacements correspond to

$$w=u_\rho\sin\theta+u_z\cos\theta=\beta t_m R_0;\quad u=u_\rho\cos\theta-u_z\sin\theta=0.$$

Thus, when a spherical shell is heated uniformly, it expands uniformly. In this case the relative change in its diameter is equal to

$$\frac{w}{R_0} = \beta t_m. \tag{4.2}$$

(b) Let us now assume that the temperature varies in a linear fashion along the vertical axis of the shell, i.e.,

$$t_0 = t_m \cos\theta, \quad t_1 = \frac{t_m}{R_0}\cos\theta. \tag{4.3}$$

In this case (as established in section 2), the stresses in the shell are absent. On the other hand, we find from relationships (3.21), by dropping the constant a_1 which characterizes the rigid axial displacement, that /295

$$\vartheta = \vartheta^T = \beta t_m \sin\theta; \tag{4.4}$$

$$u_\rho = u_\rho^T = \beta\frac{t_m R_0}{2}\sin 2\theta; \quad u_z = \beta\frac{t_m R_0}{2}\cos 2\theta. \tag{4.5}$$

A rather unique form of the deformed meridian which corresponds to displacements (4.5) is shown in figure 31 (not to scale).

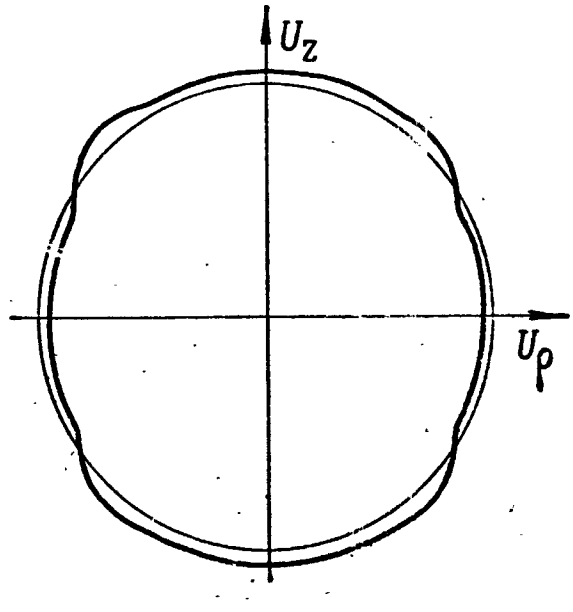

Figure 31

(c) Finally, let us consider the case when a constant temperature is maintained on the external surface and on the internal surface so that $t^+ = -t^-$. Then

$$t_0 = 0, \quad t_1 = \frac{\Delta t}{h} = \text{const}, \tag{4.6}$$

where $\Delta t = t^+ - t^-$ is the temperature drop across the thickness of the shell which is constant for the entire shell.

According to relationships (3.17)-(3.19)

$$T_1 = T_2 = 0; \quad M_1^T = M_2^T = \frac{Eh^2\beta\Delta t}{12(1-\nu)}; \qquad \sigma_{M_1} = \sigma_{M_2} = -\frac{E\beta\Delta t}{12(1-\nu)}. \tag{4.7}$$

In regard to displacements, we can utilize (3.20)-(3.21) to establish that /296

$$\vartheta = 0; \quad u_\rho = u_z = 0. \tag{4.8}$$

The result obtained is of a general nature, because, as we can see from (3.21), the middle surface does not deform for any symmetrical shell of revolution when $t_0 = 0$ ($\epsilon_T = 0$).

In order to illustrate the value of the thermal stresses and deformations we shall assume that the sphere is heated to a temperature $T = 100^\circ C$ and that it is made from conventional steel. Then

$$\beta \approx 12 \cdot 10^{-6} 1/^\circ C, \; E \approx 2.1 \cdot 10^6 \text{ kg/cm}^2, \; T = 100^\circ C,$$

from which it follows that

$$E\beta T = 2.1 \cdot 10^6 \cdot 12 \cdot 10^{-6} \cdot 100 = 2500 \text{ kg/cm}^2.$$

This simplified calculation shows that under relatively moderate temperatures high thermal stresses may occur in the shell. When these stresses are superposed on stresses due to external loading, they may have a substantial effect on the strength of the thin-walled structure.

2. Let us consider a circular cylindrical shell (tube) with free edges, such that there is a constant temperature drop along the axis

$$t_1=\frac{\Delta t}{h};\ t_0=0. \tag{4.9}$$

By applying equalities, (3.17) relationships (3.18)-(3.19) give us

$$\begin{gathered} T_1^*=T_2^*=0;\quad M_1^T=M_2^T=\frac{Eh^2\beta\Delta t}{12(1-\nu)};\\ \varkappa_1^T=\varkappa_2^T=\beta\frac{\Delta t}{h},\quad \sigma_{M_1^T}=\sigma_{M_2^T}=\frac{E\beta\Delta t}{2(1-\nu)}. \end{gathered} \tag{4.10}$$

As in the preceding case,

$$\vartheta^T=\vartheta^*=0,\quad w^T=w^*=u^T=u^*=0. \tag{4.11}$$

Thus the established particular solution leads to a constant bending moment along the axis. Since it is specified that the ends of the tube are free

$$M_0=0,\quad \sigma_{M_0}=\frac{E\beta\Delta t}{2(1-\nu)}. \tag{4.12}$$

Taking this into consideration and also considering the problem $Q_0 = Q_0^* = 0$, we obtain the following solution near the edge $\xi = 0$ from (3.24) and (3.25)

$$\begin{gathered} \sigma_{M_1}=-\frac{E\beta\Delta t}{2(1-\nu)}\{1-(\cos\beta+\sin\beta)e^{-\beta}\},\\ \sigma_{M_2}=-\frac{E\beta\Delta t}{2(1-\nu)}\{1-\nu(\cos\beta+\sin\beta)e^{-\beta}\},\\ \sigma_{T_2}=-\frac{E\beta\Delta t}{2(1-\nu)}\left\{\frac{\sqrt{3(1-\nu^2)}}{3}(\cos\beta-\sin\beta)e^{-\beta}\right\},\\ \sigma_{T_1}=0\\ \left(\beta=\sqrt{\frac{r_0}{2c}}\,\xi\right). \end{gathered} \tag{4.13}$$

It is easy to see that in this case the following stress will have a maximum value /297

$$(\sigma_2)_{\max}=(\sigma_{T_2}+\sigma_{M_2})|_{\xi=0}=-\frac{E\beta\Delta t}{2(1-\nu)}\left\{1-\nu+\frac{\sqrt{3(1-\nu^2)}}{3}\right\},$$

which occurs in the external transverse fibers of the tube along its edges. When $\nu = 0.3$

$$1-\nu+\frac{\sqrt{3(1-\nu^2)}}{3}\approx 1.25.$$

Thus the stresses at the tube's edge exceed by 25 percent their values $-\sigma^T_{M_1} = -\sigma^T_{M_2}$ away from the edge.

3. Let us assume that the same tube is subjected to nonuniform heating along the axis (fig. 32)

along the first region ($\xi < \xi_0$)

$$t_1 = 0; \quad t_0 = t^I_m; \tag{4.14}$$

along the second region ($\xi_0 < \xi < \xi_1$)

$$t_1 = 0; \quad t_0 = t^I_m + (t^{II}_m - t^I_m)\frac{\xi-\xi_0}{\xi_1-\xi_0}. \tag{4.15}$$

Let us break the tube up into two parts (as normally done in the case of discontinuous loads). For the first part ($\xi < \xi_0$)

$$\begin{aligned} &T^*_1 = T^*_2 = 0; \quad M^T_1 = M^T_2 = 0; \\ &\vartheta^{TI} = 0; \quad u^{TI}_\rho = w^{TI} = r_0\beta t^I_m. \end{aligned} \tag{4.16}$$

For the second part ($\xi > \xi_0$)

$$\begin{aligned} &T^*_1 = T^*_2 = 0; \quad M^T_1 = M^T_2 = 0; \\ &\vartheta^{TII} = -\beta\frac{dt_0}{d\xi} = \beta\frac{t^I_m - t^{II}_m}{\xi_1-\xi_0}; \\ &u^{TII}_\rho = w^{TII} = r_0\beta t_0 = \beta r_0\left\{t^I_m + (t^{II}_m - t^I_m)\frac{\xi-\xi_0}{\xi_1-\xi_0}\right\}. \end{aligned} \tag{4.17}$$

In view of the expressions obtained above, equalities (3.22) are written in the following manner for both parts of the shell

$$u_0^{\mathrm{I}} = u_0^{T\mathrm{I}} + \alpha_{11}^{\mathrm{I}} Q_0^{\mathrm{I}} + \alpha_{12}^{\mathrm{I}} M_0^{\mathrm{I}};$$
$$\vartheta_0^{\mathrm{I}} = \alpha_{12}^{\mathrm{I}} Q_0^{\mathrm{I}} + \alpha_{22}^{\mathrm{I}} M_0^{\mathrm{I}}; \tag{4.18}$$

$$u_0^{\mathrm{II}} = u_0^{T\mathrm{II}} + \alpha_{11}^{\mathrm{II}} Q_0^{\mathrm{II}} + \alpha_{12}^{\mathrm{II}} M_0^{\mathrm{II}};$$
$$\vartheta_0^{\mathrm{II}} = \vartheta_0^{T\mathrm{II}} + \alpha_{12}^{\mathrm{II}} Q_0^{\mathrm{II}} + \alpha_{22} M_0^{\mathrm{II}}. \tag{4.19}$$

Here

$$\vartheta_0^{T\mathrm{II}} = (\vartheta^{T\mathrm{II}})_{\xi=\xi_0} = \beta \frac{(t_m^{\mathrm{I}} - t_m^{\mathrm{II}})}{\xi_1 - \xi_0};$$
$$u_0^{T\mathrm{I}} = (u_\rho^{T\mathrm{I}})_{\xi=\xi_0} = u_0^{T\mathrm{II}} = \beta r_0 t_m^{\mathrm{I}}. \tag{4.20}$$

Since for the first part the normal to the boundary element (the section m-m) coincides with the direction of tne coordinate ξ, we should let $\mu = +1$ in relationships (Chapter V, 8.16) ($\sin\theta = 1$) when we consider it. Similarly for the second part we should assume $\mu = -1$. Thus /298

$$\alpha_{11}^{\mathrm{I}} = -\alpha_{11}^{\mathrm{II}} = \alpha_{11} = 2\sqrt[4]{3(1-\nu^2)}\frac{r_0^2}{Eh^2}\sqrt{\frac{h}{r_0}};$$
$$\alpha_{12}^{\mathrm{I}} = \alpha_{12}^{\mathrm{II}} = \alpha_{12} = -2\sqrt{3(1-\nu^2)}\frac{r_0}{Eh^2}; \tag{4.21}$$
$$\alpha_{22}^{\mathrm{I}} = -\alpha_{22}^{\mathrm{II}} = \alpha_{22} = 4\left[3(1-\nu^2)\right]^{\frac{3}{4}}\frac{1}{Eh^2}\sqrt{\frac{r_0}{h}}.$$

For the elastic joining of both parts we must require that

$$u_0^{\mathrm{I}} = u_0^{\mathrm{II}}, \quad \vartheta_0^{\mathrm{I}} = \vartheta_0^{\mathrm{II}}, \quad M_0^{\mathrm{I}} = M_0^{\mathrm{II}} = M_0, \quad Q_0^{\mathrm{I}} = Q_0^{\mathrm{II}} = Q_0. \tag{4.22}$$

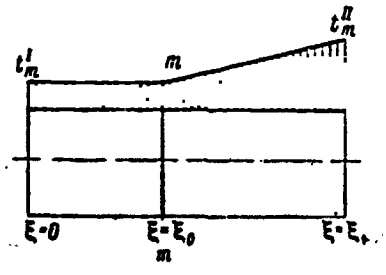

Figure 32

Equating expressions (4.18) and (4.19) and utilizing equalities (4.20)-(4.22), we obtain

$$2a_{11}Q_0=0;$$
$$2a_{22}M_0=\beta\frac{(t_m^{\mathrm{I}}-t_m^{\mathrm{II}})}{\xi_1-\xi_0}.$$

From this it follows that

$$Q_0=0;\quad M_0=\frac{Eh^2}{8\,[3(1-\nu^2)]^{\frac{3}{4}}}\sqrt{\frac{h}{r_0}}\frac{\beta(t_m^{\mathrm{I}}-t_m^{\mathrm{II}})}{\xi_1-\xi_0}.$$

It is easy to establish from equalities (3.24) that

$$\sigma_{\max}=|\sigma_{M_1}|_{\xi=\xi_0}=\frac{6}{8\,[3(1-\nu^2)]^{\frac{3}{4}}}E\sqrt{\frac{h}{r_0}}\frac{\beta|t_m^{\mathrm{I}}-t_m^{\mathrm{II}}|}{\xi_1-\xi_0}, \tag{4.23}$$

when $\nu = 0.3$ ($l = r_0(\xi_1 - \xi_0)$)

$$\sigma_{\max}=0.353\quad E\beta|t_m^{\mathrm{I}}-t_m^{\mathrm{II}}|\frac{\sqrt{hr_0}}{l}.$$

The last examples are considered in more detail in reference 41, p. 398.

CHAPTER XV. CIRCULAR TOROIDAL SHELLS

In the present chapter and in the next two chapters we consider various problems associated with the calculation of toroidal shells. When various equipments and structures are designed, we encounter the necessity of applying toroidal shells (thermal expansion tanks, curved tubes, transient regions, etc.). The design of toroidal sections involves large mathematical difficulties. This is associated with the fact that at the apexes of the meridian (fig. 33), which correspond to the values $\theta = 0$; $\theta = \pi$, the term in the basic complex equations, which has a large factor, becomes equal to zero and the elementary approach described in Chapter XIII for finding the solution becomes inapplicable. Therefore, in order to design toroidal shells which contain the above singular points of the meridian, we must use more complicated and awkward methods. In this case a complex mixed state of stress occurs in the shell which cannot be broken down into a membrane stress and into the edge effect. /299

Below we shall limit ourselves to the consideration of the more widely used circular toroidal shells. The consideration of the toroidal shells having a meridian of the general form leads to more cumbersome computations, although it does not introduce any fundamental difficulties (refs. 333, 335 and 344). /300

For a circular toroidal shell (fig. 33)

$$R_1 = b = \alpha a; \quad R_2 = \frac{a + b\sin\theta}{\sin\theta} = a\frac{1 + \alpha\sin\theta}{\sin\theta}, \tag{0.1}$$

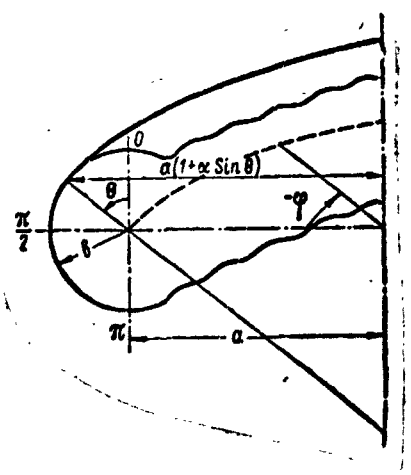

Figure 33

where

$$\alpha = \frac{b}{a}. \tag{0.2}$$

It follows from this that

$$\begin{gathered} r = R_2 \sin\theta = a(1 + \alpha \sin\theta); \\ z = -\int_{-\pi/2}^{\theta} R_1 \sin\theta d\theta = b\cos\theta = a\alpha\cos\theta. \end{gathered} \tag{0.3}$$

1. Construction of the Components of Dislocation Deformation

Let us consider part of a circular toroidal shell bounded by two meridian planes $\varphi = -\varphi_0$ and $\varphi = 0$ (fig. 34). In the case of a shell closed with respect to φ we must assume that $\varphi_0 = 2\pi$. For the dislocation function we adopt the expression

$$\Phi^1 = \frac{\varphi + \varphi_0}{\varphi_0}. \tag{1.1}$$

By replacing the dislocation function in relationships (Chapter X, 2.7)-(Chapter X, 2.10) with the one just adopted and replacing the radii of curvature with expressions (0.1), we obtain

$$\begin{gathered} v^{\text{д}} = \{-\sin\varphi U_x^1 + \cos\varphi U_y^1 - \alpha\cos\theta(\cos\varphi a\Omega_x^1 + \sin\varphi a\Omega_y^1) + \\ + (1 + \alpha\sin\theta) a\Omega_z^1\} \frac{\varphi + \varphi_0}{\varphi_0}; \\ u_\rho^{\text{д}} = \{\cos\varphi U_x^1 + \sin\varphi U_y^1 - \alpha\cos\theta(\sin\varphi a\Omega_x^1 - \cos\varphi a\Omega_y^1)\} \frac{\varphi + \varphi_0}{\varphi_0}; \\ u_z^{\text{д}} = \{U_z^1 + (1 + \alpha\sin\theta)(\sin\varphi a\Omega_x^1 - \cos\varphi a\Omega_y^1)\} \frac{\varphi + \varphi_0}{\varphi_0}; \\ \vartheta^{\text{д}} = \{-\sin\varphi\Omega_x^1 + \cos\varphi\Omega_y^1\} \frac{\varphi + \varphi_0}{\varphi_0}; \\ \psi^{\text{д}} = \{-\cos\theta(\cos\varphi\Omega_x^1 + \sin\varphi\Omega_y^1) + \sin\theta\Omega_z^1\} \frac{\varphi + \varphi_0}{\varphi_0} - \\ - \frac{1}{a\varphi_0(1 + \alpha\sin\theta)} \{\sin\theta(\cos\varphi U_x^1 + \sin\varphi U_y^1) + \cos\theta U_z^1 + \\ + \cos\theta(\sin\varphi a\Omega_x^1 - \cos\varphi a\Omega_y^1)\}; \end{gathered} \tag{1.2}$$

$$\varepsilon_1^{\text{д}} = \varkappa_1^{\text{д}} = 0; \tag{1.3}$$

$$\begin{gathered} \varepsilon_2^{\text{д}} = \frac{1}{a\varphi_0(1 + \alpha\sin\theta)} \{-\sin\varphi U_x^1 + \cos\varphi U_y^1 - \alpha\cos\theta(\cos\varphi a\Omega_x^1 + \\ + \sin\varphi a\Omega_y^1) + (1 + \alpha\sin\theta) a\Omega_z^1\}; \\ \omega^{\text{д}} = \frac{1}{a\varphi_0(1 + \alpha\sin\theta)} \{\cos\theta(\cos\varphi U_x^1 + \sin\varphi U_y^1) - \sin\theta U_z^1 - \end{gathered} \tag{1.4}$$

Note: Д = d = dislocation

$$- (\alpha + \sin\theta)(\sin\varphi a\Omega_x^1 - \cos\varphi a\Omega_y^1)];$$

$$\varkappa_2 = \frac{1}{a^2\varphi_0(1+\alpha\sin\theta)^2}[\sin\theta(\sin\varphi U_x^1 - \cos\varphi U_y^1) - (2+\alpha\sin\theta) \times$$

$$\times\cos\theta(\cos\varphi a\Omega_x^1 + \sin\varphi a\Omega_y^1) + (1+\alpha\sin\theta)\sin\theta a\Omega_z^1],$$

$$\tau^1 = \frac{1}{a^2\varphi_0(1+\alpha\sin\theta)^2}[\sin\theta\cos\theta(\cos\varphi U_x^1 + \sin\varphi U_y^1) +$$

$$+ \cos^2\theta U_z^1 - (\alpha+\sin\theta)\sin\theta(\sin\varphi a\Omega_x^1 - \cos\varphi a\Omega_y^1)].$$

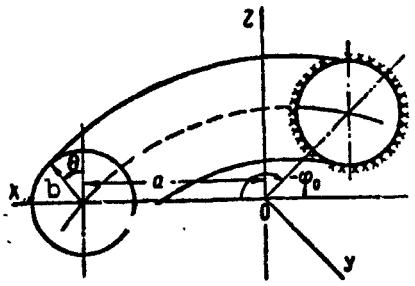

Figure 34

Assuming that $\varphi = 0$ and $\varphi = -\varphi_0$, for the dislocation displacements in the above expressions, we obtain /301

$$\begin{aligned} v^1(0) &= U_y^1 - a\cos\theta a\Omega_x^1 + (1+\alpha\sin\theta)a\Omega_z^1; \\ u_\varphi^1(0) &= U_x^1 + a\cos\theta a\Omega_y^1; \\ u_z^1(0) &= U_z^1 - (1+\alpha\sin\theta)a\Omega_y^1; \\ v^1(-\varphi_0) &= 0; \\ u_\varphi^1(-\varphi_0) &= 0; \\ u_z^1(-\varphi_0) &= 0. \end{aligned} \tag{1.5}$$

From the relationships which have been established we can see that the introduced dislocation displacements characterize the displacement as a solid section $\varphi = 0$, relative to the fixed section $\varphi = -\varphi_0$. As was to be expected, this displacement is described by six parameters

$$U_x^1,\ U_y^1,\ U_z^1,\ \Omega_x^1,\ \Omega_y^1,\ \Omega_z^1. \tag{1.6}$$

2. Construction of the Systems of Static Functions

By utilizing the static-geometric analogies it is easy to write down the static system of functions which satisfies the homogeneous equilibrium equations

$$\begin{aligned} T_2^* &= 0; \quad M_2^* = 0; \\ T_1^* &= -\frac{Ehc}{a^2\varphi_0(1+\alpha\sin\theta)^2}[\sin\theta(\sin\varphi\bar{U}_x^1 - \cos\varphi\bar{U}_y^1) - \\ &\quad - (2+\alpha\sin\theta)\cos\theta(\cos\varphi a\bar{\Omega}_x^1 + \sin\varphi a\bar{\Omega}_y^1) + \end{aligned} \tag{2.1}$$

$$+(1+\alpha\sin\theta)\sin\theta a\bar{\Omega}_z^1\Big\};$$

$$S^*=\frac{Ehc}{a^2\varphi_0(1+\alpha\sin\theta)^2}\Big\{-\sin\theta\cos\theta\left(\cos\varphi\bar{U}_x^1+\sin\varphi\bar{U}_y^1\right)-$$

$$-\cos^2\theta\bar{U}_z^1+(\alpha+\sin\theta)\sin\theta\left(\sin\varphi a\bar{\Omega}_x^1-\cos\varphi a\bar{\Omega}_y^1\right)\Big\},$$

$$M_1^*=\frac{Ehc}{a\varphi_0(1+\alpha\sin\theta)}\Big\{\sin\varphi\bar{U}_x^1-\cos\varphi\bar{U}_y^1+\alpha\cos\theta\left(\cos\varphi a\bar{\Omega}_x^1+\right.$$

$$\left.+\sin\varphi a\bar{\Omega}_y^1\right)-(1+\alpha\sin\theta)a\bar{\Omega}_z^1\Big\},$$

$$2H^*=\frac{Ehc}{a\varphi_0(1+\alpha\sin\theta)}\Big\{\cos\theta\left(\cos\varphi\bar{U}_x^1+\sin\varphi\bar{U}_y^1\right)-$$

$$-\sin\theta\bar{U}_z^1-(\alpha+\sin\theta)\left(\sin\varphi a\bar{\Omega}_x^1-\cos\varphi a\bar{\Omega}_y^1\right)\Big\}.$$

As we shall show in the next chapter, the six parameters $\bar{U}_x^1$, $\bar{U}_y^1$, $\bar{U}_z^1$, $\bar{\Omega}_x^1$, $\bar{\Omega}_y^1$, $\bar{\Omega}_z^1$ for the toroidal shell, closed with respect to φ, are expressed directly in terms of the components of the principal vector and of the principal moment of forces and moments applied to the parallel circle $\theta = \theta_0$. For shells which are closed with respect to θ (tubes) a suitable selection of these parameters provides for periodic variations in the angles of rotation (with respect to θ). /302

When surface loading is present, this static system must be supplemented with a particular solution for the system of equilibrium equations. In the symmetric case the latter have the form

$$\frac{d(1+\alpha\sin\theta)T_1^*}{d\theta}-\alpha\cos\theta T_2^*+(1+\alpha\sin\theta)Q_{1n}^*=-b(1+\alpha\sin\theta)q_1;$$

$$T_1^*+\frac{\alpha\sin\theta}{1+\alpha\sin\theta}T_2^*-\frac{1}{1+\alpha\sin\theta}\frac{d(1+\alpha\sin\theta)Q_{1n}^*}{d\theta}=bq_n, \tag{2.2}$$

where

$$Q_{1n}^*=\frac{1}{b(1+\alpha\sin\theta)}\left\{\frac{d(1+\alpha\sin\theta)M_1^*}{d\theta}-\alpha\cos\theta M_2^*\right\}.$$

We shall limit ourselves to the two most interesting practical forms of loading.

(a) Uniform pressure

$$q_n = p = \text{const}; \quad q_1 = 0. \tag{2.3}$$

It is convenient to adopt the membrane solution in this case

$$T_1^*=\frac{pb}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}, \quad T_2^*=\frac{pb}{2}$$
$$(M_1^*=M_2^*=0; \quad Q_{1n}^*=0). \tag{2.4}$$

(b) Centrifugal force

$$\begin{matrix} q_\rho = Ba(1+\alpha\sin\theta) & \left(q_1 = Ba(1+\alpha\sin\theta)\cos\theta\right) \\ q_z = 0 & \left(q_n = Ba(1+\alpha\sin\theta)\sin\theta\right) \end{matrix}. \qquad (2.5)$$

It is easy to verify by direct substitution that the membrane solution which corresponds to this loading has the form

$$T_1^* = 0; \quad T_2^* = Ba^2(1+\alpha\sin\theta)^2 \qquad (2.6)$$
$$\left(M_1^* = M_2^* = 0; \quad Q_{1n}^* = 0\right).$$

In the above relationships $B = m\omega_0^2 h$; ω_0 is the angular rate of rotation of the shell around its axis; m is the density of the shell material. /303

If we introduce the horizontal and vertical forces Q_ρ^*, Q_z^*, then

$$T_1^* = \cos\theta Q_\rho^* + \sin\theta Q_z^*, \quad Q_{1n}^* = \sin\theta Q_\rho^* - \cos\theta Q_z^* \qquad \text{(Chapter V, 1.20)}$$

and the system of equilibrium equations (2.2) may be reduced to a form which is often more convenient

$$T_2^* = \frac{1}{a}\frac{d(1+\alpha\sin\theta)Q_\rho^*}{d\theta} + a(1+\alpha\sin\theta)q_\rho; \qquad (2.7)$$
$$\frac{d(1+\alpha\sin\theta)M_1^*}{d\theta} - a\cos\theta M_2^* =$$
$$= b(1+\alpha\sin\theta)\left(\sin\theta Q_\rho^* - \cos\theta Q_z^*\right),$$

where

$$q_\rho = q_n\sin\theta + q_1\cos\theta; \quad q_z = q_n\cos\theta - q_1\sin\theta; \qquad (2.8)$$
$$Q_z^* = \frac{b\left\{\int_{\theta_0}^{\theta} q_z(1+\alpha\sin\theta)\,d\theta + C_1\right\}}{1+\alpha\sin\theta}.$$

3. The Second Symmetric Case (Symmetric Torsion)

According to the relationships in sections 1 and 2, in the second symmetric case

$$u^1_{z,0}=\frac{\varphi+\varphi_0}{\varphi_0}U^1_z;\quad u^1_{\rho,0}=v^1_{,0}=0; \tag{3.1}$$

$$w^1_{,0}=-\frac{\sin\theta}{1+\alpha\sin\theta}\cdot\frac{U^1_z}{a\varphi_0};\quad \tau^1_{,0}-\frac{w^1_{,0}}{R_2}=\frac{1}{(1+\alpha\sin\theta)^2}\frac{U^1_z}{a^2\varphi_0}; \tag{3.2}$$

$$S^*_{,0}+\frac{2H^*_{,0}}{R_2}=-\frac{1}{(1+\alpha\sin\theta)^2}\frac{EhcU^1_z}{a^2\varphi_0}. \tag{3.3}$$

By utilizing these relationships, it follows from equalities of section 3, Chapter XI that

$$S_{,0}=-\frac{1}{(1+\alpha\sin\theta)^2}\frac{EhcU^1_z}{a^2\varphi_0};$$
$$H_{,0}=\frac{1-\nu}{(1+\alpha\sin\theta)^2}\frac{Ehc^2U^1_z}{a^2\varphi_0}; \tag{3.4}$$

$$v_{,0}=v^{\text{п}}_{,0}=\alpha(1+\alpha\sin\theta)\left\{\frac{U^1_z}{\varphi_0}\int_{\theta_0}^{\theta}\frac{\sin\theta d\theta}{(1+\alpha\sin\theta)^2}-\right. \tag{3.5}$$
$$\left.-\frac{2(1+\nu)c}{a}\frac{U^1_z}{\varphi_0}\int_{\theta_0}^{\theta}\frac{d\theta}{(1+\alpha\sin\theta)^3}\right\}+a(1+\alpha\sin\theta)\,a_2.$$

Note: п = c = correction

4. The First Symmetric Case

According to the relationships in sections 1 and 2 we have

$$v^1=a\Omega^1_z(1+\alpha\sin\theta)\frac{\varphi+\varphi_0}{\varphi_0};\quad u^1_\rho=u^1_z=0; \tag{4.1}$$

$$\varepsilon^1_2=\frac{\Omega^1_z}{\varphi_0};\quad \varkappa^1_2=\frac{\Omega^1_z}{b\varphi_0}\frac{\alpha\sin\theta}{1+\alpha\sin\theta}; \tag{4.2}$$
$$\varepsilon^1_1=w^1=0;\quad \varkappa^1_1=\tau^1=0;$$

$$T^*_1=\frac{pb}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}+\frac{EhcU^1_z}{b\varphi_0}\frac{\alpha\sin\theta}{1+\alpha\sin\theta};$$
$$T^*_2=\frac{pb}{2};\quad M^*_1=-\frac{EhcU^1_z}{\varphi_0};\quad M^*_2=H^*=0;\quad S^*=0; \tag{4.3}$$
$$Q^*_{1n}=-\frac{EhcU^1_z}{b\varphi_0}\frac{\alpha\cos\theta}{1+\alpha\sin\theta};\quad \varkappa^1_{2n}=-\frac{\Omega^1_z}{b\varphi_0}\frac{\alpha\cos\theta}{1+\alpha\sin\theta}$$

(to be specific, we have selected a uniform pressure as the surface load). /304

The basic equation (Chapter XI, 4.3) assumes the form

$$\frac{d^2\tilde{\vartheta}^n}{d\theta^2}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\tilde{\vartheta}^n}{d\theta}+\left[-\left(\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\right)^2+\right.$$

$$\left.+i2k^2\frac{\sin\theta}{1+\alpha\sin\theta}\right]\tilde{\vartheta}^n=$$

$$=i2k^2\left\{\frac{\Omega_z^1+i\bar{\Omega}_z^1}{\varphi_0}\frac{\cos\theta}{1+\alpha\sin\theta}\right\}$$

$$\left(2k^2=\frac{a^2a}{c}=\frac{ab}{c}\right).$$

Since

$$\frac{\left(\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\right)^2}{2k^2\frac{\sin\theta}{1+\alpha\sin\theta}}=\frac{c}{a}\frac{\cos^2\theta}{(1+\alpha\sin\theta)\sin\theta},$$

when we consider toroidal shells which do not contain apexes ($\theta = 0$, $\theta = \pi$), we can drop the term which is underlined in the left part. However, there is some doubt as to whether we can drop it, when we consider the neighborhood of the apex point. The analysis carried out in references 320 and 344 (see also section 8 of the present chapter) has shown that when the values of the parameter $2k^2$ are sufficiently large, this term may be dropped. Thus, by dropping the underlined term we obtain /305

$$\frac{d^2\tilde{\vartheta}^n}{d\theta^2}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\tilde{\vartheta}^n}{d\theta}+i2k^2\frac{\sin\theta}{1+\alpha\sin\theta}\tilde{\vartheta}^n=$$

$$=i2k^2\left\{\frac{\Omega_z^1+i\bar{\Omega}_z^1}{\varphi_0}\frac{\cos\theta}{1+\alpha\sin\theta}\right\}. \qquad (4.4)$$

Furthermore, according to (Chapter XI, 4.5) and (4.2), (4.3)

$$\tilde{T}_1=\frac{pb}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}-i\frac{Ehc}{b}\left\{\frac{\Omega_z^1+i\bar{\Omega}_z^1}{\varphi_0}\frac{\alpha\sin\theta}{1+\alpha\sin\theta}+\right.$$

$$\left.+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\vartheta}^n\right\}; \qquad (4.5)$$

$$\tilde{T}_2=\frac{pb}{2}-i\frac{Ehc}{b}\frac{d\tilde{\vartheta}^n}{d\theta}.$$

Dropping the small terms we find from relationships (Chapter XI, 4.9)

$$u_\rho^n = -\frac{a\Omega_z^1}{\varphi_0}(1+\alpha\sin\theta) + \frac{pba}{2Eh}(1+\alpha\sin\theta)\left(1-\nu\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}\right) + \\ + \frac{b}{2k^2}(1+\alpha\sin\theta)\,\mathrm{Im}\frac{d\tilde{\vartheta}^n}{d\theta}; \qquad (4.6)$$

$$u_z^n = a_1 - b\int_0^\theta\left\{\frac{pb}{2Eh}\left(\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}-\nu\right)\sin\theta + \cos\theta\,\mathrm{Re}\,\tilde{\vartheta}^n\right\}d\theta.$$

Finally, for the complex boundary quantities (Chapter XI, 4.14) we obtain the following simplified expression

$$\tilde{Q}_\rho = \frac{pb}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}\cos\theta - i\frac{Ehc}{a}\frac{\tilde{\vartheta}^n}{1+\alpha\sin\theta};$$
$$\tilde{Q}_z = \frac{pb}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}\sin\theta - i\frac{Ehc}{a\varphi_0}\frac{\Omega_z^1+i\bar{\Omega}_z^1}{1+\alpha\sin\theta}; \qquad (4.7)$$
$$\tilde{M}_1 = \frac{Ehc^2}{b}\left\{\frac{d\tilde{\vartheta}^n}{d\theta} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\vartheta}^n\right\}.$$

5. The Antisymmetric Case

It follows from the relationships in sections 1 and 2 that /306

$$u_\rho^a = \begin{Bmatrix} 1 & \sin\varphi U_y^1 & -\alpha\cos\theta & \sin\varphi a\Omega_x^1 \\ \cos\varphi U_x^1 & & & -\cos\varphi a\Omega_y^1 \end{Bmatrix}\frac{\varphi+\varphi_0}{\varphi_0};$$
$$u_z^a = \begin{Bmatrix} (1+\alpha\sin\theta) & \sin\varphi a\Omega_x^1 \\ & -\cos\varphi a\Omega_y^1 \end{Bmatrix}\frac{\varphi+\varphi_0}{\varphi_0}; \qquad (5.1)$$
$$v^a = \begin{Bmatrix} 1 & \cos\varphi U_y^1 & -\alpha\cos\theta & \cos\varphi a\Omega_x^1 \\ -\sin\varphi U_x^1 & & & \sin\varphi a\Omega_y^1 \end{Bmatrix}\frac{\varphi+\varphi_0}{\varphi_0};$$

$$\varepsilon_2^a = \frac{1}{a\varphi_0(1+\alpha\sin\theta)}\begin{Bmatrix} 1 & \cos\varphi U_y^1 & -\alpha\cos\theta & \cos\varphi a\Omega_x^1 \\ -\sin\varphi U_x^1 & & & \sin\varphi a\Omega_y^1 \end{Bmatrix},\ \varepsilon_1^a = 0,\ \varkappa_1^1 = 0;$$
$$\omega^a = \frac{1}{a\varphi_0(1+\alpha\sin\theta)}\begin{Bmatrix} \cos\theta & \sin\varphi U_y^1 & +(\alpha+\sin\theta) & -\sin\varphi a\Omega_x^1 \\ & \cos\varphi U_x^1 & & \cos\varphi a\Omega_y^1 \end{Bmatrix}; \qquad (5.2)$$
$$\varkappa_2^a = \frac{1}{a^2\varphi_0(1+\alpha\sin\theta)^2}\begin{Bmatrix} \sin\theta & -\cos\varphi U_y^1 & -(2+\alpha\sin\theta)\cos\theta & \cos\varphi a\Omega_x^1 \\ & \sin\varphi U_x^1 & & \sin\varphi a\Omega_y^1 \end{Bmatrix};$$
$$\tau^a = \frac{1}{a^2\varphi_0(1+\alpha\sin\theta)^2}\begin{Bmatrix} \sin\theta\cos\theta & \sin\varphi U_y^1 & +(\alpha+\sin\theta)\sin\theta & -\sin\varphi a\Omega_x^1 \\ & \cos\varphi U_x^1 & & \cos\varphi a\Omega_y^1 \end{Bmatrix};$$

$$T_1^* = \frac{Ehc}{a^2\varphi_0(1+\alpha\sin\theta)^2}\begin{Bmatrix} \sin\theta & -\cos\varphi\bar{U}_y^1 & - \\ & \sin\varphi\bar{U}_x^1 & \end{Bmatrix} \qquad (5.3)$$
$$-(2+\alpha\sin\theta)\cos\theta\begin{Bmatrix} \cos\varphi a\bar{\Omega}_x^1 \\ \sin\varphi a\bar{\Omega}_y^1 \end{Bmatrix};$$

$$S^* = \frac{Ehc}{a^2\varphi_0(1+\alpha\sin\theta)^2}\left\{-\sin\theta\cos\theta\begin{matrix}\sin\varphi\overline{U}_y^1\\ \cos\varphi\overline{U}_x^1\end{matrix}+\right.$$
$$\left.+(\alpha+\sin\theta)\sin\theta\begin{matrix}\sin\varphi a\overline{\Omega}_x^1\\ -\cos\varphi a\overline{\Omega}_y^1\end{matrix}\right\};$$
$$M_1^* = \frac{Ehc}{a\varphi_0(1+\alpha\sin\theta)}\left\{1\begin{matrix}-\cos\varphi\overline{U}_y^1+\\ \sin\varphi\overline{U}_x^1\end{matrix}\right.$$
$$\left.+\alpha\cos\theta\begin{matrix}\cos\varphi a\overline{\Omega}_x^1\\ \sin\varphi a\overline{\Omega}_y^1\end{matrix}\right\};$$
$$2H^* = \frac{Ehc}{a\varphi_0(1+\alpha\sin\theta)}\left\{\cos\theta\begin{matrix}\sin\varphi\overline{U}_y^1+\\ \cos\varphi\overline{U}_x^1\end{matrix}\right.$$
$$\left.+(\alpha+\sin\theta)\begin{matrix}-\sin\varphi a\overline{\Omega}_x^1\\ \cos\varphi a\overline{\Omega}_y^1\end{matrix}\right\};$$
$$T_2^* = 0;\quad M_2^* = 0$$

Carrying out simplifications analogous to those made in the preceding section and utilizing relationships (5.2)-(5.3), we reduce the characteristic equation (Chapter XI, 5.14 and 5.15) to the following form

$$\frac{d^2\tilde{\chi}}{d\theta^2}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\frac{d\tilde{\chi}}{d\theta}+i2k^2\frac{\sin\theta}{1+\alpha\sin\theta}\tilde{\chi}=$$
$$=i\frac{4k^2}{a\varphi_0}\left\{\frac{\cos\theta}{(1+\alpha\sin\theta)^2}\begin{matrix}-(U_y^1+i\overline{U}_y^1)\\ (U_x^1+i\overline{U}_x^1)\end{matrix}+\frac{\alpha+\sin\theta}{(1+\alpha\sin\theta)^2}\begin{matrix}a(\Omega_x^1+i\overline{\Omega}_x^1)\\ a(\Omega_y^1+i\overline{\Omega}_y^1)\end{matrix}\right\}. \qquad (5.4)$$

The complex forces are expressed in terms of the basic complex function $\tilde{\chi}$ by means of relationships (Chapter XI, 5.18) /307

$$\tilde{T}_{1,1} = -i\frac{2Ehc}{a^2\varphi_0}\left\{\frac{\sin\theta}{(1+\alpha\sin\theta)^2}\begin{matrix}-(U_y^1+i\overline{U}_y^1)\\ (U_x^1+i\overline{U}_x^1)\end{matrix}-\right.$$
$$\left.-\frac{\cos\theta}{(1+\alpha\sin\theta)^2}\begin{matrix}a(\Omega_x^1+i\overline{\Omega}_x^1)\\ a(\Omega_y^1+i\overline{\Omega}_y^1)\end{matrix}\right\}-i\frac{Ehc}{b}\left\{\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}\right\}; \qquad (5.5)$$
$$\tilde{T}_{2,1} = -i\frac{Ehc}{b}\left\{\frac{d\tilde{\chi}}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}\right\};$$
$$\tilde{S}_{,1} = \mp i\frac{Ehc}{b}\left\{\frac{\alpha}{1+\alpha\sin\theta}\tilde{\chi}\right\}.$$

Furthermore, by dropping the small terms we find from equalities (Chapter XI, 5.19-5.21)

$$u^n_{z,1} = -b(1+\alpha\sin\theta)\int_{\theta_0}^{\theta}\frac{\cos v'\,\mathrm{Re}\,\tilde{\gamma}}{1+\alpha\sin\theta'}\,d\theta' + a(1+\alpha\sin\theta)\begin{matrix}a_3\\a_5\end{matrix};$$

$$\vartheta^n_{,1} = \mathrm{Re}\,\tilde{\chi} + \alpha\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,\tilde{\chi}}{1+\alpha\sin\theta'}\,d\theta' - 1\begin{matrix}a_3\\a_5\end{matrix}; \qquad (5.6)$$

$$u^n_{\rho,1} = -b\int_{\theta_0}^{\theta}\sin\theta'\left[\mathrm{Re}\,\tilde{\chi} + \underline{\left(\alpha\int_{\theta_0}^{\theta'}\frac{\cos\theta''\,\mathrm{Re}\,\tilde{\chi}\,d\theta''}{1+\alpha\sin\theta''} - 1\begin{matrix}a_3\\a_5\end{matrix}\right)}\right]d\theta' + 1\begin{matrix}a'_4\\a'_6\end{matrix}.$$

Integrating the underlined term by parts, we obtain a more convenient expression for $u^c_{\rho,1}$

$$u^n_{\rho,1} = b\left\{\alpha\cos\theta\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,\tilde{\chi}}{1+\alpha\sin\theta'}\,d\theta' - \right.$$
$$\left. -\int_{\theta_0}^{\theta}\frac{(\sin\theta'+\alpha)\,\mathrm{Re}\,\tilde{\chi}}{1+\alpha\sin\theta'}\,d\theta'\right\} - b\cos\theta\begin{matrix}a_3\\a_5\end{matrix} + 1\begin{matrix}a_4\\a_5\end{matrix}. \qquad (5.7)$$

Now, by dropping the small terms in (Chapter XI, 5.22), we find the displacement

$$\pm v^n_{,1} = \frac{1}{\varphi_0}\left\{1\begin{matrix}-U^1_y\\U^1_x\end{matrix} + \alpha\cos\vartheta\begin{matrix}a\Omega^1_x\\a\Omega^1_y\end{matrix}\right\} - u^n_{\rho,1}. \qquad (5.8)$$

Finally we have the following simplified expressions for the complex boundary quantities (Chapter XI, 5.26) /308

$$\tilde{Q}_{\rho,1} = i\frac{Ehc}{a\varphi_0}\frac{2}{(1+\alpha\sin\theta)}\begin{matrix}(\Omega^1_x+i\bar{\Omega}^1_x)\\(\Omega^1_y+i\bar{\Omega}^1_y)\end{matrix} - i\frac{Ehc}{a}\left\{\frac{\tilde{\chi}}{1+\alpha\sin\theta}\right\};$$

$$\tilde{Q}_{12,1} = \mp i\frac{Ehc}{a}\left\{\frac{\tilde{\chi}}{1+\alpha\sin\theta}\right\};$$

$$\tilde{Q}_{z,1} = i\frac{Ehc}{a^2\varphi_0}\frac{2}{(1+\alpha\sin\theta)^2}\left\{1\begin{matrix}(U^1_y+i\bar{U}^1_y)\\-(U^1_x+i\bar{U}^1_x)\end{matrix} - b\cos\theta\begin{matrix}(\Omega^1_x+i\bar{\Omega}^1_x)\\(\Omega^1_y+i\bar{\Omega}^1_y)\end{matrix}\right\} - \qquad (5.9)$$
$$- \frac{Ehc^2}{ba(1+\alpha\sin\theta)}\left\{\frac{d\tilde{\chi}}{d\theta} + 2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}\right\};$$

$$\tilde{M}_{1,1} = \frac{Ehc^2}{b}\left\{\frac{d\tilde{\chi}}{d\theta} + 2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}\right\}.$$

We recall that (Chapter V, 9.10)

$$\begin{aligned}\widetilde{Q}_{z,1} &\equiv Q_{z,1} - i\frac{Ehc}{a}\frac{\varepsilon_{2,1}}{1+\alpha\sin\theta}; \quad \widetilde{M}_{1,1} \equiv M_{1,1} + iEhc\varepsilon_{2,1}\\ \widetilde{Q}_{\rho,1} &\equiv Q_{\rho,1} - i\frac{Ehc}{a}\frac{\chi}{1+\alpha\sin\theta}; \quad \widetilde{Q}_{12,1} \equiv Q_{12,1} \mp i\frac{Ehc}{a}\frac{\chi}{1+\alpha\sin\theta}\end{aligned} \tag{5.10}$$

6. Periodic Solutions of the Basic Equations

Let us obtain certain relationships which will be required in the future. For this purpose we express the periodic solutions of the following equations

$$L\{X\} + i2k^2\sin\theta X = -k^2\cos\theta; \tag{6.1}$$

$$L\{Y\} + i2k^2\sin\theta Y = -k^2\frac{\cos\theta}{1+\alpha\sin\theta}; \tag{6.2}$$

$$L\{Z\} + i2k^2\sin\theta Z = -k^2\frac{\sin\theta+\alpha}{1+\alpha\sin\theta}; \tag{6.3}$$

$$L\{W\} + i2k^2\sin\theta W = -k^2. \tag{6.4}$$

in terms of X, Y, Z, W. Here we have used the following representation

$$L\{\ldots\} = \frac{d}{d\theta}\left\{(1+\alpha\sin\theta)\frac{d(\ldots)}{d\theta}\right\}. \tag{6.5}$$

First of all we note the following easily verified even and odd properties of the introduced functions

$$\begin{aligned}&X(\pi-\theta) = -X(\theta); \quad \frac{dX}{d\theta}(\pi-\theta) = \frac{dX}{d\theta}(\theta);\\ &Y(\pi-\theta) = -Y(\theta); \quad \frac{dY}{d\theta}(\pi-\theta) = \frac{dY}{d\theta}(\theta);\\ &Z(\pi-\theta) = Z(\theta); \quad \frac{dZ}{d\theta}(\pi-\theta) = -\frac{dZ}{d\theta}(\theta);\\ &W(\pi-\theta) = W(\theta); \quad \frac{dW}{d\theta}(\pi-\theta) = -\frac{dW}{d\theta}(\theta).\end{aligned} \tag{6.6}$$

We shall show that within the accuracy common to the theory of thin shells, the functions Y and Z may be expressed quite simply in terms of X and W, respectively. For this purpose we utilize the obvious equalities /309

$$\frac{d}{d\theta}\left(\frac{\cos\theta}{1+\alpha\sin\theta}\right)=-\frac{\sin\theta+\alpha}{(1+\alpha\sin\theta)^2};\quad \frac{d}{d\theta}\left(\frac{\sin\theta}{1+\alpha\sin\theta}\right)=\frac{\cos\theta}{(1+\alpha\sin\theta)^2};$$
$$\frac{d}{d\theta}\left(\frac{\sin\theta+\alpha}{1+\alpha\sin\theta}\right)=(1-\alpha^2)\frac{\cos\theta}{(1+\alpha\sin\theta)^2};\quad \frac{d}{d\theta}\left(\frac{1}{1+\alpha\sin\theta}\right)= \tag{6.7}$$
$$=-\alpha\frac{\cos\theta}{(1+\alpha\sin\theta)^2},$$

from which it follows that

$$L\left\{\frac{\cos\theta}{1+\alpha\sin\theta}\right\}=-(1-\alpha^2)\frac{\cos\theta}{(1+\alpha\sin\theta)^2};\quad L\left\{\frac{\sin\theta}{1+\alpha\sin\theta}\right\}=$$
$$=-\frac{\sin\theta+\alpha}{(1+\alpha\sin\theta)^2};$$
$$L\left\{\frac{\sin\theta+\alpha}{1+\alpha\sin\theta}\right\}=-(1-\alpha^2)\frac{\sin\theta+\alpha}{(1+\alpha\sin\theta)^2};\quad L\left\{\frac{1}{1+\alpha\sin\theta}\right\}= \tag{6.8}$$
$$=\alpha\frac{\sin\theta+\alpha}{(1+\alpha\sin\theta)^2}.$$

We let $Y = X + Y^{(1)}$ and substitute Y into equation (6.2). Since the function X satisfies equation (6.1), we obtain the following equation for the second term

$$L\{Y^{(1)}\}+i2k^2\sin\theta\,Y^{(1)}=k^2\frac{\alpha\sin\theta\cos\theta}{1+\alpha\sin\theta}. \tag{6.9}$$

Furthermore, let us assume that

$$Y^{(1)}=-i\frac{\alpha}{2}\frac{\cos\theta}{1+\alpha\sin\theta}+Y^{(2)}, \tag{6.10}$$

where the first term represents the membrane solution of equation (6.9), while $Y^{(2)}$ represents the correction to this term. Substituting $Y^{(1)}$ into (6.9) and utilizing (6.2) we obtain the following characteristic equation for the correction $Y^{(2)}$

$$L\{Y^{(2)}\}+i2k^2\sin\theta\,Y^{(2)}=-i\frac{\alpha(1-\alpha^2)}{2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}=$$
$$=-i\frac{\alpha(1-\alpha^2)}{2}\left\{\frac{\cos\theta}{1+\alpha\sin\theta}-\frac{\alpha\sin\theta\cos\theta}{(1+\alpha\sin\theta)^2}\right\}.$$

Furthermore, by applying the above approach of combining the membrane solution and its correction, we find

$$Y^{(2)}=i\frac{\alpha(1-\alpha^2)}{2k^2}Y+\frac{\alpha^2(1-\alpha^2)}{4k^2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}+\dots,$$

where the dots represent very small terms. Dropping these and collecting the computed parts of Y we obtain

$$\left(1-\underline{i\frac{\alpha(1-\alpha^2)}{2k^2}}\right)Y=X-i\frac{\alpha}{2}\frac{\cos\theta}{1+\alpha\sin\theta}+\underline{\frac{\alpha^2(1-\alpha^2)}{4k^2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}}.$$

Dropping the small underlined terms we obtain the unknown approximate expression /310

$$\dot{Y}=X-i\frac{\alpha}{2}\frac{\cos\theta}{1+\alpha\sin\theta}. \tag{6.11}$$

In exactly the same manner we find the following expression for the periodic solution of equation (6.3)

$$Z=\tau W+i\frac{1-\alpha^2}{2}\frac{1}{1+\alpha\sin\theta}. \tag{6.12}$$

When we determine the rigidity of bent tubes, we shall have to deal with the quantities

$$J_k=\frac{1}{2\pi}\oint\frac{d\theta}{(1+\alpha\sin\theta)^k}, \tag{6.13}$$

where the $\oint$ sign means that integration is carried out in the limits from 0 to 2π. Integrals of this type can be easily evaluated. Indeed, by integrating the identity

$$\frac{1}{(1+\alpha\sin\theta)^{k+1}}=\frac{1}{(1+\alpha\sin\theta)^k}-\frac{\alpha\sin\theta}{(1+\alpha\sin\theta)^{k+1}}, \tag{6.14}$$

we obtain recurrent relationships

$$J_{k+1}(\alpha)=\frac{\alpha}{k}J_k'(\alpha)+J_k(\alpha), \tag{6.15}$$

and when k = 1 these take the form

$$J_2(\alpha)=\alpha J_1'(\alpha)+J_1(\alpha). \tag{6.16}$$

We note the validity of equality

$$J_1'(\alpha)=\alpha J_2(\alpha)\left(\frac{1}{2\pi}\oint\frac{(\alpha+\sin\theta)\,d\theta}{(1+\alpha\sin\theta)^2}=0\right), \tag{6.17}$$

which follows from identity $(6.7)_1$. By eliminating quantity $J_1'(\alpha)$ from (6.16) and (6.17) we obtain

$$J_1(\alpha)=(1-\alpha^2)J_2(\alpha). \tag{6.18}$$

Comparing the resulting equality with (6.16), we obtain the differential equation for $J_1(\alpha)$

$$J_1'(\alpha)-\frac{\alpha}{1-\alpha^2}J_1(\alpha)=0. \tag{6.19}$$

Integrating this quantity and noting that $J_1(0) = 1$, we obtain the following expression for $J_1(\alpha)$

$$J_1(\alpha)=\frac{1}{\sqrt{1-\alpha^2}}. \tag{6.20}$$

The subsequent functions are obtained from the recurrent relationships (6.15). In particular,

$$J_2(\alpha)=\frac{1}{(1-\alpha^2)^{\frac{3}{2}}},\quad J_3(\alpha)=\frac{1+\frac{1}{2}\alpha^2}{(1-\alpha^2)^{\frac{5}{2}}}. \tag{6.21}$$

Furthermore, by using the identity /311

$$\frac{\cos^2\theta}{(1+\alpha\sin\theta)^2}=-\frac{1-\alpha^2}{\alpha^2}\frac{1}{(1+\alpha\sin\theta)^2}+\frac{2}{\alpha^2}\frac{1}{1+\alpha\sin\theta}-\frac{1}{\alpha^2} \tag{6.22}$$

which can be easily proven, we obtain the following by means of equalities (6.20) and (6.21)

$$\frac{1}{2\pi}\oint\frac{\cos^2\theta d\theta}{(1+\alpha\sin\theta)^2}=\frac{J_1(\alpha)-1}{\alpha^2}. \tag{6.23}$$

In the same way we find

$$\frac{1}{2\pi}\oint\frac{\cos^2\theta d\theta}{1+\alpha\sin\theta}=\frac{1-(1-\alpha^2)J_1(\alpha)}{\alpha^2};\ \frac{1}{2\pi}\oint\frac{\sin\theta d\theta}{1+\alpha\sin\theta}=\frac{1-J_1(\alpha)}{\alpha}. \tag{6.24}$$

In Chapter XVII we utilized the quantities

$$I_1=\frac{1}{2\pi}\oint\cos\theta X d\theta;\quad I_2=\frac{1}{2\pi}\oint\frac{\cos\theta Y}{1+\alpha\sin\theta}d\theta;$$
$$I_3=\frac{1}{2\pi}\oint\frac{Z d\theta}{1+\alpha\sin\theta},\quad I=\frac{1}{2\pi}\oint W d\theta; \tag{6.25}$$

$$\Lambda_1=2\frac{(\operatorname{Re}I_1)^2+(\operatorname{Im}I_1)^2}{\operatorname{Re}I_1};\quad \Lambda_2=2\frac{(\operatorname{Re}I_2)^2+(\operatorname{Im}I_2)^2}{\operatorname{Re}I_2};$$
$$\Lambda_3=2\alpha\frac{(\operatorname{Re}I_3)^2+(\operatorname{Im}I_3)^2}{\operatorname{Re}I_3}. \tag{6.26}$$

Let us show that when the values of $2k^2$ are sufficiently large they can all be expressed approximately in terms of I.

It follows from (6.11) that

$$I_3=\frac{1}{2\pi}\oint\frac{\cos\theta X d\theta}{1+\alpha\sin\theta}-i\frac{\alpha}{2}\frac{1}{2\pi}\oint\frac{\cos^2\theta d\theta}{(1+\alpha\sin\theta)^2}. \tag{6.27}$$

First of all,

$$\frac{1}{2\pi}\oint\frac{\cos\theta X d\theta}{1+\alpha\sin\theta}=\frac{1}{2\pi}\oint\left\{\cos\theta-\frac{\alpha\sin\theta\cos\theta}{1+\alpha\sin\theta}\right\}X d\theta=$$
$$=I_1-\alpha\frac{1}{2\pi}\oint\frac{\sin\theta\cos\theta X}{1+\alpha\sin\theta}d\theta. \tag{6.28}$$

If we multiply equation (6.1) by $\frac{1}{2\pi}\frac{\cos\theta}{1+\alpha\sin\theta}$ and integrate it over θ from 0 to 2π, we obtain the equality

$$\frac{1}{2\pi}\oint\frac{\cos\theta}{1+\alpha\sin\theta}\frac{d}{d\theta}\left\{(1+\alpha\sin\theta)\frac{dX}{d\theta}\right\}d\theta - \\ + i2k^2\frac{1}{2\pi}\oint\frac{\sin\theta\cos\theta X}{1+\alpha\sin\theta}d\theta = -k^2\frac{1}{2\pi}\oint\frac{\cos^2\theta d\theta}{1+\alpha\sin\theta}. \tag{6.29}$$

Integrating the first integral in parts and taking into account equalities (6.7) as well as the periodic properties of function X, we obtain

$$\underline{-(1-\alpha^2)\frac{1}{2\pi}\oint\frac{\cos\theta X d\theta}{(1+\alpha\sin\theta)^2}} + i2k^2\frac{1}{2\pi}\oint\frac{\sin\theta\cos\theta X d\theta}{1+\alpha\sin\theta} = \\ = -k^2\frac{1}{2\pi}\oint\frac{\cos^2\theta d\theta}{1+\alpha\sin\theta}.$$

Dropping the underlined terms, which are small compared with the other terms, we find

$$\frac{1}{2\pi}\oint\frac{\sin\theta\cos\theta X}{1+\alpha\sin\theta}d\theta = \frac{i}{2}\frac{1}{2\pi}\oint\frac{\cos^2\theta d\theta}{1+\alpha\sin\theta}.$$

By taking into account equalities (6.20), (6.23) and (6.24) we can write

$$I_2 = I_1 - i\frac{\alpha}{2\sqrt{1-\alpha^2}}. \tag{6.30}$$

By using equalities (6.12) and (6.25) we express I_3 in terms of

$$I_3 = \frac{1}{2\pi}\oint\frac{Zd\theta}{1+\alpha\sin\theta} = \alpha\left\{\frac{1}{2\pi}\oint Wd\theta - \alpha\frac{1}{2\pi}\oint\frac{\sin\theta Wd\theta}{1+\alpha\sin\theta}\right\} + \\ + i\frac{1-\alpha^2}{2}\frac{1}{2\pi}\oint\frac{d\theta}{(1+\alpha\sin\theta)^4}. \tag{6.31}$$

By utilizing the "membrane" solution of equation (6.4) and following the procedure used above, we find

$$\frac{1}{2\pi}\oint\frac{\sin\theta Wd\theta}{1+\alpha\sin\theta} = \frac{i}{2}\frac{1}{2\pi}\oint\frac{d\theta}{(1+\alpha\sin\theta)} = \frac{i}{2}J_1(\alpha).$$

Substituting the resulting expression into the preceding one, we obtain

$$I_3 = \alpha I + i\frac{1-\alpha^2}{2} J_1(\alpha) = \alpha I + i\frac{\sqrt{1-\alpha^2}}{2}. \tag{6.32}$$

Furthermore, by using the "membrane" solution of equations (6.1) and (6.4) we obtain the equality

$$i2k^2\left\{\frac{1}{2\pi}\oint \cos\theta X d\theta - \frac{1}{2\pi}\oint W d\theta\right\} = k^2\frac{1}{2\pi}\oint \frac{(1-\cos^2\theta)}{\sin\theta} d\theta =$$
$$= k^2\frac{1}{2\pi}\oint \sin\theta d\theta = 0,$$

from which we obtain the following approximate equality

$$I_1 = I.$$

Thus for large values $2k^2$ we have the following approximate equalities

$$I_1 = I;\ I_2 = I - i\frac{\alpha}{2\sqrt{1-\alpha^2}};\ I_3 = \alpha I + i\frac{\sqrt{1-\alpha^2}}{2}, \tag{6.34}$$

which make it possible to determine the introduced integral quantities as soon as we can compute, for example, the quantity

$$I = \frac{1}{2\pi}\oint W d\theta.$$

In conclusion we shall present one other approximate equality

$$\frac{1}{2\pi}\oint \frac{\sin\theta Z d\theta}{1+\alpha\sin\theta} = 0, \tag{6.35}$$

which is established by integrating equation (6.3) and utilizing the identity $(6.7)_1$.

7. Determination of the Periodic Solutions of the Basic Equations by Means of Continued Fractions (the V. V. Novozhilov Method)

It is convenient to apply the method of Fourier series to find the periodic solutions of equations (6.1) and (6.4), because each term of the series is a periodic function of θ. Thus we seek the solution of equation /313

$$(1+\alpha\sin\theta)\frac{d^2X}{d\theta^2}+\alpha\cos\theta\frac{dX}{d\theta}+i2k^2\sin\theta X=-k^2\cos\theta \tag{7.1}$$

in the form

$$X=\sum_{l=0}^{\infty}(a_l\cos l\theta+b_l\sin l\theta). \tag{7.2}$$

Substituting this expression into (7.1) and taking into account the well-known reduction equations

$$\begin{aligned}
\sin\theta\sin l\theta&=\tfrac{1}{2}[\cos(l-1)\theta-\cos(l+1)\theta],\\
\cos\theta\sin l\theta&=\tfrac{1}{2}[\sin(l-1)\theta+\sin(l+1)\theta],\\
\sin\theta\cos l\theta&=\tfrac{1}{2}[-\sin(l-1)\theta+\sin(l+1)\theta],\\
\cos\theta\cos l\theta&=\tfrac{1}{2}[\cos(l-1)\theta+\cos(l+1)\theta],
\end{aligned} \tag{7.3}$$

we obtain

$$\begin{aligned}
\sum_{l=0}^{\infty}\Big\{&-l^2(a_l\cos l\theta+b_l\sin l\theta)-\tfrac{\alpha l^2}{2}[(-\sin(l-1)\theta+\\
&+\sin(l+1)\theta)a_l+(\cos(l-1)\theta-\cos(l+1)\theta)b_l]+\\
&+\tfrac{\alpha l}{2}[-(\sin(l-1)\theta+\sin(l+1)\theta)a_l+(\cos(l-1)\theta+\\
&+\cos(l+1)\theta)b_l]+ik^2[(-\sin(l-1)\theta+\sin(l+1)\theta)a_l+\\
&+(\cos(l-1)\theta-\cos(l+1)\theta)b_l]\Big\}=-k^2\cos\theta.
\end{aligned}$$

Changing the summation signs and thereby equating the factors before the functions $\cos l\theta$ and $\sin l\theta$, we obtain an infinite system of equations

$$
\begin{aligned}
&-l^2 a_l + \left[ik^2 - \tfrac{\alpha}{2}(l+1)l\right] b_{l+1} - \left[ik^2 - \tfrac{\alpha}{2} l(l-1)\right] b_{l-1} = \\
&\qquad = -k^2 \delta_l^1; \\
&-l^2 b_l - \left[ik^2 - \tfrac{\alpha}{2}(l+1)l\right] a_{l+1} + \left[ik^2(1+\delta_1^l) - \right. \\
&\qquad \left. - \tfrac{\alpha}{2} l(l-1)\right] a_{l-1} = 0 \\
&\qquad \left(\delta_i^j = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}\right).
\end{aligned} \tag{7.4}
$$

It is easy to notice that in the above we can set /314

$$
\begin{aligned}
&a_0 = a_2 = \ldots = a_{2l} = \ldots = 0; \\
&b_1 = b_3 = \ldots = b_{2l+1} = \ldots = 0.
\end{aligned}
$$

Expanding the remaining equations, we obtain the system

$$
\begin{aligned}
&-1^2 a_1 + \left[ik^2 - \tfrac{\alpha}{2} 2 \cdot 1\right] b_2 = -k^2, \\
&-2^2 b_2 - \left[ik^2 - \tfrac{\alpha}{2} 3 \cdot 2\right] a_3 + \left[ik^2 - \tfrac{\alpha}{2} 2 \cdot 1\right] a_1 = 0, \\
&-3^2 a_3 + \left[ik^2 - \tfrac{\alpha}{2} 4 \cdot 3\right] b_4 - \left[ik^2 - \tfrac{\alpha}{2} 3 \cdot 2\right] b_2 = 0, \\
&-4^2 b_4 - \left[ik^2 - \tfrac{\alpha}{2} 5 \cdot 4\right] a_5 + \left[ik^2 - \tfrac{\alpha}{2} 4 \cdot 3\right] a_3 = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

which may be conveniently written in the form

$$
\begin{aligned}
&\frac{1^2}{k^2} - \left(1 + i\frac{\alpha}{2k^2} 2 \cdot 1\right) i \frac{b_2}{a_1} = \frac{1}{a_1}; \\
&\frac{2^2}{k^2} + \left(1 + i\frac{\alpha}{2k^2} 3 \cdot 2\right) i \frac{a_3}{b_2} - \left(1 + i\frac{\alpha}{2k^2} 2 \cdot 1\right) i \frac{a_1}{b_2} = 0; \\
&\frac{3^2}{k^2} - \left(1 + i\frac{\alpha}{2k^2} 4 \cdot 3\right) i \frac{b_4}{a_3} + \left(1 + i\frac{\alpha}{2k^2} 3 \cdot 2\right) i \frac{b_2}{a_3} = 0; \\
&\frac{4^2}{k^2} + \left(1 + i\frac{\alpha}{2k^2} 5 \cdot 4\right) i \frac{a_5}{b_4} - \left(1 + i\frac{\alpha}{2k^2} 4 \cdot 3\right) i \frac{a_3}{b_4} = 0; \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

By eliminating the ratios of the coefficients successively we obtain equalities which make it possible for us to obtain the unknown coefficients successively

$$a_1 = \cfrac{1}{\cfrac{1^2}{k^2} + \cfrac{\left(1 + i\frac{\alpha}{2k^2}\,2\cdot 1\right)^2}{\cfrac{2^2}{k^2} + \cfrac{\left(1 + i\frac{\alpha}{2k_2}\,3\cdot 2\right)^2}{\cfrac{3^2}{k^2} + \dots}}}\;;$$

$$-i\frac{b_2}{a_1} = -\cfrac{\left(1 + i\frac{\alpha}{2k^2}\,2\cdot 1\right)}{\cfrac{2^2}{k^2} + \cfrac{\left(1 + i\frac{\alpha}{2k^2}\,3\cdot 2\right)^2}{\cfrac{3^2}{k^2} + \dots}}\;; \tag{7.6}$$

$$i\frac{a_3}{b_2} = \cfrac{\left(1 + i\frac{\alpha}{2k^2}\,3\cdot 2\right)}{\cfrac{3^2}{k^2} + \cfrac{\left(1 + i\frac{\alpha}{2k^2}\,4\cdot 3\right)^2}{\cfrac{4^2}{k^2} + \dots}}\;;$$

. . . . . .

Let us introduce the designations /315

$$K_m^n = \frac{n^2}{k^2} + \cfrac{\left(1 + i\frac{\alpha}{2k^2}(n+1)\,n\right)^2}{\cfrac{(n+1)^2}{k^2} + \dots \cfrac{}{\dots + \cfrac{m^2}{k^2}}}\,. \tag{7.7}$$

It is easy to establish the validity of the recurrent relationships

$$K_m^n = \frac{n^2}{k^2} + \frac{\left(1 + i\frac{\alpha}{2k^2}(n+1)\,n\right)^2}{K_m^{n+1}}\,. \tag{7.8}$$

By using the introduced designations, the equalities obtained above can be written in the form

$$a_1 = \frac{1}{K_\infty^1}\,,\qquad b_{2l} = ia_{2l-1}\frac{1 + i\frac{\alpha}{2k^2}\,2l(2l-1)}{K_\infty^{2l}}\,,$$
$$a_{2l+1} = -ib_{2l}\frac{1 + i\frac{\alpha}{2k^2}\,(2l+1)\,2l}{K_\infty^{2l+1}}\,. \tag{7.9}$$

Now let us consider the second basic equation

$$(1+\alpha\sin\theta)\frac{d^2W}{d\theta^2}+\alpha\cos\theta\frac{dW}{d\theta}+i2k^2\sin\theta\,W=-k^2. \tag{7.10}$$

Seeking its solution in the same form, we obtain the system

$$\begin{aligned}
&-l^2a_l+\left[ik^2-\frac{\alpha}{2}(l+1)\,l\right]b_{l+1}-\left[ik^2-\frac{\alpha}{2}l(l-1)\right]b_{l-1}=\\
&\qquad=-k^2\delta_l^0;\\
&-l^2b_l-\left[ik^2-\frac{\alpha}{2}(l+1)\,l\right]a_{l+1}+\left[ik^2(1+\delta_1^l)-\right.\\
&\qquad\left.-\frac{\alpha}{2}l(l-1)\right]a_{l-1}=0.
\end{aligned} \tag{7.11}$$

In this system we may set

$$\begin{aligned}
b_2=b_4=b_6=\ldots=0;\\
a_1=a_3=a_5=\ldots=0.
\end{aligned}$$

The remaining equations give us a chain of equalities from which we successively find /316

$$a_0=\frac{1}{2}K_\infty^1;\quad b_1=i. \tag{7.12}$$

The remaining coefficients are determined from the recurrent relationships

$$\begin{aligned}
a_{2l}&=-ib_{2l-1}\frac{1+i\frac{\alpha}{2k^2}2l(2l-1)}{K_\infty^{2l}};\\
b_{2l+1}&=ia_{2l}\frac{1+i\frac{\alpha}{2k^2}(2l+1)2l}{K_\infty^{2l+1}}.
\end{aligned} \tag{7.13}$$

When the values $2k^2\,a_1=\frac{1}{K_\infty^1}\approx 1$. Since the recurrent relationships (7.13) and (7.9) are identical and $b_1 = i$, it follows that for large values $2k^2$ the following approximate relationships between the coeffficients of the expansion are valid

$$b_{2l+1}=i\,a_{2l+1};\ a_{2l}=-\,b_{2l}\qquad (l>1) \tag{7.14}$$

8. Asymptotic Integration. The Method of Standard Equations

As the parameter $2k^2$ increases, the continued fractions considered in the preceding paragraph converge at an ever-decreasing rate. For this reason it becomes necessary to retain a substantial number of terms in expansion (7.2). Therefore, for large values $2k^2$ it is more convenient to apply the asymptotic methods which utilize this large parameter. We cannot seek the solution in the form given in section 7 of Chapter V, because at the transient points of the meridian ($\theta = 0$, $\theta = \pi$) the term in the equation containing the large factor becomes equal to zero. Therefore we shall utilize a more complex method of "standard" equations (refs. 328 and 320). In this method the approximate solution is constructed by utilizing the solution of a simpler standard equation containing the same singularities in the coefficients (poles and zeros).

In the problem under consideration it is convenient to take the Stokes equation as the standard equation

$$\frac{d^2U}{d\psi^2}+\psi U=0, \tag{8.1}$$

since it has a simple zero in the coefficient before the unknown function. The solutions of this equation are called the Airy functions.

Let us write the basic equations (6.1) and (6.4) in the form /317

$$\frac{d^2X_k}{d\theta^2}+p(\theta)\frac{dX_k}{d\theta}+[r(\theta)+(-i\lambda)^3 q(\theta)]\,X_k=(-i\lambda)^3 f_k(\theta). \tag{8.2}$$

Here

$$\begin{aligned} &X_1=X;\ X_2=W;\ \lambda=\sqrt[3]{2k^2}=\sqrt[3]{\frac{ab}{c}};\\ &p(\theta)=\frac{\alpha\cos\theta}{1+\alpha\sin\theta};\ q(\theta)=\frac{\sin\theta}{1+\alpha\sin\theta};\\ &f_1(\theta)=\frac{i}{2}\frac{\cos\theta}{1+\alpha\sin\theta};\ f_2(\theta)=\frac{i}{2}\frac{1}{1+\alpha\sin\theta}. \end{aligned} \tag{8.3}$$

In the simplified equations (6.1)-(6.4) $r(\theta)\equiv 0$, because terms of this type were dropped in transformations carried out in sections 4 and 5. Nevertheless,

we have again introduced $r(\theta)$ to show that during asymptotic integration their magnitude has no effect on the resulting approximate solution. This will serve as one additional argument in favor of adopting the basic equations in the simplified form (6.1) and (6.4).

We seek the particular solutions of the above equations in the following form

$$X_k = \varphi(\theta) H[\psi(\theta)]. \tag{8.4}$$

From this we obtain

$$\frac{dX_k}{d\theta} = (\varphi\psi') H' + (\varphi') H;$$
$$\frac{d^2X_k}{d\theta^2} = (\varphi\psi'^2) H'' + (2\varphi'\psi' + \varphi\psi'') H' + (\varphi'') H.$$

Substituting the resulting expressions into (8.2) we obtain the equality

$$\varphi\{\psi'^2 H'' + (-i\lambda)^3 qH\} + \{(2\varphi' + p\varphi)\psi' + \varphi\psi''\} H' + \\ + \{\varphi'' + p\varphi' + r\varphi\} H = (-i\lambda)^3 f_k(\theta). \tag{8.5}$$

We shall first consider the homogeneous equation setting $H = U(\psi)$. By using the standard equation (8.1), which is satisfied by the Airy function, we write the expression in the first brackets as follows

$$\varphi\{-\psi'^2\psi + (-i\lambda)^3 q\} U.$$

Setting it equal to 0, we obtain

$$\sqrt{\psi}\psi' = \pm (-i\lambda)^{\frac{3}{2}} \sqrt{\operatorname{sign} q}\, \sqrt{|q|}, \tag{8.6}$$

where

$$\operatorname{sign} f = \begin{cases} +1 & \text{when } f > 0 \\ -1 & \text{when } f < 0. \end{cases}$$

Integrating this equation in the limits from 0 to θ, we find /318

$$\frac{2}{3}(\psi)^{\frac{3}{2}} = \pm (-i\lambda)^{\frac{3}{2}} \sqrt{\operatorname{sign} q} \int_0^\theta \sqrt{|q|}\, d\theta,$$

from which

$$\psi(\theta) = -i\lambda\omega(\theta). \tag{8.7}$$

Here

$$\omega(\theta) = \operatorname{sign} q\left(\frac{3}{2}\int_0^{|\theta|}\sqrt{|q|}\,d\theta\right)^{\frac{2}{3}} = \operatorname{sign}\theta\left(\frac{3}{2}\int_0^{|\theta|}\sqrt{\left|\frac{\sin\theta}{1+\alpha\sin\theta}\right|}\,d\theta\right)^{\frac{2}{3}}. \tag{8.8}$$

The values of functions $\omega(\theta)$ and $\omega'(\theta)$ are shown in tables 1 and 2.

Next, by equating the expression in the second brackets of (8.5) to 0 and taking into account relationships (8.3), we obtain the following expression for the function $\varphi(\theta)$

$$\varphi(\theta) = \frac{1}{\sqrt{\omega'(\theta)(1+\alpha\sin\theta)}} = \sqrt[4]{\frac{\omega(\theta)}{\sin\theta(1+\alpha\sin\theta)}}. \tag{8.9}$$

Here we have not utilized the equality which follows from (8.6)

$$\omega'(\theta) = \sqrt{\frac{\sin\theta}{\omega(\theta)(1+\alpha\sin\theta)}}. \tag{8.10}$$

When the values of θ are small, the following approximate equations are valid

$$\begin{aligned}\omega(\theta) &= \theta(1-0{,}2\alpha\theta);\\ \varphi(\theta) &= 1-0.3\alpha\theta.\end{aligned} \tag{8.11}$$

Now the general solution of the homogeneous equation may be written in the form

$$X_k(\theta) = \varphi(\theta)\{\widetilde{C}_1 U_1(-i\lambda\omega) + \widetilde{C}_2 U_2(-i\lambda\omega)\}. \tag{8.12}$$

For the functions U_1 and U_2 we can select any pair of linearly independent solutions of the Stokes equation (8.1). In this connection the most convenient ones are the modified Hankel functions $h_1(z)$ and $h_2(z)$, which have the following asymptotic representation ($y > 0$) when the moduli of purely imaginary values of the argument are large

$$h_1(iy) = \gamma (iy)^{-\frac{1}{4}} e^{-\left[\frac{2}{3}\frac{1+i}{\sqrt{2}} y^{\frac{3}{2}} + i\frac{5\pi}{12}\right]};$$
$$h_2(iy) = \gamma (iy)^{-\frac{1}{4}} e^{\left[\frac{2}{3}\frac{1+i}{\sqrt{2}} y^{\frac{3}{2}} + i\frac{5\pi}{12}\right]} \tag{8.13}$$
$$(\gamma = 2^{\frac{1}{3}} 3^{\frac{1}{6}} \pi^{-\frac{1}{2}} = 0.8537).$$

We note that the latter expression could have been obtained directly, if /319 we had sought the asymptotic solution of (8.2) in the form $X_k(\theta) = \varphi(\theta) e^{\psi(\theta)}$. Incidentally, this form of the solution corresponds to the following standard equation

$$\frac{d^2U}{d\psi^2} - U = 0 \tag{8.14}$$

The series, corresponding to the Hankel functions (which converge over the entire complex plane) for a purely imaginary argument, has the form

$$h_1(iy) = g + i\frac{\sqrt{3}}{2}(g - 2f); \; h_2(iy) = g - i\frac{\sqrt{3}}{2}(g - 2f), \tag{8.15}$$

where

$$f(iy) = \frac{2^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m (3m-2)(3m-5)\ldots 4\cdot 1}{(3m)!} (iy)^{3m}\right];$$
$$g(iy) = \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma\left(\frac{4}{3}\right)}\left[iy + \sum_{m=1}^{\infty} \frac{(-1)^m (3m-1)(3m-4)\ldots 5\cdot 2}{(3m)!} (iy)^{3m+1}\right]$$
$$\left(\frac{2^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} = 0.9304, \; \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma\left(\frac{4}{3}\right)} = 0.6783\right).$$

The functions $h_1(z)$ and $h_2(z)$ are tabulated for complex argument values $|z| = \sqrt{x^2 + y^2} \leq 6.0$ (ref. 354). Selections from these tables for purely imaginary values of the argument when $0 \leq y \leq 6.0$ are shown in table 3. When the values of the argument are negative, the odd and even properties

$$h_1(-iy)=\overline{h_2(iy)};\ h_1'(-iy)=\overline{h_2'(iy)};$$
$$h_2(-iy)=\overline{h_1(iy)},\ h_2'(-iy)=\overline{h_1'(iy)}. \tag{8.16}$$

from relationships (8.15) should be utilized for negative values of the argument. For $y > 6.0$ the values of the functions are determined from expressions (8.13). When $y < -6.0$ we should also utilize equality (8.16).

As we can see from expression $(8.11)_2$, when the values of θ are close to zero the function $\varphi(\theta)$ is smooth and does not much vary. It differs substantially from unity only when the values of θ are close to $\pm\frac{\pi}{2}$. However, at the point on the meridian removed from the apex of the torus, as we can see from (8.13), the solution is transformed into the conventional asymptotic solution which has been considered in detail in section 7, Chapter V. We recall the following: it was established in section 7 of Chapter V that /320 we can neglect the variation in the amplitude factor of the solution compared with the variation of the exponential factor, while maintaining an accuracy associated with the dropping of terms of order $\frac{1}{\sqrt{\lambda^2}}$ compared to unity.

In view of the above, by stretching the point slightly, we can utilize the solution in the following form, which is cruder, but substantially simpler

$$X_k(\theta)=\tilde{C}_1 h_1(-i\lambda\omega)+\tilde{C}_2 h_2(-i\lambda\omega). \tag{8.17}$$

Since $\tilde{\vartheta}^c$ and $\tilde{\chi}$ are obtained from X and W by multiplication of constants, including the latter, into the arbitrary constants $\tilde{C}_1$ and $\tilde{C}_2$, we obtain the same expression for $\tilde{\vartheta}^c$ and $\tilde{\chi}$.

Let us now seek the particular solution of the nonhomogeneous equation. For this purpose, in equation (8.5) we let

$$\psi(\theta)=\omega(\theta);\ \varphi(\theta)=\sqrt[4]{\frac{\omega(\theta)}{\sin\theta(1+a\sin\theta)}}. \tag{8.18}$$

Since the expression for the function $\psi(\theta)$ differs from the old value only by the constant factor, while $\varphi(\theta)$ remains the same, the term in the second brackets of equality (8.5) vanishes. The terms contained in the third brackets

are small because the quantities in those brackets are smooth. Dropping these we obtain the following equation

$$\frac{d^2H}{d\omega^2} + i\lambda^3 \omega H = i\lambda^3 g(\omega), \tag{8.19}$$

where

$$g(\omega) = \frac{\sqrt{1+a\sin\theta}}{[(\omega'(\theta)]^{\frac{3}{2}}} f_k(0) = \sqrt[4]{\frac{\omega^3(\theta)(1+a\sin\theta)^5}{\sin^3\theta}} f_k(0). \tag{8.20}$$

In the expression for $g(\omega)$ we must consider θ a function of ω implicity assigned by relationship (8.8). It will be shown below that it is not necessary to establish this relationship in explicit form.

Let us rewrite the last equation in a slightly different manner

$$\frac{d^2H}{d\omega^2} + i\lambda^3 \omega H = i\lambda^3 [g(\omega) - g(0)] + i\lambda^3 g(0). \tag{8.21}$$

For the solution corresponding to the first term in the right side we take the "membrane" solution

$$H^{(1)} = \frac{g(\omega) - g(0)}{\omega}. \tag{8.22}$$

The validity of this approach away from the "point" $\omega = 0$ should raise no doubt. We should only clarify its error in the neighborhood $\omega = 0$. We shall show that this error is not large. For this purpose we expand $g(\omega)$ in a series

$$g(\omega) = g(0) + \frac{g'(0)}{1!}\omega + \frac{g''(0)}{2!}\omega^2 + \dots$$

Then

$$H^{(1)} = \frac{g'(0)}{1!} + \frac{g''(0)}{2!}\omega + \frac{g'''(0)}{3!}\omega^2 + \dots$$

Substituting this expression into the left side of the equation, we obtain

$$\left[\frac{g''(0)}{3}+\left(\frac{g^{IV}(0)}{4}\omega+\ldots\right)\right]+i\lambda^3\left\{\frac{g'(0)}{1!}\omega+\frac{g''(0)}{2!}\omega^2+\ldots\right\}=\ldots$$

The terms which determine the error (discrepancy) in solution (8.22), are contained within the square brackets. First of all, since $g(\omega)$ is smooth, the coefficients of these terms are not large. Therefore the second and subsequent terms which have been written out and which, in the neighborhood $\omega = 0$, behave like the large basic terms (in the figure brackets) may be dropped. We are left with the first term which, unlike the basic terms, does not become equal to zero when $\omega = 0$. However, this term is also insignificant. Indeed, if we retain it and transfer it to the first part, we obtain a modified constant

$$\lambda^3\left[g(0)+\frac{i}{\lambda^3}\frac{g''(0)}{3}\right],$$

which differs very little from the old value, because $g(\omega)$ is smooth. Thus we may assume that the solution of form (8.22) is acceptable.

Let us proceed with the determination of the particular solution corresponding to the second term in the right side. Again, by introducing the variable $\psi = -i\lambda\omega$ we write the equation for the second particular solution $H^{(2)}$ in the form

$$\frac{d^2H^{(2)}}{d\psi^2}+\psi H^{(2)}=-i\lambda g(0). \tag{8.23}$$

We utilize the variational method for the arbitrary constants. As applied to this equation, the method consists of seeking the particular solution in the form

$$H^{(2)}=C_1(\psi)U_1(\psi)+C_2(\psi)U_2(\psi), \tag{8.24}$$

where $U_1(\psi)$, $U_2(\psi)$ are the solutions of the homogeneous Stokes equation (Airy functions) mentioned above. Let us differentiate the last expression with respect to ψ

$$\frac{dH^{(2)}}{d\psi}=C_1U_1'+C_2U_2'+[C_1'U_1+C_2'U_2]$$

and equate to zero the expression in the square brackets

$$C_1' U_1 + C_2' U_2 = 0. \tag{8.25}$$

Taking into account the last equality we differentiate $H^{(2)}$ once again

$$\frac{d^2 H^{(2)}}{d\psi^2} = C_1 U_1' + C_2 U_2' + [C_1' U_1' + C_2' U_2'].$$

Substituting the resulting expression into equation (8.23), we obtain the equality /322

$$[C_1 (U_1' + \psi U_1) + C_2 (U_2' + \psi U_2)] + C_1' U_1' + C_2' U_2' = -i\lambda g(0).$$

In view of equation (8.1) the terms contained in the square brackets vanish. Therefore we obtain the following system of equations for determining the varied constants $C_1(\psi)$, $C_2(\psi)$

$$\begin{aligned} U_1(\psi) C_1'(\psi) + U_2(\psi) C_2'(\psi) &= 0; \\ U_1'(\psi) C_1'(\psi) + U_2'(\psi) C_2'(\psi) &= -i\lambda g(0), \end{aligned} \tag{8.26}$$

whose solution consists of expressions

$$C_1'(\psi) = i\lambda g(0) \frac{U_2}{W}; \quad C_2'(\psi) = -i\lambda g(0) \frac{U_1}{W}. \tag{8.27}$$

The quantity

$$W = \begin{vmatrix} U_1(\psi) & U_2(\psi) \\ U_1'(\psi) & U_2'(\psi) \end{vmatrix} \tag{8.28}$$

is called the Wronskian in the theory of differential equations. When the Wronskian is equal to zero, the functions are linearly dependent. Since in our case they are linearly independent, the Wronskian does not vanish. Furthermore, it can be shown that it is a constant quantity. Indeed, let us write two relationships which are identical for the Airy function

$$U_1' + \psi U_1 = 0, \quad U_2' + \psi U_2 = 0.$$

Multiplying the first of these by $(-U_2)$ in the second by U_1 and adding them, we obtain

$$0=(U_2'U_1-U_1'U_2)=(U_2'U_1-U_1'U_2)'=\frac{dW}{d\psi},$$

from which

$$W=C=\begin{vmatrix} U_1(\psi_0) & U_2(\psi_0) \\ U_1'(\psi_0) & U_2'(\psi_0) \end{vmatrix},$$

where ψ_0 is any point from the interval over which the variable ψ changes its value.

Now it follows from (8.27) that

$$C_1(\psi)=i\lambda\frac{g(0)}{W}\int_{\psi_2}^{\psi}U_2(\tau)\,d\tau,\quad C_2(\psi)=-i\lambda\frac{g(0)}{W}\int_{\psi_1}^{\psi}U_1(\tau)\,d\tau \tag{8.29}$$

and

$$H^{(2)}(\psi)=-i\frac{\lambda g(0)}{W}\left\{U_1(\psi)\int_{\psi}^{\psi_2}U_2(\tau)\,d\tau+U_2(\psi)\int_{\psi_1}^{\psi}U_1(\tau)\,d\tau\right\}.$$

Assuming that in the above

$$\psi_1=-i\infty,\ \psi_2=i\infty,\ U_1=h_1,\ U_2=h_2,$$

we obtain

$$H^{(2)}(\psi)=-i\lambda g(0)\,e_0(\psi), \tag{8.30}$$

where

$$e_0(\psi)=\frac{1}{W}\left\{h_2(\psi)\int_{-i\infty}^{\psi}h_1(\tau)\,d\tau+h_1(\psi)\int_{\psi}^{i\infty}h_2(\tau)\,d\tau\right\}$$

is a function which has been considered in detail by S. A. Tumarkin (ref. 328). Substituting expression (8.30) into equation (8.23), we find that $e_0(\psi)$ satisfies the equation

$$\frac{d^2e_0}{d\psi^2}+\psi e_0=1. \tag{8.32}$$

We shall represent the function e_0 by a series which converges over the entire complex plane and which assumes the following form for purely imaginary values of the argument

$$\begin{aligned} e_0(iy)=e_0(0)\left[1-\frac{1}{3!}(iy)^3+\frac{4}{6!}(iy)^6-\frac{4\cdot7}{9!}(iy)^9+\right.\\ \left.+\frac{4\cdot7\cdot10}{12!}(iy)^{12}-\ldots\right]+e_0'(0)\,iy\left[1-\frac{2}{4!}(iy)^3+\frac{2\cdot5}{7!}(iy)^6-\right.\\ \left.-\frac{2\cdot5\cdot8}{10!}(iy)^9+\ldots\right]+(iy)^2\left[\frac{1}{2!}-\frac{3}{5!}(iy)^3+\frac{3\cdot6}{8!}(iy)^6-\right.\\ \left.-\frac{3\cdot6\cdot9}{11!}(iy)^9+\ldots\right], \end{aligned} \tag{8.33}$$

where

$$e_0(0)=3^{-\frac{2}{3}}\Gamma\left(\frac{1}{3}\right)=1.288;\ e_0'(0)=-3^{-\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)=$$
$$=-0.9389.$$

Table 4, taken from reference 332, presents the values of the function

$$E(y) = -e_0(iy) \tag{8.34}$$

when $0 \le y \le 6.0$. For negative values of y we should use the following even and odd properties of the introduced function

$$\begin{gathered} \operatorname{Re}E(-y)=\operatorname{Re}E(y);\ \operatorname{Im}E(-y)=-\operatorname{Im}E(y);\\ \operatorname{Re}E'(-y)=-\operatorname{Re}E'(y);\ \operatorname{Im}E'(-y)=\operatorname{Im}E'(y);\\ \left(E(-y)=\overline{E(y)};\ E'(-y)=-\overline{E'(y)}\right). \end{gathered} \tag{8.35}$$

For $|y| > 6.0$ the values of the function are computed by means of its asymptotic representation

$$E(y)=-\frac{1}{iy}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(3k-1)!}{(k-1)!3^{k-1}(iy)^{3k+1}}=i\frac{1}{y}+\frac{2}{y^4}-i\frac{40}{y^7}+\ldots \tag{8.36}$$

By using relationships (8.32) and (8.34) it is easy to see that the function E (y) is a solution of the equation /324

$$\frac{d^2E(y)}{dy^2}-iyE(y)=1. \tag{8.37}$$

Noting that according to (8.20) and (8.3) g (0) = $f_k(0)=\frac{i}{2}$, we write the particular solution H = $H^{(1)}$ + $H^{(2)}$ in the final form

$$H(\theta)=\frac{i}{2}\left\{\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)}{\sin^3\theta}}l_k-1\right\}\frac{1}{\omega(\theta)}-\frac{\lambda}{2}E\{-\lambda\omega(\theta)\} \tag{8.38}$$
$$(l_1=\cos\theta,\ l_2=1),$$

which, in essence, coincides with the first approximation of S. A. Tumarkin (ref. 328).

For large values of $|\lambda\omega(\theta)|$ the function $E\{-\lambda\omega(\theta)\}$ may be replaced by the first term of its expansion (8.36)

$$E\{-\lambda\omega(\theta)\}\approx\frac{i}{(-\lambda\omega(\theta))}. \tag{8.39}$$

In this case the particular solution takes the form

$$H(\theta)\approx\sqrt[4]{\frac{(1+\alpha\sin\theta)^5}{\omega(\theta)\sin^3\theta}}f_k(\theta)=\frac{g(\omega)}{\omega}, \tag{8.40}$$

which, as we can easily see, corresponds to the adoption of a "membrane" solution for equation (8.19). Thus, the constructed solution, which coincides with the "membrane" solution for large values of the argument, unlike the preceding one, is valid in the neighborhoods θ = 0.

Reference 328 presents more exact expressions for the particular solution. We should note, however, that for practical purposes the approximate solution presented above is quite sufficient.

The particular solution presented in the works of Clark and Reissner (refs. 332, 336) has a slightly different form. When applied to the above equation it has the following form

$$H(\theta)=i\lambda g(\omega)E\{-\lambda\omega\}=i\lambda\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)^5}{\sin^3\theta}}f_k(\theta)E\{-\lambda\omega(\theta)\}. \tag{8.41}$$

In view of relationships (8.39) this expression tends asymptotically to the membrane expression, and when $\theta \to 0$ $\left(\text{because } \begin{matrix} g(\omega) \to g(0) \\ \omega \to 0 \end{matrix}\right)$, it approaches solution (8.30), corresponding to equation (8.19), if the right side of the latter is replaced with its value when $\theta = 0$ ($\omega = 0$). This solution is somewhat less accurate than (8.38). It has been refined in a later work by Clark (ref. 333).

Let us return to expression (8.38). According to (8.4) it is associated with the following solutions of equations (6.1) and (6.4) /325

$$X(\theta)=iL_1(\theta)-\frac{\lambda}{2}\varphi(\theta)E\{-\lambda\omega(\theta)\}; \tag{8.42}$$

$$W(\theta)=iL_2(\theta)-\frac{\lambda}{2}\varphi(\theta)E\{-\lambda\omega(\theta)\}, \tag{8.43}$$

where

$$\begin{aligned} L_1(\theta)&=\frac{1}{2}\left\{\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)}{\sin^3\theta}}\cos\theta-1\right\}\frac{1}{\sqrt[4]{\sin\theta(1+\alpha\sin\theta)\omega^3(\theta)}};\\ L_2(\theta)&=\frac{1}{2}\left\{\sqrt[4]{\frac{\omega^3(\theta)(1+\alpha\sin\theta)}{\sin^3\theta}}-1\right\}\frac{1}{\sqrt[4]{\sin\theta(1+\alpha\sin\theta)\omega^3(\theta)}}. \end{aligned} \tag{8.44}$$

The values of the functions $L_1(\theta)$, $L_2(\theta)$ and their first derivatives are shown in tables 5 and 6. It follows from (8.42) and (8.43) that

$$\frac{dX}{d\theta}=iL_1'(\theta)+\frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)E'(-\lambda\omega(\theta))-\frac{\lambda}{2}\varphi'(\theta)E(-\lambda\omega(\theta)); \tag{8.45}$$

$$\frac{dW}{d\theta}=iL_2'(\theta)+\frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)E'(-\lambda\omega(\theta))-\frac{\lambda}{2}\varphi'(\theta)E(-\lambda\omega(\theta)). \tag{8.46}$$

In order to accelerate calculations, tables 7-9 show the values of the functions $\varphi(\theta)$, $\varphi'(\theta)$, $\varphi(\theta)\omega'(\theta)$. We should point out that even for values λ, which are not too large, the following simplified expressions yield results which are of sufficient accuracy for practical purposes

$$X = W = -\frac{\lambda}{2}\varphi(\theta)E(-\lambda\omega(\theta));$$

$$\frac{dX}{d\theta} = \frac{dW}{d\theta} = \frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)E'(-\lambda\omega(\theta)) \qquad (8.47)$$

$$\left(\lambda = \left(\frac{ab}{c}\right)^{\frac{1}{3}}\right).$$

CHAPTER XVI. TOROIDAL EXPANSION TANKS

In the present chapter we shall consider circular toroidal shells closed with respect to φ and open with respect to another coordinate (θ). Such shells are widely used in machine building as thin-walled structures having the form of shells of revolution. Thus thermal expansion vessels frequently contain toroidal regions. Smooth transitions from one diameter to another are also usually achieved by using a section of a toroidal surface. The wide application of toroidal shells and the specific difficulties associated with their design (we have discussed these in the preceding chapter) have attracted a large number of research people to their investigation, both in the USSR and abroad. A detailed list of references and the critical survey of basic works in this field is given in reference 331. /326

We refer the reader to this work, and below we shall limit ourselves to the consideration of a few problems of prime interest. The Soviet work in this field consists primarily of further development and expansion of the V. V. Novozhilov method presented in reference 33. The works of Clark, which are also of interest, are less known. Below we shall present some of his results with insignificant changes. We shall briefly consider also the problem of computing toroidal shells with antisymmetric loading.

1. Axial Extension of a Tubular Expansion Tank

Let us consider a complete toroidal shell which is cut along the parallel circle $\theta = -\frac{\pi}{2}$ and whose edges are welded to cylindrical tubes (fig. 35). The design load is an axial force P^0.

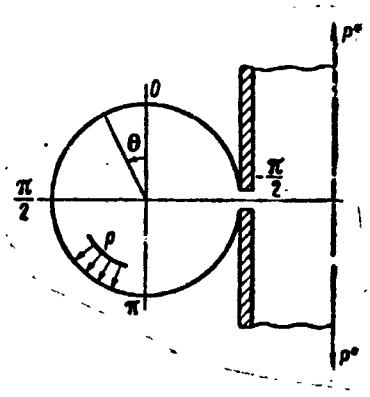

Figure 35

First of all, we shall not consider dislocation displacements in this chapter, because in particular $\Omega_z^1 = 0$. Furthermore, in the present section /327

$p = 0$. Therefore, according to (Chapter XV, $4.7)_2$ we have ($\varphi_0 = 2\pi$)

$$Q_z = \frac{Ehc\bar{\Omega}_z^1}{2\pi a(1+\alpha\sin\theta)}.$$

Using this expression to compute the axial force we find

$$P^0 = \int_0^{2\pi} Q_z a(1+\alpha'\sin\theta)\,d\varphi = Ehc\bar{\Omega}_z^1, \text{ i.e., } \bar{\Omega}_z^1 = \frac{P^0}{Ehc}. \tag{1.1}$$

For reasons which will be discussed below we limit ourselves to a particular solution thereby assuming that (compare (Chapter XV, 4.4) with (Chapter XV, 6.1))

$$\tilde{\vartheta}^n = \frac{2\bar{\Omega}_z^1}{2\pi} X = \frac{P^0}{\pi Ehc} X. \tag{1.2}$$

By introducing the resulting expressions into the relationships in section 4, Chapter XV we obtain, first of all,

$$\frac{\sigma_{T_1}}{\sigma_V} = \frac{\sin\theta}{1+\alpha\sin\theta} + 2\frac{\cos\theta}{1+\alpha\sin\theta}\operatorname{Im}X;$$

$$\frac{\sigma_{T_2}}{\sigma_V} = \frac{2}{\alpha}\operatorname{Im}\frac{dX}{d\theta}; \tag{1.3}$$

$$\frac{\sigma_{M_1}}{\sigma_V} = \sqrt{\frac{12}{1-\nu^2}}\frac{1}{\alpha}\operatorname{Re}\left(\frac{dX}{d\theta} + \nu\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right);$$

$$\frac{\sigma_{M_2}}{\sigma_V} = \sqrt{\frac{12}{1-\nu^2}}\frac{1}{\alpha}\operatorname{Re}\left(\nu\frac{dX}{d\theta} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right). \tag{1.4}$$

Here $\sigma_V = \frac{P^0}{2\pi ah}$ is a stress occurring in the cylindrical tube of radius a and thickness h, which is subjected to tension by the same force P^0.

Furthermore, for the complex boundary quantities

$$\tilde{Q}_\rho \equiv Q_\rho - i\frac{Ehc}{a}\frac{\vartheta}{1+\alpha\sin\theta}, \quad \tilde{M}_1 = M_1 + i\frac{Ehc}{a}\frac{u_\rho}{1+\alpha\sin\theta} \tag{1.5}$$

we have the following simplified expressions

$$\widetilde{Q}_\rho = -i\frac{P^0}{\pi a}\frac{1}{1+\alpha\sin\theta}X;\quad \widetilde{M}_1 = \frac{cP^0}{\pi b}\left(\frac{dX}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right). \tag{1.6}$$

The vertical mutual separation of the edges is determined by means of the following equation

$$\delta_V = 2\left[u_z\left(-\frac{\pi}{2}\right)-u_z\left(\frac{\pi}{2}\right)\right] = \frac{2bP^0}{\pi Ehc}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,\mathrm{Re}\,X d\theta. \tag{1.7}$$

The latter can be written in a slightly different form /328

$$\delta_V = \delta_0\left(\frac{4}{\pi 2k^2}\right)\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}}\cos\theta\,\mathrm{Re}\,X d\theta, \tag{1.8}$$

where

$$\delta_0 = \frac{b^3P^0}{2aD}\quad\left(D=Ehc^3=\frac{Eh^3}{12(1-\nu^2)},\ 2k^2=\frac{ab}{c}\right) \tag{1.9}$$

is the separation of the edges of the cylindrical tube with radius b, thickness h and length 2πa, which has been cut under the action of the same tension force P^0 (i.e., a tube obtained by cutting the above torus along the meridian followed by its unbending). In this case it is assumed that the edge of the tube does not rotate during the separation.

From the symmetry of the problem in question we have the following boundary conditions

$$\vartheta = 0;\quad Q_\rho = 0 \text{ when } \theta = \frac{\pi}{2}. \tag{1.10}$$

We set the same boundary conditions along the edge of the shell

$$\vartheta = 0,\quad Q_\rho = 0 \text{ when } \theta = -\frac{\pi}{2}. \tag{1.11}$$

The first of these is justified by the fact that a tube which contains a torus along its edge usually has a greater flexural rigidity. The second condition

is usually not satisfied. However, as calculations have shown, for moderate values of $2k^2$ this has a very small effect on the magnitude of the maximum stresses and on shell rigidity. The first relationships from (1.5) and (1.6) show that the conditions (1.10) and (1.11) are equivalent to the following

$$X\left(-\frac{\pi}{2}\right)=X\left(\frac{\pi}{2}\right)=0. \tag{1.12}$$

For the adopted boundary conditions we can limit ourselves to the consideration of the particular solution. Indeed, relationships (Chapter XV, 8.40 and 8.3) show that for large $2k^2$ $H(\theta)$ satisfies conditions (1.12), and it is not necessary to bring the solution of the homogeneous equation into the picture. Thus we can assume

$$X=\varphi(\theta)H(\theta)=iL_1(\theta)-\frac{\lambda}{2}\varphi(\theta)E(-\lambda\omega(\theta)); \tag{1.13}$$

$$\frac{dX}{d\theta}=iL_1'(\theta)+\underline{\frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)E'(-\lambda\omega(\theta))}-\underline{\frac{\lambda}{2}\varphi'(\theta)E(-\lambda\omega(\theta))} \tag{1.14}$$

$$\left(\lambda=\sqrt[3]{2k^2}=\left(\frac{\sqrt{12(1-\nu^2)}ab}{h}\right)^{\frac{1}{3}}\right).$$

Substituting X and $\frac{dX}{d\theta}$ into expressions (1.3) we determine the values /329 of stresses in the shell. Figures 36 and 37 represent the most interesting stresses: the ring tangential stress σ_{T_2} and the meridian flexural stress σ_{M_1} computed for a shell with parameters

$$\begin{gathered}a = 8.49";\ E = 29.0\cdot10^6 \text{ psi}\\ h = 0.066";\ b = 2.16";\ \nu = 0.3\end{gathered} \tag{1.15}$$

(1 inch = 2.54 cm; 1 pound = 0.454 kg; 1 psi = 0.0703 $\frac{kg}{cm^2}$)

$$2k^2 = 27.5;\ \alpha = 0.254;\ P = -1000 \text{ lbs} \tag{1.16}$$

The solid line shows the values computed by means of the total equations, while the broken line shows results obtained by means of simplified equations (when only the principal underlined terms are retained in relationships (1.3) and (1.14)). The circles show the extremal values of the stresses (ref. 337).

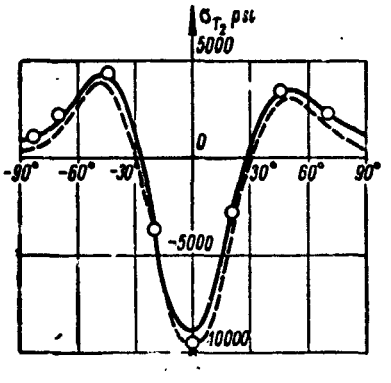

Figure 36

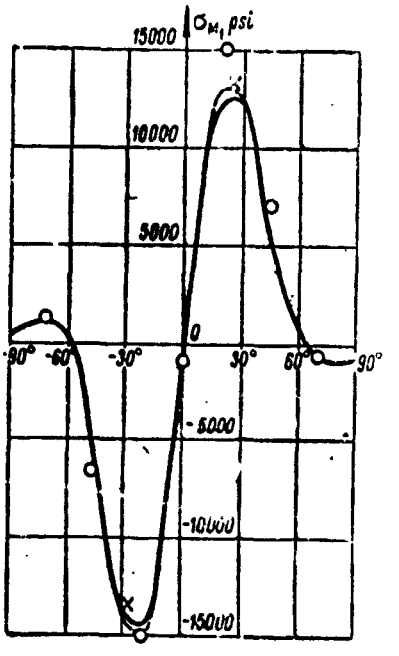

Figure 37

As we can see from the graphs, the simplified equations give us a good approximation for the particular value of the principal parameter ($2k^2 = 27.5$).

Using the trapezoidal equation to compute the integral in (1.7) we obtain /330

$$\frac{\delta_V}{\delta_0} = 0.0726. \tag{1.17}$$

The corresponding extremal value is $\frac{\delta_V}{\delta_0} = 0.0825$.

The asymptotic method makes it possible for us to obtain simple equations for the quantities of prime interest. Let us obtain these equations by following Clark (ref. 332). First of all, from $(1.3)_2$ and table 4 it follows that σ_{T_2} assumes a maximum value when $\theta_* = 0$. In this case, according to (1.14) and $(1.3)_2$

$$\left.\frac{dX}{d\theta}\right|_{\theta_*} = \frac{\lambda^2}{2} E'(0);$$
$$\frac{(\sigma_{T_2})_{max}}{\sigma_V} = \frac{1}{\alpha}\left(\gamma\sqrt{12(1-\nu^2)}\,\frac{ab}{h}\right)^{\frac{2}{3}} \cdot 0.939. \tag{1.18}$$

From this we obtain the first equation of Clark

$$\frac{(\sigma_r)_{max}}{\sigma_V} = 2.15\,(1-\nu^2)^{\frac{1}{3}}\alpha^{-\frac{1}{3}}\beta^{\frac{2}{3}}, \tag{1.19}$$

where

$$\alpha = \frac{b}{a}, \quad \beta = \frac{a}{h}. \tag{1.20}$$

Furthermore, by taking into account the fact that $\frac{dX}{d\theta} \gg X$, we should adopt the following simplified expression for σ_{M_1}

$$\frac{\sigma_{M_1}}{\sigma_V} = \sqrt{\frac{12}{1-\nu^2}}\,\frac{1}{\alpha}\,\mathrm{Re}\,\frac{dX}{d\theta}. \tag{1.21}$$

This quantity reaches a maximum value when

$$\mathrm{ReE}'(y) = \max, \quad \text{i.e.} \quad \mathrm{ReE}''(y) = 0. \tag{1.22}$$

According to (Chapter XV, 8.37) such points correspond to the roots of equation

$$y\,\mathrm{Im}\,E(y) = 1. \tag{1.23}$$

The first root is equal

$$y_* = (-\lambda\omega_*) = \pm 1.225; \quad \omega_* = \mp\frac{1.225}{\lambda}. \tag{1.24}$$

By using table 1 it is easy to find $\vartheta_* = \theta(\omega_*)$. For large values of λ we can assume (Chapter XV, 8.11) approximately that

$$\theta_* \approx \omega_* = \mp\frac{1.225}{\lambda}. \tag{1.25}$$

Since

$$\mathrm{ReE}'\,(-\lambda\omega_*) = 0.753,$$

we find, from (1.21) and (1.14)

$$\frac{\sigma_{max}}{\sigma_V} = \sqrt{\frac{12}{1-\nu^2}}\,\frac{1}{2\alpha}\left(\frac{\sqrt{12(1-\nu^2)}\,\alpha b}{h}\right)^{\frac{2}{3}} 0.753\,\varphi(\theta_*)\,\omega'(\theta_*). \tag{1.26}$$

Thus we obtained (with an insignificant refinement) the second equation of Clark /331

$$\frac{(\sigma_{M_1})_{max}}{\sigma_V} = \pm 2.99(1-\nu^2)^{-\frac{1}{6}}\alpha^{-\frac{1}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\,\omega'(\theta_*). \tag{1.27}$$

Finally, let us consider equation (1.8). For large values of $2k^2$ we may assume that

$$\mathrm{Re}X = -\frac{\lambda}{2}\varphi(\theta)\,\mathrm{Re}E(-\lambda\omega).$$

Then

$$J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,\mathrm{Re}\,X d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\omega'(\theta)}\varphi(\theta)\,\mathrm{Re}\,E(-\lambda\omega)\,d(-\lambda\omega). \tag{1.28}$$

Since $\lambda \gg 1$, we can see from table 4 that the quantity $\mathrm{Re}E(-\lambda\omega)$ is significant only for small value of $|\theta|$. Therefore we may assume that $\frac{\cos\theta}{\omega'(\theta)}\varphi(\theta) \approx 1$ in the expression under the integral sign. Let us now proceed with the integration of $y = -\lambda\omega$. Since the limits of integration $-\frac{\pi}{2}, \frac{\pi}{2}$ correspond to large quantities $-\lambda\omega\left(-\frac{\pi}{2}\right) = \lambda\left|\omega\left(-\frac{\pi}{2}\right)\right|$ and $-\lambda\omega\left(\frac{\pi}{2}\right) = -\lambda\left|\omega\left(\frac{\pi}{2}\right)\right|$, we may assume approximately that the integral has infinite limits. Then, by taking into account the even properties (Chapter XV, 8.35) we obtain

$$J = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{Re}\,E(y)\,dy = -\int_{0}^{\infty}\mathrm{Re}\,E(y)\,dy. \tag{1.29}$$

The following equality follows from the integral representation of function E(y) (ref. 328)

$$\int_{0}^{\infty}\mathrm{Re}\,E(y)\,dy = -\frac{\pi}{2}. \tag{1.30}$$

Taking this into account, we obtain the third equation of Clark

$$\delta_V = \delta_0 \frac{2}{2k^2} \quad \left(\delta_0 = \frac{b^3 p_0}{2aD}\right). \tag{1.31}$$

The separation of the edges computed by means of this equation for parameters of (1.16) gives us $\delta_V = 0.0727\ \delta_0$--a quantity which is very close to (1.17). The asymptotic values of the maximum stresses computed by means of the first and second equation of Clark are shown by crosses in figures 37 and 36. /332

2. Tubular Expansion Tank Under the Action of an Axial Force and Normal Pressure. Closed Torus

Let us assume that the shell considered in the preceding section is loaded with normal pressure in addition to the axial force. Neglecting the small terms (which ones?) in the relationships in section 4, Chapter XV we obtain the following expression for the stresses

$$\begin{aligned}
\sigma_{T_1} &= \sigma_p \left\{\frac{1+0{,}5\alpha\sin\theta}{1+\alpha\sin\theta}\right\} + \sigma_V \left\{\frac{\sin\theta}{1+\alpha\sin\theta} + 2\frac{\cos\theta}{1+\alpha\sin\theta}\operatorname{Im} X\right\};\\
\sigma_{T_2} &= \sigma_p \left\{\frac{1}{2}\right\} + \sigma_V \left\{\frac{2}{\alpha}\operatorname{Im}\frac{dX}{d\theta}\right\};\\
\sigma_{M_1} &= \sigma_V \left\{\sqrt{\frac{12}{1-\nu^2}}\operatorname{Re}\left(\frac{1}{\alpha}\frac{dX}{d\theta} + \frac{\nu\cos\theta}{1+\alpha\sin\theta}X\right)\right\},\\
\sigma_{M_2} &= \sigma_V \left\{\sqrt{\frac{12}{1-\nu^2}}\operatorname{Re}\left(\frac{\nu}{\alpha}\frac{dX}{d\theta} + \frac{\cos\theta}{1+\alpha\sin\theta}X\right)\right\}
\end{aligned} \tag{2.1}$$

Here, in addition to $\sigma_V = \dfrac{P^0}{2\pi ah}$ we have introduced the quantity

$$\sigma_p = \frac{pb}{h}, \tag{2.2}$$

which represents a ring stress and which corresponds to the torus of the cylindrical tube. The quantity P^0 contained in σ_V is easily associated with the axial tension force in the tube

$$F_z = \{2\pi a(1+\alpha\sin\theta)\,Q_z\}_{\theta=-\frac{\pi}{2}}. \tag{2.3}$$

Indeed, if we substitute according to (Chapter XV, $4.7)_2$ and (1.1) the expression

$$Q_z = Q_z^* = \frac{pb}{2}\frac{2+\alpha\sin\theta}{1+\alpha\sin\theta}\sin\theta + \frac{P^0}{2\pi a}\frac{1}{1+\alpha\sin\theta}, \tag{2.4}$$

in the above equation, we obtain

$$F_z = P^0 - (2-\alpha)\,\pi abp. \tag{2.5}$$

We can also see from (2.4) that P^0 is an axial force acting on the parallel circle $\theta = 0$.

Furthermore, by using equalities (Chapter XV, 4.6) and (1.2) we obtain/333

$$\begin{aligned}\delta_V &= 2\left[u_z\left(-\frac{\pi}{2}\right) - u_z\left(\frac{\pi}{2}\right)\right] = \\ &= 2b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\frac{pb}{2Eh}\left[(1-\nu)\sin\theta + \frac{\sin\theta}{1+\alpha\sin\theta}\right] + \frac{P^0}{\pi Ehc}\cos\theta\,\mathrm{Re}\,X\right\}d\theta.\end{aligned} \tag{2.6}$$

However, according to (Chapter XV, 6.24 ai 5.20)

$$\begin{aligned}\frac{2bpb}{2Eh}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[(1-\nu)\sin\theta + \frac{\sin\theta}{1+\alpha\sin\theta}\right]d\theta &= \frac{pb^2}{Eh}\,\frac{2\pi}{2}\,\frac{1}{2\pi}\oint\frac{\sin\theta\,d\theta}{1+\alpha\sin\theta} = \\ &= \frac{\pi pb^2}{Eh}\,\frac{1-J_1(\alpha)}{\alpha} = \frac{\pi pb^2}{Eh}\left(\frac{\sqrt{1-\alpha^2}-1}{\alpha\sqrt{1-\alpha^2}}\right)\end{aligned}$$

and

$$\delta_V = \left(\frac{\sqrt{1-\alpha^2}-1}{\alpha\sqrt{1-\alpha^2}}\right)\frac{\pi pb^2}{Eh} + \frac{2bP^0}{\pi Ehc}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,\mathrm{Re}\,X\,d\theta. \tag{2.7}$$

It is easy to obtain asymptotic equations which are analogous to those in the preceding section. As before, for large values of $2k^2$

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,\mathrm{Re}\,X\,d\theta = \frac{\pi}{2}. \tag{2.8}$$

Now equality (2.7) may be represented in the form

$$\delta_V = \left(\frac{\sqrt{1-x^2}-1}{x\sqrt{1-x^2}} \right) \frac{\pi p h^2}{Eh} + \frac{bf^0}{Ehc} \tag{2.9}$$

or

$$\delta_V = \frac{\pi b}{E} \left\{ \left(\frac{\sqrt{1-x^2}-1}{\alpha\sqrt{1-\alpha^2}} \right) \sigma_p + 4\sqrt{3(1-\nu^2)}\, \frac{a}{h}\, \sigma_V \right\}. \tag{2.10}$$

In the case of the combined action of the axial force and of the normal pressure it is necessary, generally speaking, to retain both terms containing p

as well as the terms with p^0 in these relationships. However, depending on their relative magnitudes, we can adopt certain simplifications. Let us consider three basic cases

$$\text{(a)}\ \sigma_V \gg \sigma_p.$$

In this case we can neglect all of the terms which depend on p and utilize /334 the relationships of the preceding section.

$$\text{(b)}\ \sigma_V \sim \sigma_p,$$

i.e., the case when both stresses are quantities of the same order. In this case, as we can easily see, the equations for stresses (2.1) are retained, while (2.9) and (2.10) are transformed into (1.31).

$$\text{(c)}\ \sigma_V \ll \sigma_p.$$

In this case the stresses should be established by means of the simplified equations

$$\frac{\sigma_{T_1}}{\sigma_p} = \frac{1+0{,}5\alpha\sin\theta}{1+\alpha\sin\theta};\quad \frac{\sigma_{T_2}}{\sigma_p} = \frac{1}{2}; \tag{2.11}$$

$$\frac{\sigma_{M_1}}{\sigma_V} = \sqrt{\frac{12}{1-\nu^2}}\,\frac{1}{\alpha}\,\mathrm{Re}\frac{dX}{d\theta};\quad \sigma_{M_2} = \nu\sigma_{M_1} \tag{2.12}$$

while relationships (2.9) and (2.10) are retained.

An example of the latter case is a closed toroidal shell loaded with normal pressure. Since in this case there is no separation of the edges

$$\delta_V = 0 = \frac{\pi b}{E}\left\{\frac{\sqrt{1-a^2}-1}{a\sqrt{1-a^2}}\sigma_p + 4\sqrt{3(1-\nu^2)}\frac{a}{h}\sigma_V\right\},$$

i.e.,

$$\sigma_V = \left(\frac{1-\sqrt{1-a^2}}{a\sqrt{1-a^2}}\right)\frac{1}{4\sqrt{3(1-\nu^2)}}\frac{h}{a}\sigma_p. \tag{2.13}$$

Then

$$\frac{\sigma_{M_1}}{\sigma_p} = \left(\frac{1-\sqrt{1-a^2}}{a\sqrt{1-a^2}}\right)\frac{1}{2(1-\nu^2)}\frac{h}{b}\operatorname{Re}\frac{dX}{d\theta}; \tag{2.14}$$
$$\sigma_{M_2} = \nu\sigma_{M_1}.$$

From the resulting expression we can see that in the considered problem the role of bending moments is small. It is also easy to carry out their asymptotic evaluation. Thus, by using table 4 we find from (2.14) and (1.14)

$$\frac{(\sigma_{M_1})_{\max}}{\sigma_p} \left(\frac{1-\sqrt{1-a^2}}{a\sqrt{1-a^2}}\right)\frac{1}{2(1-\nu^2)}\frac{h}{b}\left|\operatorname{Re}\frac{dX}{d\theta}(0)\right|;$$

$$\frac{dX}{d\theta}(0) \approx \frac{\lambda^2}{2}(-1{,}288) = -\frac{1{,}288}{2}\left(\frac{\sqrt{12(1-\nu^2)}ab}{h}\right)^{\frac{2}{3}} = \tag{2.15}$$

$$= -1{,}472(1-\nu^2)^{\frac{1}{3}}\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}$$

$$\left(\beta = \frac{b}{h}\right).$$

Substituting the resulting expression into (2.15), we obtain /335

$$\frac{(\sigma_{M_1})_{\max}}{\sigma_p} = 0.736\left(\frac{1-\sqrt{1-a^2}}{a^2\sqrt{1-a^2}}\right)(1-\nu^2)^{-\frac{2}{3}}\alpha^{\frac{5}{3}}\beta^{-\frac{1}{3}}. \tag{2.16}$$

The values of the quantity

$$\gamma(a) = \frac{1-\sqrt{1-a^2}}{a^2(1-a^2)} \tag{2.17}$$

are shown in figure 38.

Let us consider two examples.

1. $$\alpha=0.1;\quad \beta=20;\quad \nu=0.3\ (2k^2=3.305\,\alpha^2=6.61),$$
$$\gamma=0.5;\quad \frac{(\sigma_{M_1})_{max}}{\sigma_p}=\frac{0.736\cdot0.5\cdot0.1\cdot0.25}{0.94\cdot2.71}=0.003.$$

The small value obtain for σ_{M_1} is associated basically with the small value of parameter α.

2. $$\alpha=0.254;\quad \beta=32.8;\quad \nu=0.3\ (2k^2=27.5),$$
$$\gamma=0.517;\quad \frac{(\sigma_{M_1})_{max}}{\sigma_p}=$$
$$=\frac{0.736\cdot0.517\cdot0.254\cdot0.402}{0.94\cdot3.2}=0.013.$$

These examples show that for moderate values of α and a closed toroidal shell, we can use the Föppl membrane solution

$$\sigma_{T_1}=\frac{1+0.5\alpha\sin\theta}{1+\alpha\sin\theta},\quad \sigma_{T_2}=\frac{1}{2}\frac{pb}{h}, \tag{2.18}$$

which is given in handbooks.

3. Flexure of a Quarter of a Torus by a Distributed Edge Moment

Let us consider the problem of flexure of a quarter toroid produced by the edge distribution of moments (fig. 39) applied at the section $\theta = 0$.

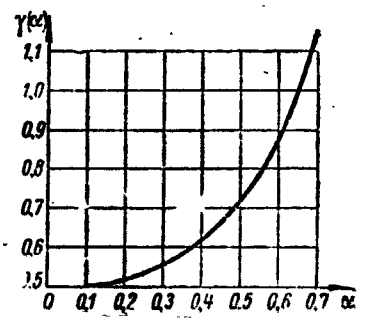

Figure 38

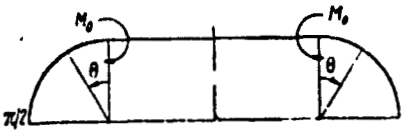

Figure 39

Since surface loading is absent, while the edge loading is self-balanced, we need only consider the homogeneous problem. The relationships in section 4, Chapter XV are written in the following manner

$$T_1 = \frac{Ehc}{b}\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Im}\tilde{\vartheta}^n \quad ; \quad M_1 = \frac{Ehc^2}{b}\operatorname{Re}\left\{\frac{d\tilde{\vartheta}}{d\theta} + \nu\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\vartheta}^n\right\};$$
$$T_2 = \frac{Ehc}{b}\operatorname{Im}\frac{d\tilde{\vartheta}^n}{d\theta} \qquad\qquad M_2 = \frac{Ehc^2}{b}\operatorname{Re}\left\{\nu\frac{d\tilde{\vartheta}^n}{d\theta} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\vartheta}^n\right\} \tag{3.1}$$

$$u_\rho = \frac{b}{2k^2}(1+\alpha\sin\theta)\operatorname{Im}\frac{d\tilde{\vartheta}^n}{d\theta}, \quad u_z = u_z(0) - b\int_0^\theta \cos\theta\operatorname{Re}\tilde{\vartheta}^n d\theta; \tag{3.2}$$

$$\tilde{Q}_\rho \equiv Q_\rho - i\frac{Ehc}{a}\frac{\vartheta}{1+\alpha\sin\theta} = -i\frac{Ehc}{a}\frac{\tilde{\vartheta}^n}{1+\alpha\sin\theta};$$
$$\tilde{M}_1 \equiv M_1 + i\frac{Ehc}{a}\frac{u_\rho}{1+\alpha\sin\theta} = \frac{Ehc^2}{b}\left(\frac{d\tilde{\vartheta}^n}{d\theta} + \frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\vartheta}^n\right). \tag{3.3}$$

Note: П = c = correction

We adopt a simple expression (Chapter XV, 8.17) for X

$$\left(\tilde{C}_1 = A_1 + iB_1, \quad \tilde{C}_2 = A_2 + iB_2\right) \tag{3.4}$$
$$\tilde{\vartheta}^n = (A_1 + iB_1)h_1(-i\lambda\omega) + (A_2 + iB_2)h_2(-i\lambda\omega),$$

from which we obtain

$$\frac{d\tilde{\vartheta}^n}{d\theta} = -i\lambda\omega'(\theta)\left\{(A_1 + iB_1)h_1'(-i\lambda\omega) + (A_2 + iB_2)h_2'(-i\lambda\omega)\right\}. \tag{3.5}$$

In order to be able to use table 3 which was prepared for the positive values of the arguments, we rewrite the last expression in a slightly different form by utilizing (Chapter XV, 8.16)

$$\tilde{\vartheta}^n = (A_1 + iB_1)\,\overline{h_2(i\lambda\omega)} + (A_2 + iB_2)\,\overline{h_1(i\lambda\omega)};$$
$$\frac{d\tilde{\vartheta}^n}{d\theta} = -i\lambda\omega'(\theta)\left\{(A_1 + iB_1)\,\overline{h_2'(i\lambda\omega)} + (A_2 + iB_2)\,\overline{h_1'(i\lambda\omega)}\right\}. \tag{3.6}$$

The boundary conditions of the considered problem

$$\begin{aligned} &M_1 = M_0, \quad Q_\rho = 0 \text{ when } \theta = 0, \\ &M_1 = 0, \quad Q_\rho = 0 \text{ when } \theta = \frac{\pi}{2}, \end{aligned} \tag{3.7}$$

may be written in the following manner, if we drop the small underlined terms in (3.3)

$$\begin{aligned} &\frac{Ehc^2}{b}\operatorname{Re}\frac{d\tilde{\vartheta}^n}{d\theta} = M_0, \quad \operatorname{Im}\tilde{\vartheta}^n = 0 \text{ when } \theta = 0, \\ &\operatorname{Re}\frac{d\tilde{\vartheta}^n}{d\theta} = 0, \quad \operatorname{Im}\tilde{\vartheta}^n = 0 \text{ when } \theta = \frac{\pi}{2}. \end{aligned} \tag{3.8}$$

Substituting expressions (3.6) and utilizing table 3, we obtain a system of equations for determining the arbitrary constants

$$\begin{aligned} &A_1 = A_2; \quad B_1 + B_2 = \frac{bM_0}{Ehc^2\lambda}\frac{1}{0.6783}; \\ &\operatorname{Im}\left\{(A_1 + iB_1)\,\overline{h_2\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)} + (A_2 + iB_2)\,\overline{h_1\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)}\right\} = 0; \\ &\operatorname{Im}\left\{(A_1 + iB_1)\,\overline{h_2'\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)} + (A_2 + iB_2)\,\overline{h_1'\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)}\right\} = 0. \end{aligned} \tag{3.9}$$

Substituting the established values of the constants in this system into expressions presented above, we obtain equations for quantities which are of interest to us. /337

If $2k^2$ is sufficiently large, we may use the approach of satisfying the boundary conditions separately by using h_1 to satisfy the boundary conditions at the edge $\theta = 0$ and using h_2 to satisfy them at the edge $\theta = \frac{\pi}{2}$. This separation of the problem is based on the exponential nature of the damping associated with the Hankel function when we move away from the respective edge.

Since the edge $\theta = \frac{\pi}{2}$ is free we obtain, first of all,

$$A_1 = B_1 = 0. \tag{3.10}$$

Then

$$A_2 = 0, \quad B_2 = \frac{bM_0}{Ehc^2\lambda}\frac{1}{0.6783}. \tag{3.11}$$

Now

$$\begin{aligned} &\operatorname{Re}\tilde{\vartheta}^n = B_2 \operatorname{Im} h_1(i\lambda\omega); \qquad \operatorname{Im}\tilde{\vartheta}^n = B_2 \operatorname{Re} h_1(i\lambda\omega); \\ &\operatorname{Re}\frac{d\tilde{\vartheta}^n}{d\theta} = B_2\lambda\omega' \operatorname{Re} h_1'(i\lambda\omega); \quad \operatorname{Im}\frac{d\tilde{\vartheta}^n}{d\theta} = -B_2\lambda\omega' \operatorname{Im} h_1'(i\lambda\omega). \end{aligned} \tag{3.12}$$

Substituting these expressions into equations (3.1), (3.2) we obtain

$$\begin{aligned} \frac{\sigma_{M_1}}{\sigma_{M_0}} &= 1.474\left\{\omega' \operatorname{Re} h_1'(i\lambda\omega) + \underline{\frac{\nu\alpha}{\lambda}\frac{\cos\theta}{1+\alpha\sin\theta}\operatorname{Im} h_1(i\lambda\omega)}\right\}, \\ \frac{\sigma_{M_2}}{\sigma_{M_0}} &= 1.474\left\{\nu\omega' \operatorname{Re} h_1'(i\lambda\omega) + \underline{\frac{\alpha}{\lambda}\frac{\cos\theta}{1+\alpha\sin\theta}\operatorname{Im} h_1(i\lambda\omega)}\right\}, \\ \frac{\sigma_{T_2}}{\sigma_{M_0}} &= 0.852\sqrt{1-\nu^2}\{-\omega' \operatorname{Im} h_1'(i\lambda\omega)\}, \\ \frac{\sigma_{T_1}}{\sigma_{M_0}} &= 0.852\sqrt{1-\nu^2}\left\{\underline{\frac{\alpha}{\lambda}\frac{\cos\theta}{1+\alpha\sin\theta}\operatorname{Re} h_1(i\lambda\omega)}\right\}. \end{aligned} \tag{3.13}$$

The underlined terms are small and may be dropped. The quantity $\omega'(\theta)$ contained here is determined from table 2. On the other hand, the quantity $\sigma_{M_0} = \frac{6M_0}{h^2}$ represents the maximum flexural stress which takes place at the edge $\theta = 0$.

Furthermore,

$$u_\rho = -5.11\sqrt{1-\nu^2}\frac{aM_0}{Eh^2}\omega'(\theta)(1+\alpha\sin\theta)\operatorname{Im} h_1'(i\lambda\omega); \tag{3.14}$$

$$\delta_V = u_z(0) - u_z\left(\frac{\pi}{2}\right) = \frac{M_0b^2}{Ehc^2\lambda\ 0.6783}\int_0^{\frac{\pi}{2}}\cos\theta \operatorname{Im} h_1(i\lambda\omega)\,d\theta. \tag{3.15}$$

From the relationships which have been presented we can see that $(\sigma_{T_2})_{max}$ = $\sigma_{T_2}(0)$. In this case, according to table 2.3, /338

$$\frac{\sigma_{T_2}(0)}{\sigma_{M_0}} = -0.333\sqrt{1-\nu^2}. \tag{3.16}$$

After minor transformations it follows from (3.14) that

$$(u_\rho)_{max} = u_\rho(0) = -0.333\sqrt{1-\nu^2}\left(\frac{a\sigma_{M_0}}{E}\right). \tag{3.17}$$

Finally

$$I = \int_0^{\frac{\pi}{2}} \cos\theta \operatorname{Im} h_1(i\lambda\omega)\, d\theta = \frac{1}{\lambda}\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\omega'(\theta)} \operatorname{Im} h_1(i\lambda\omega)\, d(\lambda\omega),$$

where $h_1(i\lambda\omega(\theta))$ is a rapidly decreasing function, so that the principal part of the integral is due to the region which adjoins $\theta = 0$. Along this region $\frac{\cos\theta}{\omega'(\theta)} \approx 1$.

Therefore

$$\begin{aligned} J &\approx \frac{1}{\lambda}\int_0^{\frac{\pi}{2}} \operatorname{Im} h_1(i\lambda\omega)\, d(\lambda\omega) = \\ &= \frac{1}{\lambda}\int_0^{\lambda\omega\left(\frac{\pi}{2}\right)} \operatorname{Im} h_1(iy)\, dy \approx \frac{1}{\lambda}\int_0^{\infty} \operatorname{Im} h_1(iy)\, dy. \end{aligned} \tag{3.18}$$

Substituting this expression into (3.15) we obtain

$$\delta_\nu = 0.852\sqrt{1-\nu^2}\,(2k^2)^{\frac{1}{3}}\frac{a\sigma_{M_0}}{E}\int_0^{\infty} \operatorname{Im} h_1(iy)\, dy. \tag{3.19}$$

Using table 3 and the trapezoidal equation to carry out the calculations, obtain

$$\int_{\gamma}^{\infty} \operatorname{Im} h_1(iy) \approx -0.873. \tag{3.20}$$

Now we have the following simple equation for the value of edge separation

$$\delta_v = -0.743 \sqrt{1-\nu^2}(2k^2)^{\frac{1}{3}} \frac{a\sigma_{M_0}}{E}. \tag{3.21}$$

By using the equations presented above we have carried out calculations for shells with the same parameters as those given in the preceding section. /340

In figure 40 the line with the points shows the stresses computed for the case which has a mutual interaction of the edges for $2k^2 = 6.61$ and $\alpha = 0.1$. The line with the crosses shows the same quantities computed by means of equations (3.13), which correspond to the separate fulfillment of the boundary conditions. As we can see by comparing both solutions, an insufficiently large value of the parameter $2k^2$ does not permit us to use the method for the separate fulfillment of the boundary conditions.

The values of the stresses (taking into account the interaction of the edges) in figure 41 are the same as in the preceding drawing and are compared

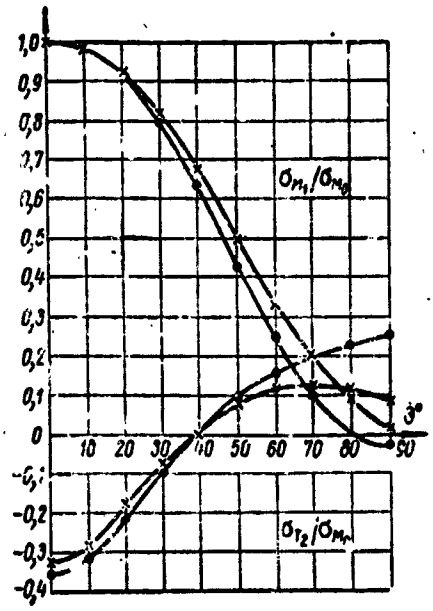

Figure 40

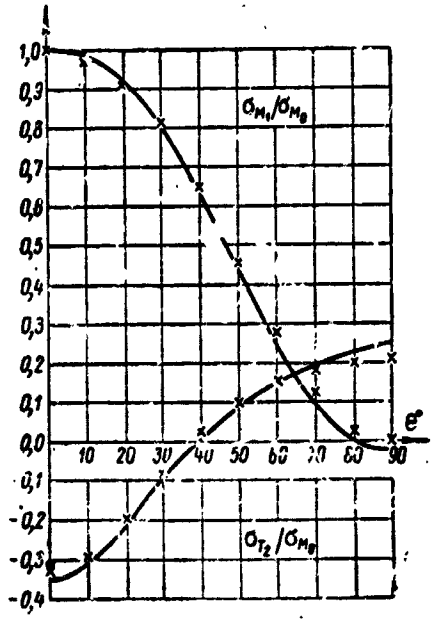

Figure 41

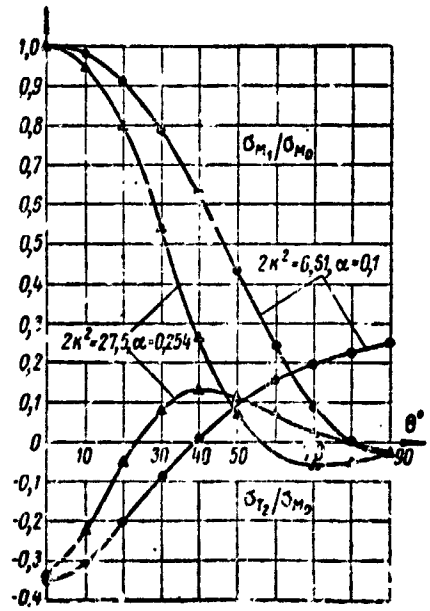

Figure 42

with those computed by means of the Clark equations (ref. 332) (crosses). As was to be expected, the solutions practically coincide. The circle shows the asymptotic value of the quantity

$$\frac{(\sigma_{T_2})_{max}}{\sigma_{M_0}}. \tag{3.16}$$

The nature of stress damping as we move away from the edge as a function of the basic parameter $2k^2$ is seen in figure 42.

4. Lenticular Expansion Tank Without a Ringplate. Perforated Tube

A lenticular expansion tank without a ring plate (fig. 43a) and the perforated tube (fig. 43b) consisting of circular toroidal shells are made up of sections of two forms

$$\text{I. for which } 0 \leq \theta \leq \frac{\pi}{2};$$
$$\text{II. for which } -\frac{\pi}{2} \leq \theta \leq 0.$$

Sections of the first type have already been considered above. Section of the second type, on the other hand, differ from those considered above only by the fact that they are rotated with respect to the horizontal. Therefore, following the designations adopted earlier we should utilize, for the considered composite shell, the following positive directions for the static and geometric quantities in line with figure 44. According to this figure the conditions of the elastic joining of both sections are written in the form

$$Q_\rho^{\mathrm{I}} = Q_\rho^{\mathrm{II}}; \quad M_1^{\mathrm{I}} = -M_1^{\mathrm{II}}; \tag{4.1}$$
$$u_\rho^{\mathrm{I}} = u_\rho^{\mathrm{II}}; \quad \vartheta^{\mathrm{I}} = -\vartheta^{\mathrm{II}};$$
$$(P^0)^{\mathrm{I}} = P^0 = -(P^0)^{\mathrm{II}} \quad \text{при } \theta^{\mathrm{I}} = \theta^{\mathrm{II}} = 0. \tag{4.2}$$

Relationships (3.3) show that the real conditions at the joints of the shells (4.1) may be replaced by two complex conditions /342

$$\widetilde{Q}_\rho^{\mathrm{I}} = \overline{\widetilde{Q}}_\rho^{\mathrm{II}}; \quad \widetilde{M}_1^{\mathrm{I}} = -\overline{\widetilde{M}}_1^{\mathrm{II}}. \tag{4.3}$$

Here the bar over the wave indicates the joining of a complex quantity.

Taking into account equality (4.2) and the fact that

$$p^{I} = -p^{II} = p, \tag{4.4}$$

we can use relationships (1.2), (Chapter XV, 8.42, 8.45, 8.17, 8.35 and 8.16) to obtain

$$\begin{aligned}\tilde{\vartheta}^{I} &= \frac{p_0}{\pi Ehc}\left[iL_1(\theta) - \frac{\lambda}{2}\varphi(\theta)\overline{E(\lambda\omega(\theta))}\right] + \\ &\quad + \tilde{C}_1'\overline{h_2(i\lambda\omega)} + \tilde{C}_2'\overline{h_1(i\lambda\omega)}; \\ \frac{d\tilde{\vartheta}^{I}}{d\theta} &= \frac{p_0}{\pi Ehc}\left[iL_1'(\theta) - \frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)\overline{E'(\lambda\omega(\theta))} - \right. \\ &\left. - \frac{\lambda}{2}\varphi'(\theta)\overline{E(\lambda\omega(\theta))}\right] - i\lambda\omega'(\theta)\left[\tilde{C}_1'\overline{h_2'(i\lambda\omega)} + \tilde{C}_2'\overline{h_1'(i\lambda\omega)}\right];\end{aligned} \tag{4.5}$$

$$\begin{aligned}\tilde{\vartheta}^{II} &= -\frac{p_0}{\pi Ehc}\left[iL_1(\theta) - \frac{\lambda}{2}\varphi(\theta)E(-\lambda\theta)\right] + \\ &\quad + \tilde{C}_1'h_1(-i\lambda\omega) + \tilde{C}_2'h_2(-i\lambda\omega); \\ \frac{d\tilde{\vartheta}^{II}}{d\theta} &= -\frac{p_0}{\pi Ehc}\left[iL_1'(\theta) + \frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)E'(-\lambda\omega(\theta)) - \right. \\ &\left. - \frac{\lambda}{2}\varphi'(\theta)E(-\lambda\omega(\theta))\right] - i\lambda\omega'(\theta)\left[\tilde{C}_1'h_1'(-i\lambda\omega) + \tilde{C}_2'h_2'(-i\lambda\omega)\right].\end{aligned} \tag{4.6}$$

By using equalities (Chapter XV, 4.7) (dropping the small underlined terms in the latter) we write out the conditions at the joints (4.3) in an

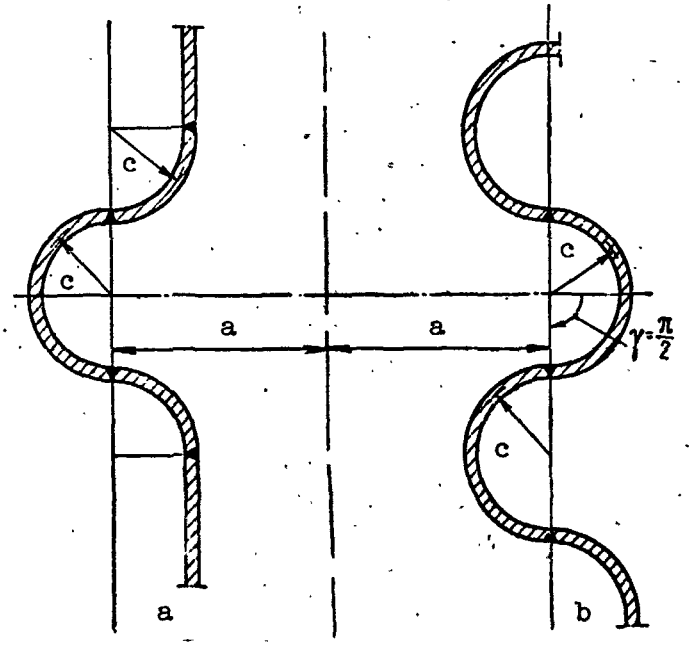

Figure 43

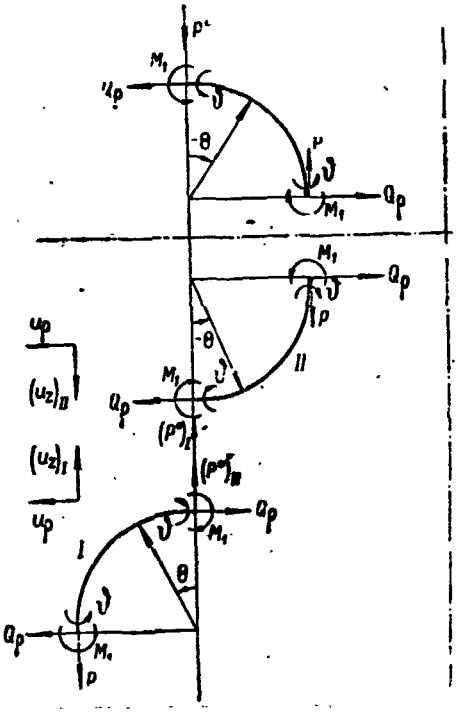

Figure 44

expanded form (the values of functions E(y), $h_1(iy)$, $h_2(iy)$ are borrowed from tables 3 and 4)

$$pb - i\frac{Ehc}{a}\left\{\frac{P^0}{\pi Ehc}\left[\underline{iL_1(0)} - \frac{\lambda}{2}(-1.288)\right] + \right.$$
$$\left. + \widetilde{C}_1'(-i1.074) + \widetilde{C}_2'(i1.074)\right\} = -pb + i\frac{Ehc}{a} \times$$
$$\times\left\{-\frac{P^0}{\pi Ehc}\left[-\underline{iL_1(0)} - \frac{\lambda}{2}(-1.288)\right] + \right.$$
$$\left. + \widetilde{\widetilde{C}}_1'(i1.074) + \widetilde{\widetilde{C}}_2'(-i1.074)\right\};$$
$$\frac{P^0}{\pi Ehc}\left[\underline{iL_1'(0)} - \frac{\lambda^2}{2}(-i0.939) + \frac{\lambda}{2}\underline{0.3\alpha(-1.288)}\right] -$$
$$- i\lambda\left[\widetilde{C}_1'(0.6783 + i0.3916) + \widetilde{C}_2'(0.6783 - i0.3916)\right] =$$
$$= \frac{P^0}{\pi Ehc}\left[-\underline{iL_1'(0)} + \frac{\lambda^2}{2}(-i0.939) + \frac{\lambda}{2}\underline{0,3\alpha(-1.288)}\right] -$$
$$- i\lambda\left[\widetilde{\widetilde{C}}_1'(0.6783 - i0.3916) + \widetilde{\widetilde{C}}_2'(0.6783 + i0.3916)\right].$$

Dropping the small underlined terms, we obtain two equations for the four arbitrary constants /343

$$-\widetilde{C}_1' + \widetilde{C}_2' + \widetilde{\widetilde{C}}_1' - \widetilde{\widetilde{C}}_2' = \frac{pba}{Ehc}\{-1.86\};$$
$$(0.6783 + i0.3916)\left(\widetilde{C}_1' - \widetilde{\widetilde{C}}_2'\right) + (0.6783 - i0.3916)\left(\widetilde{C}_2' - \widetilde{\widetilde{C}}_1'\right) = \quad (4.7)$$
$$= \frac{P^0\lambda}{\pi Ehc}0.939.$$

The remaining two equations are established form the boundary conditions when $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$. In the problem which we are considering concerning the calculation of a perforated tube, the following boundary conditions follow from considerations of symmetry

$$Q_\rho = 0, \quad \vartheta = 0 \quad \text{when } \theta = \pm\frac{\pi}{2}, \quad (4.8)$$

which, according to (1.5), may be represented in the complex form

$$\widetilde{Q}_\rho = 0 \quad \text{when } \theta = \pm\frac{\pi}{2}.$$

By means of relationships (Chapter XV, 4.7_1) we can write these in a more convenient form

$$\widetilde{\vartheta}'\left(\frac{\pi}{2}\right) = 0; \quad \widetilde{\vartheta}''\left(-\frac{\pi}{2}\right) = 0. \quad (4.9)$$

Substituting expressions (4.5) and (4.7) in the above and solving the resulting equations with respect to $\widetilde{C}_1'$ and $\widetilde{C}_2''$, we obtain

$$\widetilde{C}_1' = -\frac{\overline{h_1\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)}}{\overline{h_2\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)}}\widetilde{C}_2' + \frac{P^0}{\pi Ehc}\frac{\frac{\lambda}{2}\varphi\left(\frac{\pi}{2}\right)E\left\{\lambda\omega\left(\frac{\pi}{2}\right)\right\} - iL_1\left(\frac{\pi}{2}\right)}{\overline{h_2\left(i\lambda\omega\left(\frac{\pi}{2}\right)\right)}};$$

$$\widetilde{C}_2'' = -\frac{h_1\left(i\lambda\left|\omega\left(-\frac{\pi}{2}\right)\right|\right)}{h_2\left(i\lambda\left|\omega\left(-\frac{\pi}{2}\right)\right|\right)}\widetilde{C}_1'' - \frac{P^0}{\pi Ehc}\frac{\frac{\lambda}{2}\varphi\left(-\frac{\pi}{2}\right)E\left\{\lambda\left|\omega\left(-\frac{\pi}{2}\right)\right|\right\} - iL_1\left(-\frac{\pi}{2}\right)}{h_2\left\{\lambda\left|\omega\left(-\frac{\pi}{2}\right)\right|\right\}}. \quad (4.10)$$

(teble 3 shows that even for moderate values of λ we may assume that /344

$$\widetilde{C}_1' = \widetilde{C}_2'' = 0, \quad (4.11)$$

thereby neglecting the mutual interaction of the edges).

Thus the design of a perforated tube is reduced to the solution of the system of equations (4.7)-(4.10). After the constants are determined, the displacements and stresses are obtained from equations (Chapter XV, 4.5 and 4.6).

Following Clark (ref. 332) we obtain simple approximation equations for the maximum stresses and for the separation of the perforation wave edges. We begin by considering the case of axial symmetry.

Axial force

In this case we assume that the interaction of the edges may be neglected, i.e., we apply equalities (4.11). Solving system (4.7) with respect to the remaining constants, we obtain

$$\widetilde{C}_2' = (0.517 + i0.298)\frac{P^0\lambda}{\pi Ehc};\quad \widetilde{C}_1'' = (-0.517 + i0.298)\frac{P^0\lambda}{\pi Ehc}. \quad (4.12)$$

With a desire to obtain extremely simple equations, we retain only the basic terms in expressions (4.5)-(4.6). Then

$$\tilde{\vartheta}^{I} = \frac{P_0\lambda}{2\pi Ehc}\{-\varphi(\theta)\overline{E(\lambda\omega)} + 1.04(1+i0.578)\overline{h_1(i\lambda\omega)}\};$$

$$\frac{d\tilde{\vartheta}^{I}}{d\theta} = \frac{P_0\lambda^2\omega'(\theta)}{2\pi Ehc}\{-\varphi(\theta)\overline{E'(\lambda\omega)} - i1{,}04(1+i0.578)\overline{h_1'(i\lambda\omega)}\};$$

$$\tilde{\vartheta}^{II} = \frac{P_0\lambda}{2\pi Ehc}\{\varphi(\theta)E(-\lambda\omega) - 1.04(1-i0.578)h_1(-i\lambda\omega)\}; \qquad (4.13)$$

$$\frac{d\tilde{\vartheta}^{II}}{d\theta} = \frac{P_0\lambda^2\omega'(\theta)}{2\pi Ehc}\{-\varphi(\theta)E'(-\lambda\omega) + i1.04(1-i0{,}578)h_1'(-i\lambda\omega)\}.$$

From this and from (Chapter XV, 4.5) we see that the stresses σ_{T_2} and σ_{M_1}, are also characteristic in this case. In this case the following simplified expressions are valid for these stresses

$$\sigma_{T_2} = \pm\frac{P_0\lambda^2}{2\pi bh}\omega'(\theta)\{\varphi(\theta)\operatorname{Im}E'(\lambda|\omega|) - 1.04[\operatorname{Re}h_1'(i\lambda|\omega|) + 0{,}578\operatorname{Im}h_1'(i\lambda|\omega|)]\}; \qquad (4.14)$$

$$\sigma_{M_1} = \mp\frac{6\lambda^2}{\sqrt{12(1-\nu^2)}}\frac{P_0\omega'(\theta)}{2\pi bh}\{\varphi(\theta)\operatorname{Re}E'(\lambda|\omega|) + 1.04[\operatorname{Im}h_1'(i\lambda|\omega|) - 0{,}578\operatorname{Re}h_1'(i\lambda|\omega|)]\}. \qquad (4.15)$$

Here the upper sign should be taken for the first region $0 \le \theta \le \frac{\pi}{2}$, /345

and the lower one for the second region $-\frac{\pi}{2} \le \theta \le 0$. We note that equation (4.15) has been written down taking into account the situation that the external wall of the second region corresponds to the negative direction of the normal, so that if we retain the minus sign along the first region and a plus sign along the second, we obtain the values of flexural stresses at the external wall of the expansion tank.

First let us determine $(\sigma_{T_2})_{max}$. The analysis conducted by Clark (ref. 332) has shown that the expression contained in the figure brackets of (4.14) reaches its maximum value when

$$\lambda\omega_* = 1.89, \quad \omega_*(\theta_*) = \frac{1.89}{\lambda}. \qquad (4.16)$$

Using tables 3 and 4 we find ($\varphi(\theta_*) \approx 1$)

$$\operatorname{Im} E'(1{,}89) = -0.240; \quad \operatorname{Re} h_1'(1.89) = 0.252;$$
$$\operatorname{Im} h_1'(1.89) = -0.171.$$

Substituting these values into (4.14) and remembering that

$$\lambda = (\sqrt{12(1-\nu^2)}\,\alpha\beta)^{\frac{1}{3}} \quad \left(\beta = \frac{b}{h}\right), \tag{4.17}$$

we obtain

$$\frac{\sigma_{T_1}(\theta_*)}{\sigma_V} = -0.91\,(1-\nu^2)^{\frac{1}{3}}\alpha^{-\frac{1}{3}}\beta^{\frac{2}{3}}\omega'(\theta_*), \tag{4.18}$$

where (see (1.4))

$$\sigma_V = \frac{pQ}{2\pi ah}. \tag{1.4}$$

The expression contained in the figure brackets of (4.15) reaches its maximum value when

$$\omega_*(\theta_* = \frac{1.35}{\lambda}. \tag{4.19}$$

Following the same procedure as above we obtain

$$\frac{\sigma_{M_1}(\theta_*)}{\sigma_V} = -1.615\,(1-\nu^2)^{-\frac{1}{6}}\alpha^{-\frac{1}{3}}\beta^{\frac{2}{3}}\omega'(\theta_*). \tag{4.20}$$

The approximate expressions for the maximum values along the second region are computed: by means of equation (4.20) for the flexural stresses and by means of an equation which differs from (4.18) only in its sign for the ring tangential forces. This can easily be established from the second equality

$$\tilde{\bar{A}}^{II} = -\tilde{\bar{b}}^{I}, \quad \frac{d\tilde{\bar{\vartheta}}^{II}}{d\theta} = \frac{d\tilde{\bar{\vartheta}}^{I}}{d\theta}, \tag{4.21}$$

which, like the first, follows from relationships (4.13).

Furthermore, from (Chapter XV, 4.6_2) we obtain the following expression for the divergence of one perforation wave /346

$$\frac{\delta_V}{2} = \left[u_z(0) - u_z\left(\frac{\pi}{2}\right)\right] - \left[u_z\left(-\frac{\pi}{2}\right) - u_z(0)\right] =$$
$$= b\left\{\int_0^{\frac{\pi}{2}} \cos\theta \operatorname{Re}\tilde{\vartheta}^{I} d\theta - \int_{-\frac{\pi}{2}}^{0} \cos\theta \operatorname{Re}\tilde{\vartheta}^{II} d\theta\right\} = \frac{P_0 b}{2\pi E h c}(J_1 + J_2), \tag{4.22}$$

where

$$J_1 = -\lambda\int_0^{\frac{\pi}{2}} \cos\theta \{\varphi(\theta) \operatorname{Re} E(\lambda\omega) - 1{,}04 [\operatorname{Re} h_1(i\lambda\omega) + 0{,}578 \operatorname{Im} h_1 + (i\lambda\omega)]\} d\theta,$$
$$J_2 = \lambda\int_0^{-\frac{\pi}{2}} \cos\theta \{\varphi(\theta) \operatorname{Re} E(-\lambda\omega) - 1{,}04 [\operatorname{Re} h_1(-i\lambda\omega) + 0{,}578 \operatorname{Im} h_1(-i\lambda\omega)]\} d\theta. \tag{4.23}$$

In addition we have

$$J_1 = -\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\varphi(\theta)} \{\underline{\varphi(\theta)} \operatorname{Re} E(\lambda\omega) - 1{,}04 [\operatorname{Re} h_1(i\lambda\omega) + 0{,}578 \operatorname{Im} h_1(i\lambda\omega)]\} d(\lambda\omega).$$

As in section 1, we can replace the smooth underlined cofactors by the value of this cofactor when $\theta = 0$. Then, proceeding with the integration over $y = \lambda\omega$ and replacing $\lambda\omega\ (\frac{\pi}{2})$ by ∞, we find

$$J_1 = -\int_0^{\infty} \{\operatorname{Re} E(y) - 1{,}0_4 [\operatorname{Re} h_1(iy) + 0{,}578 \operatorname{Im} h_1(iy)]\} dy. \tag{4.24}$$

By using the substitution $y = \lambda\omega$ we obtain the same expression for J_2. Now equation (4.22) may be represented in the form

$$\frac{\delta_V}{4} = 3{,}46\sqrt{1-\nu^2}\frac{bu}{Eh}\sigma_V J_1. \tag{4.25}$$

By using designations (1.9) it may be rewritten as

$$\frac{\delta_V}{\delta_0} = \frac{2}{k^2}\frac{1}{\pi} J_1. \tag{4.26}$$

Numerical integration gives us

$$J_1 = 0.526. \tag{4.27}$$

Then

$$\frac{\delta_V}{4} = 1.825\sqrt{1-\nu^2}\,\frac{ba}{Eh}\,\sigma_V \tag{4.27}$$

or

$$\frac{\delta_V}{\delta_0} = \frac{0.670}{2k^2}. \tag{4.28}$$

Internal pressure

Let us now assume that $P^0 = 0$, $p \neq 0$. The "exact" solution should be sought in the same way as in the case of the axial force. Since we are interested in obtaining approximate equations, we neglect the interaction of the edges (by dropping the constants $\tilde{C}_1'$ and $\tilde{C}_2''$ in (4.7)). As a result we obtain equations which determine the constant $\tilde{C}_1'$ and $\tilde{C}_2''$ /347

$$\begin{aligned} &\tilde{C}_2' + \bar{\tilde{C}}_1' = \frac{pba}{Ehc}\{-1.86\}; \\ &(0.6783 - i0.3916)\left(\tilde{C}_2' - \bar{\tilde{C}}_1'\right) = 0. \end{aligned} \tag{4.29}$$

Solving these we obtain

$$\begin{aligned} &\tilde{C}_2' = \bar{\tilde{C}}_1' = \frac{pba}{Ehc}\{-0.930\}; \\ &\tilde{\delta}^1 = \frac{pba}{Ehc}\{-0.930\overline{h_1(i\lambda\omega)}\}; \\ &\frac{d\tilde{\delta}^1}{d\theta} = \frac{pba}{Ehc}\{i0.930\lambda\omega'(\theta)\,\overline{h_1'(i\lambda\omega)}\}; \end{aligned} \tag{4.30}$$

$$\tilde{\vartheta}^{II} = \frac{pba}{Ehc}\{-0.930h_1(-i\lambda\omega)\};$$

$$\frac{d\tilde{\vartheta}^{II}}{d\theta} = \frac{pba}{Ehc}\{i0.930\lambda\omega'(\theta)h_1'(-i\lambda\omega)\}.$$

Carrying out transformations analogous to those conducted above we obtain the Clark equations

$$\frac{(\sigma_{T_1})_{max}}{p} = 0.955(1-\nu^2)^{\frac{1}{6}}\alpha^{-\frac{2}{3}}\beta^{\frac{4}{3}};$$

$$\frac{(\sigma_{M_1})_{max}}{p} = 0.955(1-\nu^2)^{-\frac{1}{3}}\alpha^{-\frac{2}{3}}\beta^{\frac{4}{3}}; \qquad (4.31)$$

$$\frac{\delta_V}{4} = 2.13(1-\nu^2)^{\frac{1}{3}}\alpha^{-\frac{4}{3}}\beta^{\frac{5}{3}}J, \qquad (4.32)$$

where

$$J = -\int_0^\infty \mathrm{Re}\,h_1(iy)\,dy = 0{,}504. \qquad (4.33)$$

Equations (4.14), (4.15) were used to construct the diagrams shown in figure 45. The crosses in these diagrams show the maximum stresses computed from the asymptotic equations (4.18) and (4.20). The shell parameters were $2k^2 = 27.5$; $\alpha = 0.254$. /348

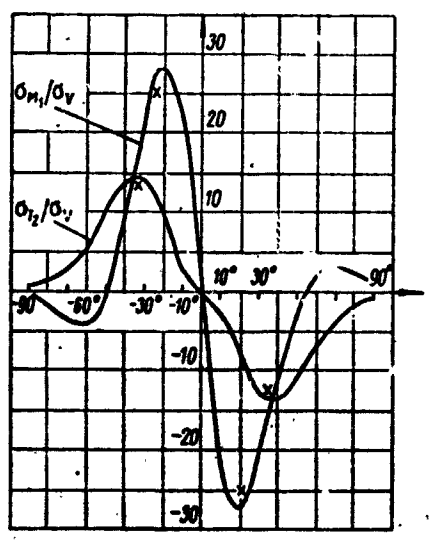

Figure 45

5. The Case of Antisymmetric Loading

The symmetric case is followed, in significance, by the case of antisymmetric loading considered in the work of the author (Some Problems of Complex Transformation in the Theory of Shells. Candidate dissertation LGU, 1955). The survey of work in this field is given in reference 331.

Below we shall consider only briefly the toroidal expansion tanks loaded antisymmetrically; we will place prime emphasis on the achievement of simple asymptotic relationships.

First of all, by separating the real parts in relationships (Chapter XV, (5.9)) and using expressions $M_1 = \mathrm{Re} M_{1,1} \begin{matrix}\cos\varphi \\ \sin\varphi\end{matrix}$

$$Q_\rho = \mathrm{Re}\,\widetilde{Q}_{\rho,1} \begin{matrix}\cos\varphi \\ \sin\varphi,\end{matrix} \quad Q_z = \mathrm{Re}\,\widetilde{Q}_{z,1} \begin{matrix}\cos\varphi \\ \sin\varphi,\end{matrix} \quad Q_{12} = \mathrm{Re}\,\widetilde{Q}_{12,1} \begin{matrix}\sin\varphi \\ \cos\varphi\end{matrix} \tag{5.1}$$

to compute the components of the principal moment and of the principal vector we obtain, for example (fig. 46),

$$B_x = \int_0^{2\pi} \{(-\sin\varphi) M_1 - [a(1+\alpha\sin\theta)\sin\varphi Q_z]\}\, a(1+\alpha\sin\theta)\, d\varphi = \\ = Ehc\{-\bar{U}_x^1 - b\cos\theta\,\bar{\Omega}_y^1\}.$$

In the same way we compute the other components, so that /349

$$\begin{aligned} B_x &= Ehc\{-\bar{U}_x^1 - b\cos\theta\bar{\Omega}_y^1\}; \quad F_x = Ehc\{-\bar{\Omega}_x^1\}; \\ B_y &= Ehc\{-\bar{U}_y^1 + b\cos\theta\bar{\Omega}_x^1\}; \quad F_y = Ehc\{-\bar{\Omega}_y^1\}, \end{aligned} \tag{5.2}$$

We introduce the designations

$$F_x^0 = -Ehc\bar{\Omega}_x^1; \quad F_y^0 = -Ehc\bar{\Omega}_y^1; \quad B_x^0 = -Ehc\bar{U}_x^1; \\ B_y^0 = -Ehc\bar{U}_y^1, \tag{5.3}$$

which are used to write the preceding equalities in the form

$$\begin{aligned} F_x &= F_x^0; \quad B_x = B_x^0 + b\cos\theta F_y^0; \\ F_y &= F_y^0; \quad B_y = B_y^0 - b\cos\theta F_x^0. \end{aligned} \tag{5.4}$$

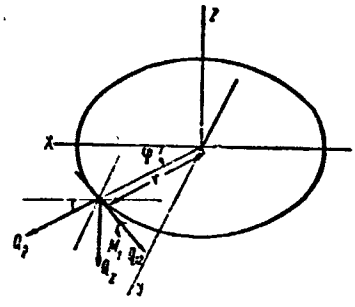

Figure 46

From this we can see that F_x^0 and F_y^0 may be identified with the constant shear forces along the axis of the shell (surface loading is absent). The meaning of B_x^0 and B_y^0, according to (5.4), is seen from the following equalities

$$B_x^0 = B_x\left(-\frac{\pi}{2}\right) = B_x\left(\frac{\pi}{2}\right); \quad B_y^0 = B_y\left(-\frac{\pi}{2}\right) = B_y\left(\frac{\pi}{2}\right). \tag{5.5}$$

The meaning of parameters $\overline{U}_x^1$, $\overline{U}_y^1$, $\overline{\Omega}_x^1$, $\overline{\Omega}_y^1$ may also be established directly from (Chapter VIII, 8.34).

Now, by comparing equations (Chapter XV, 5.4) with (Chapter XV, 6.2-6.3) we obtain, by considering only the particular solution at this time and by dropping the dislocation displacements,

$$\tilde{\chi} = \frac{4}{2\pi a}\left\{-\begin{matrix}\overline{U}_y^1 \\ \overline{U}_x^1\end{matrix}\cdot Y + \begin{matrix}\overline{\Omega}_x^1 \\ \overline{\Omega}_y^1\end{matrix}\cdot aZ\right\} = -\frac{2}{\pi a E h c}\left\{-\begin{matrix}B_y^0 \\ B_x^0\end{matrix}\cdot Y + \begin{matrix}F_x^0 \\ F_y^0\end{matrix}\cdot aZ\right\}. \tag{5.6}$$

We recall that according to (Chapter XV, 6.11 and 6.12)

$$Y = X - i\frac{\alpha}{2}\frac{\cos\theta}{1+\alpha\sin\theta}; \quad Z = \alpha W + i\frac{1-\alpha^2}{2}\frac{1}{1+\alpha\sin\theta}. \tag{5.7}$$

By means of the above expressions we reduce relationships (Chapter XV, 5.5) to the following form

$$\begin{aligned}\tilde{T}_{1,1} = &-\begin{matrix}B_y^0 \\ B_x^0\end{matrix}\left\{i\frac{2}{\pi a^2}\frac{\cos\theta}{1+\alpha\sin\theta}X - \frac{1}{\pi a^2}\frac{\sin\theta-\alpha\cos^2\theta}{(1+\alpha\sin\theta)^2}\right\} + \\ &+\begin{matrix}F_x^0 \\ F_y^0\end{matrix}\left\{i\frac{2\alpha}{\pi a}\frac{\cos\theta}{1+\alpha\sin\theta}W + \frac{\alpha^2}{\pi a}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}\right\} - i\frac{Ehc}{b}\left\{\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}^0\right\};\end{aligned} \tag{5.8}$$

$$\widetilde{T}_{2,1}=-\frac{B_y^0}{E_x}\left\{i\frac{2}{\pi ab}\left(\frac{dX}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right)-\frac{1}{\pi a^2}\frac{\sin\theta}{1+\alpha\sin\theta}\right\}+$$
$$+\frac{F_x^0}{F_y^0}\left\{i\frac{2}{\pi a}\left(\frac{dW}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}W\right)\right\}-i\frac{Ehc}{b}\left\{\frac{d\widetilde{\chi}^0}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\cos\theta}\widetilde{\chi}^0\right\};$$
$$\widetilde{S}_{,1}=\frac{B_y^0}{B_x^0}\left\{-i\frac{2}{\pi a^2}\frac{1}{1+\alpha\sin\theta}X-\frac{1}{\pi a^2}\frac{\alpha\cos\theta}{(1+\alpha\sin\theta)^2}\right\}+$$
$$+\frac{-F_x^0}{F_y^0}\left\{-i\frac{2}{\pi a}\frac{\alpha}{1+\alpha\sin\theta}W+\frac{1-\alpha^2}{\pi a}\frac{1}{(1+\alpha\sin\theta)^2}\right\}\mp$$
$$\mp i\frac{Ehc}{b}\left\{\frac{\alpha}{1+\alpha\sin\theta}\widetilde{\chi}^0\right\},$$

where according to (Chapter XV, 8.17) the general solution of the homogeneous equation $\widetilde{\chi}^0$ may be taken in the form /350

$$\widetilde{\chi}^0=\widetilde{C}_1h_1(-i\lambda\omega)+\widetilde{C}_2h_2(-i\lambda\omega). \tag{5.9}$$

Furthermore, from (Chapter XV, 5.7 and 5.6_2) we find

$$u_{p,1}=\frac{-B_y^0}{B_x^0}\left(-\frac{2\alpha}{\pi Ehc}\right)\left[\alpha\cos\theta\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,X}{1+\alpha\sin\theta'}d\theta'-\right.$$
$$\left.-\int_{\theta_0}^{\theta}\frac{(\sin\theta'+\alpha)\,\mathrm{Re}\,X}{1+\alpha\sin\theta'}d\theta'\right]+\frac{F_x^0}{F_y^0}\left(-\frac{2\alpha b}{\pi Ehc}\right)\left[\alpha\cos\theta\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,W}{1+\alpha\sin\theta'}d\theta'-\right.$$
$$\left.-\int_{\theta_0}^{\theta}\frac{(\sin\theta'+\alpha)\,\mathrm{Re}\,W}{1+\alpha\sin\theta'}d\theta'\right]+b\left\{\alpha\cos\theta\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,\widetilde{\chi}^0}{1+\alpha\sin\theta'}d\theta'-\right.$$
$$\left.-\int_{\theta_0}^{\theta}\frac{(\sin\theta'+\alpha)\,\mathrm{Re}\,\widetilde{\chi}^0}{1+\alpha\sin\theta'}d\theta'\right\}-b\cos\theta\cdot\frac{a_3}{a_5}+1\cdot\frac{a_4}{a_6}; \tag{5.10}$$

$$\vartheta_{,1}=\frac{-B_y^0}{B_x^0}\left(-\frac{2}{\pi aEhc}\right)\left[\mathrm{Re}\,X+\alpha\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,X}{1+\alpha\sin\theta'}d\theta'\right]+$$
$$+\frac{F_x^0}{F_y^0}\left(-\frac{2\alpha}{\pi Ehc}\right)\left[\mathrm{Re}\,W+\alpha\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,W}{1+\alpha\sin\theta'}d\theta'\right]+$$
$$+\left\{\mathrm{Re}\,\widetilde{\chi}^0+\alpha\int_{\theta_0}^{\theta}\frac{\cos\theta'\,\mathrm{Re}\,\widetilde{\chi}^0}{1+\alpha\sin\theta'}d\theta'\right\}-1\cdot\frac{a_3}{a_5}. \tag{5.11}$$

Finally, it follows from (Chapter XV, 5.9) that

$$\widetilde{Q}_{p,1}=\frac{-B_y^0}{B_x^0}\left\{i\frac{2}{\pi a^2}\frac{X}{1+\alpha\sin\theta}+\frac{\alpha}{\pi a^2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}\right\}+$$

$$+\frac{F_x^0}{F_y^0}\left\{i\frac{2\alpha}{\pi a}\frac{W}{1+\alpha\sin\theta}+\frac{\alpha}{\pi a}\frac{\alpha+\sin\theta}{(1+\alpha\sin\theta)^2}\right\}-i\frac{Ehc}{a}\left\{\frac{\tilde{\chi}^0}{1+\alpha\sin\theta}\right\};$$

$$\tilde{Q}_{12,1}=\frac{B_y^0}{B_x^0}\left\{-i\frac{2}{\pi a^2}\frac{X}{1+\alpha\sin\theta}-\frac{\alpha}{\pi a^2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}\right\}+$$

$$+\frac{-F_x^0}{F_y^0}\left\{-i\frac{2\alpha}{\pi a}\frac{W}{1+\alpha\sin\theta}+\frac{1-\alpha^2}{\pi a}\frac{1}{(1+\alpha\sin\theta)^2}\right\}\mp$$

$$\mp i\frac{Ehc}{a}\left\{\frac{\tilde{\chi}^0}{1+\alpha\sin\theta}\right\};$$

$$\tilde{Q}_{s,1}=\frac{-B_y^0}{B_x^0}\left\{\frac{2c}{\pi a^2 b}\frac{1}{1+\alpha\sin\theta}\left(\frac{dX}{d\theta}+2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right\}-\right.$$

$$\left.-\frac{1}{\pi a^2}\frac{1}{(1+\alpha\sin\theta)^2}\right\}+ \qquad (5.12)$$

$$+\frac{F_x^0}{F_y^0}\left\{\frac{2c}{\pi a^2}\frac{1}{1+\alpha\sin\theta}\left(\frac{dW}{d\theta}+2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}W\right)-\frac{\alpha}{\pi a}\frac{\cos\theta}{(1+\alpha\sin)^2}\right\}-$$

$$-\frac{Ehc^2}{ab(1+\alpha\sin\theta)}\left\{\frac{d\tilde{\chi}^0}{d\theta}+2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}^0\right\};$$

$$\tilde{M}_{1,1}=\frac{-B_y^0}{B_x^0}\left\{-\frac{2c}{\pi ab}\left(\frac{dX}{d\theta}+2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right)\right\}+$$

$$+\frac{F_x^0}{F_y^0}\left\{-\frac{2c}{\pi a}\left(\frac{dW}{d\theta}+2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}W\right)\right\}+$$

$$+\frac{Ehc^2}{b}\left\{\frac{d\tilde{\chi}^0}{d\theta}+2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\tilde{\chi}^0\right\}.$$

6. Tubular Expansion Tank Under Conditions of Flexure and Shear

Flexure of a Tubular Expansion Tank Produced by a Moment

Let us assume that a complete tubular expansion tank is bent by a moment B_y^0 (fig. 47). To consider this problem in terms of the relationships presented above, we should assume that only the constant B_y^0 is different from zero. We adopt the following boundary conditions /351

$$Q_{\rho,1}\left(\frac{\pi}{2}\right)=0;\quad \chi\left(\frac{\pi}{2}\right)=0; \qquad (6.1)$$

$$Q_{\rho,1}\left(-\frac{\pi}{2}\right)=0;\quad \chi\left(-\frac{\pi}{2}\right)=0. \qquad (6.2)$$

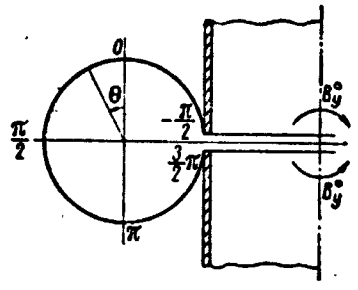

Figure 47

The boundary conditions are satisfied at the section $\theta = \frac{\pi}{2}$, because according to the symmetry of the problem, the cutting forces at this point are absent and there is no elastic rotation χ. The second condition in (6.2) is also satisfied approximately, because the edge of the thicker tube has a much higher flexural rigidity compared with the edge of the expansion tank. To support the remaining condition we present the following argument: the edge of the cylindrical tube which operates in flexure cannot produce a substantial thrust. Therefore the violation of the first condition apparently is not very substantial and (for excessively small $2k^2$), does not introduce a significant error into the solution. /352

According to (Chapter XV, 5.10), conditions (6.1) and (6.2) are equivalent to the following complex conditions

$$\widetilde{Q}_{\rho,1}\left(-\frac{\pi}{2}\right)=\widetilde{Q}_{\rho,1}\left(\frac{\pi}{2}\right)=0. \tag{6.3}$$

In view of this

$$\frac{\sigma_{T_1,1}}{\sigma_0}=\frac{\operatorname{Re}\widetilde{T}_{1,1}}{\sigma_0 h}= =2\frac{\cos\theta}{1+\alpha\sin\theta}\operatorname{Im}X+ +\frac{\sin\theta-\alpha\cos^2\theta}{(1+\alpha\sin\theta)^2}; \tag{6.4}$$

$$\frac{\sigma_{T_2,1}}{\sigma_0}=\frac{\operatorname{Re}\widetilde{T}_{2,1}}{\sigma_0 h}=\frac{2}{\alpha}\operatorname{Im}\left\{\frac{dX}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right\}+\frac{\sin\theta}{1+\alpha\sin\theta}; \tag{6.5}$$

$$\frac{\sigma_{S,1}}{\sigma_0}=\frac{\operatorname{Re}\widetilde{S}_{,1}}{\sigma_0 h}=2\frac{1}{1+\alpha\sin\theta}\operatorname{Im}X-\frac{\alpha\cos\theta}{+\alpha\sin\theta)^2};$$

$$\frac{\sigma_{M_1,1}}{\sigma_0}=-\frac{6c\operatorname{Im}\left[\widetilde{T}_{2,1}+\nu\widetilde{T}_{1,1}\right]}{\sigma_0 h^2}=\frac{12}{\mu\alpha}\operatorname{Re}\left\{\frac{dX}{d\theta}+(1+\nu)\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right\}; \tag{6.6}$$

$$\frac{\sigma_{M_2,1}}{\sigma_0}=-\frac{6c\operatorname{Im}\left[\widetilde{T}_{1,1}+\nu\widetilde{T}_{2,1}\right]}{\sigma_0 h^2}=\frac{12}{\mu\alpha}\operatorname{Re}\left\{\nu\frac{dX}{d\theta}+(1+\nu)\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right\};$$

$$\frac{\sigma_{H,1}}{\sigma_0}=\frac{6(1-\nu)c\operatorname{Im}\widetilde{S}_{,1}}{\sigma_0 h^2}=-\frac{12(1-\nu)}{\mu}\frac{1}{1+\alpha\sin\theta}\operatorname{Re}X$$

$$\left(\mu=\sqrt{12(1-\nu)^2}\right).$$

Here

$$\sigma_0 = \frac{B_y^0}{\pi h a^2} \tag{6.7}$$

is the maximum tensile stress in the cylindrical shell (of radius a and thickness h), which is bent by the same moment B_y^0.

Furthermore, it follows from equalities (5.10)-(5.11) that

$$\begin{aligned} \theta_{,1} &= \frac{2B_y^0}{\pi a E h c}\left[\operatorname{Re} X + \alpha \int_{\theta_0}^{\theta} \frac{\cos\theta' \operatorname{Re} X}{1+\alpha\sin\theta'}\,d\theta'\right] - a_3; \\ u_{\rho,1} &= \frac{2\alpha B_y^0}{\pi E h c}\left[\alpha\cos\theta \int_{\theta_0}^{\theta} \frac{\cos\theta' \operatorname{Re} X}{1+\alpha\sin\theta'}\,d\theta' - \int_{\theta_0}^{\theta} \frac{(\sin\theta'+\alpha)\operatorname{Re} X}{1+\alpha\sin\theta'}\,d\theta'\right] - \\ &\qquad - ba_3\cos\theta + a_4. \end{aligned} \tag{6.8}$$

In the resulting equations $X(\theta)$ and $X'(\theta)$ should be interpreted as expressions (Chapter XV, § 42 and 8.45). The simplified expressions (Chapter XV, 8.47) also give a good approximation. Adopting these and retaining only the principal terms in (6.5) and (6.6) /353

$$\frac{\sigma_{T_{2,1}}}{\sigma_0} = \frac{1}{\alpha}\lambda^2\varphi(\theta)\omega'(\theta)\operatorname{Im} E'\{-\lambda\omega(\theta)\}; \tag{6.9}$$

$$\frac{\sigma_{M_{1,1}}}{\sigma_0} = \frac{6}{\mu\alpha}\lambda^2\varphi(\theta)\omega'(\theta)\operatorname{Re} E'\{-\lambda\omega(\theta)\}. \tag{6.10}$$

It is also easy to obtain the asymptotic equations for the maximum values of the stresses which are analogous to the Clark equations (section 1).

Thus, since (table 4)

$$\begin{aligned} &\max\{\operatorname{Im} E'\{-\lambda\omega(\theta)\}\} = \operatorname{Im} E'(0) = 0.939, \\ &\lambda = \sqrt[6]{12(1-\nu^2)}\sqrt[3]{\alpha\beta}\ \left(\alpha = \frac{b}{a},\quad \beta = \frac{b}{h}\right), \end{aligned} \tag{6.11}$$

we have

$$\frac{\sigma_{T_{2,1}}}{\sigma_0} = 2.15\,(1-\nu^2)^{\frac{1}{3}}\alpha^{-\frac{1}{3}}\beta^{\frac{2}{3}}. \tag{6.12}$$

Comparing the resulting expression with (1.19) we see that they coincide within an accuracy governed by the substituting of (6.7) for σ_v.

Using the same approach as in section 1 we obtain the analog of the second Clark equation

$$\frac{\sigma_{M_{1,1}}(\theta_*)}{\sigma_0} = \pm 2.99(1-\nu^2)^{-\frac{1}{6}}\alpha^{-\frac{1}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\omega'(\theta_*), \tag{6.13}$$

where

$$\theta_* \approx \omega(\theta_*) \approx \mp\frac{1{,}225}{\lambda}. \tag{6.14}$$

Let us now turn to the determination of the expansion tank rigidity. From figure 47 and from equations $(6.8)_1$, (Chapte XV, 6.6)

$$Q_B = \left[\vartheta_{,1}\left(\frac{3}{2}\pi\right) - \vartheta_{,1}\left(-\frac{\pi}{2}\right)\right] = \frac{4\alpha B_y^0}{\pi a E h c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta \operatorname{Re} X\, d\theta}{1+\alpha\sin\theta} =$$

$$= \frac{2\alpha B_y^0}{\pi a E h c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{-\frac{\lambda\varphi(\theta)\cos\theta}{1+\alpha\sin\theta}\operatorname{Re} E(-\dot{\lambda}\omega(\theta))\, d\theta.\right.$$

The last integral may be written in the form /354

$$J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\varphi(\theta)\cos\theta}{\omega'(\theta)(1+\alpha\sin\theta)} \operatorname{Re} E(-\lambda\omega(\theta))\, d(-\lambda\omega(\theta)).$$

Following exactly the procedure given at the end of section 1, we find $J \approx \pi$. Then

$$Q_B = \frac{2\alpha}{aEhc} B_y^0 \tag{6.15}$$

and since

$$2k^2 = \frac{ab}{c}, \quad D = Ehc^2 = \frac{Eh^3}{12(1-\nu^2)},$$

$$\Omega_B = \frac{2}{2k^2}\Omega_0, \tag{6.16}$$

where

$$\Omega_0 = \frac{a^2}{D} B_y^0.$$

By using expression $(6.8)_2$ and the property established at the beginning of section 6, Chapter V concerning the odd property of function $X(\theta)$, it is easy to see that there is no mutual horizontal shear displacement of the expansion tank edges.

Shear of a Tubular Expansion Tank Produced by a Force

Let us assume that the same tubular expansion tank is cut by a force F_x^0 (fig. 48). We adopt the following boundary conditions

$$M_{1,1}\left(\frac{\pi}{2}\right) = 0; \quad \varepsilon_{2,1}\left(\frac{\pi}{2}\right) = 0; \tag{6.18}$$

$$M_{1\,1}\left(-\frac{\pi}{2}\right) = 0; \quad \varepsilon_{2,1}\left(-\frac{\pi}{2}\right) = 0. \tag{6.19}$$

The first two conditions follow from the asymptotic nature of the state of stress with respect to the plane $\theta = -\frac{\pi}{2}$ $-\theta = \frac{\pi}{2}$. The fourth condition is justified by the fact that the thickness of the tube is substantially greater

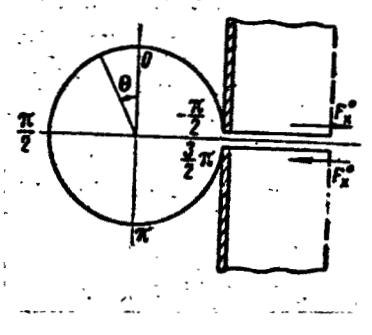

Figure 48

than the thickness of the expansion tank. In trying to justify the third condition we encounter difficulties. It appears that although it is not satisfied when the values of $2k^2$ are not too small, it cannot have a substantial effect on the value of stresses, at least on the value of the maximum stresses.

According to (Chapter XV, 5.10), conditions (6.18) and (6.19) are equivalent to the following complex conditions /355

$$\widetilde{M}_{1,1}\left(\frac{\pi}{2}\right)=\widetilde{M}_{1,1}\left(-\frac{\pi}{2}\right)=0. \tag{6.20}$$

Examining expression $(5.12)_4$ and remembering that according to (Chapter XV, 6.6) $\frac{dW}{d\theta}\left(-\frac{\pi}{2}\right)=\frac{dW}{d\theta}\left(\frac{\pi}{2}\right)=0$, we establish the validity of the following proposition: the particular solution $W(\theta)$ satisfies the boundary conditions (6.20).

Therefore we can drop $\widetilde{\chi}^0$ in the relationships presented above. Then from (5.8) we obtain the following expressions for the stresses

$$\begin{aligned}
\frac{\sigma_{T_{1,1}}}{\sigma_0}&=2\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Im}W-\frac{\alpha^2\cos\theta}{(1+\alpha\sin\theta)^2};\\
\frac{\sigma_{T_{2,1}}}{\sigma_0}&=2\operatorname{Im}\left(\frac{dW}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}W\right);\\
\frac{\sigma_{S,1}}{\sigma_0}&=2\frac{\alpha}{1+\alpha\sin\theta}\operatorname{Im}W+\frac{1-\alpha^2}{(1+\alpha\sin\theta)^2};
\end{aligned} \tag{6.21}$$

$$\begin{aligned}
\frac{\sigma_{M_{1,1}}}{\sigma_0}&=\frac{12}{\mu}\operatorname{Re}\left(\frac{dW}{d\theta}+(1+\nu)\frac{\alpha\cos\theta}{1+\alpha\sin\theta}W\right);\\
\frac{\sigma_{M_{2,1}}}{\sigma_0}&=\frac{12}{\mu}\operatorname{Re}\left(\nu\frac{dW}{d\theta}+(1+\nu)\frac{\alpha\cos\theta}{1+\alpha\sin\theta}W\right);\\
\frac{\sigma_{H,1}}{\sigma_0}&=-\frac{12(1-\nu)}{\mu}\frac{\alpha}{1+\alpha\sin\theta}\operatorname{Re}W\\
&\qquad\left(\mu=\sqrt{12(1-\nu^2)}\right).
\end{aligned} \tag{6.22}$$

Here

$$\sigma_0=-\frac{F_x^0}{\pi ah} \tag{6.23}$$

is the maximum shear stress computed for the cylindrical shell (of radius a and thickness h), which is cut by the force F_x^0.

If we adopt the simple expression (Chapter XV, 8.47) for W, we obtain the following expressions for the basic stresses

$$\frac{\sigma_{T_{2,1}}}{\sigma_0} = \lambda^2 \varphi(\theta)\,\omega'(\theta)\,\operatorname{Im} E'(-\lambda\omega(\theta));$$
$$\frac{\sigma_{M_{1,1}}}{\sigma_0} = \frac{6\lambda^2}{\mu}\varphi(\theta)\,\omega'(\theta)\operatorname{Re} E'(-\lambda\omega(\theta)). \tag{6.24}$$

By completely repeating the calculation steps presented above we obtain the asymptotic equations for the maximum values of the stresses

$$\frac{\sigma_{T_{2,1}}^{(0)}}{\sigma_0} = 2{,}15\,(1-\nu^2)^{\frac{1}{3}}\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}; \tag{6.25}$$

$$\frac{\sigma_{M_{1,1}}(\theta_*)}{\sigma_0} = \pm\, 2{,}99\,(1-\nu^2)^{-\frac{1}{6}}\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\,\omega'(\omega_*), \tag{6.26}$$

where, as before,

$$\theta_* \approx \omega(\theta_*) \approx \mp\frac{1{,}225}{\lambda}. \tag{6.27}$$

Let us determine the rigidity of the expansion tank. First of all, /356 from (5.10)

$$u_{\rho,1} = \frac{2\alpha h F_x^0}{\pi E h c}\left[-\alpha\cos\theta\int_{\theta_0}^{\theta}\frac{\cos\theta'\operatorname{Re} W}{1+\alpha\sin\theta'}\,d\theta' + \int_{\theta_0}^{\theta}\frac{(\sin\theta'+\alpha)\operatorname{Re} W}{1+\alpha\sin\theta'}\,d\theta'\right] - a_2 b\cos\theta + a_4;$$
$$\theta_{,1} = -\frac{2\alpha F_x^0}{\pi E h c}\left[\operatorname{Re} W + \alpha\int_{\theta_0}^{\theta}\frac{\cos\theta'\operatorname{Re} W}{1+\alpha\sin\theta'}\,d\theta'\right] - a_3. \tag{6.28}$$

We introduce the following quantity, which characterizes the compliance of the expansion tank to shear (cut),

$$U_F = u_{\rho,1}\left(\frac{3\pi}{2}\right) - u_{\rho,1}\left(-\frac{\pi}{2}\right). \tag{6.29}$$

With the aid of expressions $(6.28)_1$ and (Chapter XV, 6.6) we have

$$U_F = \frac{2abF_x^0}{\pi Ehc}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{(\sin\theta+\alpha)\,\mathrm{Re}\,W}{1+\alpha\sin\theta}\,d\theta \approx$$

$$\approx \frac{2abF_x^0}{\pi Ehc}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\sin\theta+\alpha)\,\varphi(\theta)}{(1+\alpha\sin\theta)\omega'(\theta)}\,\mathrm{Re}\,E(-\lambda\omega)\,d(-\lambda\omega).$$

Using exactly the same procedure as in section 1, we obtain

$$U_F = \frac{2}{2k^2}U_0, \tag{6.30}$$

where

$$U_0 = \frac{a^2b^2F_x^0}{D}\quad\left(D = \frac{Eh^3}{12(1-\nu^2)} = Ehc^2\right). \tag{6.31}$$

Taking into account the even property of the function $W(\theta)$ (Chapter XV, 6.6), we become convinced, with the aid of $(6.28)_2$, that in the considered case there is no mutual rotation of the expansion tank edges.

We now consider the flexure of a complete toroidal shell (fig. 47) /3[illegible] produced by the moment B_y^0 ($F_x^0 = 0$) as an example of analyzing a toroidal shell for an antisymmetric load. The shell and load parameters are as follows

$$a = 233.5 \text{ metric ton}; \quad \nu = 0.3; \qquad 2k^2 = 4.75;$$

$$h = 2.0 \text{ metric ton}; \quad B_y^0 = -14.4\cdot10^3 \text{ kg·cm}; \quad \alpha = 0.111.$$

The values of the meridian ($\sigma_{1,1}^+$) and ring ($\sigma_{2,1}^+$) total stresses in the external fibers (section $\varphi = 0$) are shown in figures 49 and 50. The solid lines show the stresses computed by using the complete equations (6.4)-(6.5), while the broken lines show those computed by means of simplified equations (6.9)-(6.10). The crosses show the experimental stress values obtained in 1959 at the Leningrad Metallic Plant under the direction of V. K. Naumov.

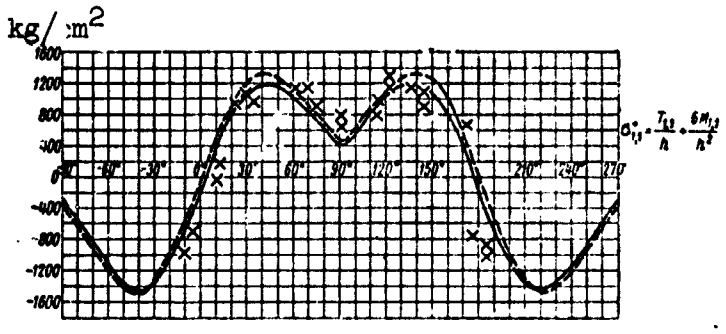

Figure 49

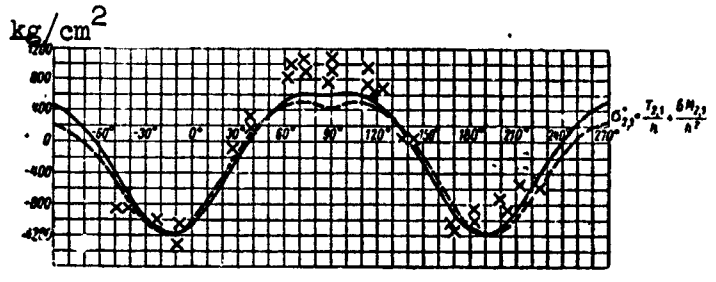

Figure 50

These graphs show that the simplified equations (6.9)-(6.10) give a good approximation even when the values of the basic parameter are relatively small ($2k^2 = 4.75$).

CHAPTER XVII. THE SAINT-VENANT PROBLEM FOR TUBES WITH A CIRCULAR AXIS

Another important application of the theory of toroidal shells is /358
the design of thin-walled tubes with a circular axis. In 1910 Batlin determined experimentally that thin-walled tubes with a curvilinear axis have a substantially smaller flexural rigidity than tubes of the same cross section with a rectilinear axis. A year later Karman (ref. 371) explained this phenomenon by the compression of the transverse section. It became clear that the compression of the transverse section produces high transverse bending stresses, which frequently exceed in value the basic tangential stresses. On the other hand, the law governing the variation of the latter over the transverse section is quite different from the linear law characteristic for tubes with a rectilinear axis.

These special features associated with the operation of thin tubes with curvilinear axis attracted a great deal of attention. The large number of reports which followed (ref. 371) (a critical survey is given in reference 378) were devoted either to the refinement of the results obtained by Karman as they apply to a wider range of parameter variations, or to the consideration of other specific forms of loading. The approach common to all of them involves the utilization of variational principles after the assignment of approximate expressions for displacements and stresses.

The work of Clark and Reissner (ref. 361) is unique in that the problem of tube torsion by a moment in the bending plane (the first symmetric case) is reduced to an equation which resembles the Meissner-Clark equation.

In the present chapter we shall use the relationships obtained in Chapter XV to obtain a solution to the deformation of a tube with circular cross section under the action of edge loading, of the general type, applied along the edges of the tube sections. The boundary conditions along the edges will follow the Saint-Venant conditions, i.e., the constructed solution produces the required principal vector and principal moment at the edges of the tube. /359

1. The Second Symmetric Case (Symmetric Torsion)

Let us consider two operations of the shell which are substantially different in nature.

1. A tube with a closed transverse section.

According to equations (section 3, Chapter XV) the periodicity conditions for v^c are written in the form

$$\frac{U_z^1}{\varphi_0}\oint\frac{\sin\theta\,d\theta}{(1+\alpha\sin\theta)^2}-2(1+\nu)\frac{c}{a}\frac{\overline{U}_z^1}{\varphi_0}\oint\frac{d\theta}{(1+\alpha\sin\theta)^3}=0. \tag{1.1}$$

From this, by using equalities (Chapter XV, 6.17_2) we find

$$\overline{U}_z^1=\frac{1}{2(1+\nu)}\frac{a}{c}\left(-\frac{\alpha J_2(\alpha)}{J_3(\alpha)}\right)U_z^1. \tag{1.2}$$

Now

$$v^c=\alpha(1+\alpha\sin\theta)\left\{\left[\int_{-\frac{\pi}{2}}^{\theta}\frac{\sin\tau\,d\tau}{(1+\alpha\sin\tau)^2}+\frac{\alpha J_2(\alpha)}{J_3(\alpha)}\int_{-\frac{\pi}{2}}^{\theta}\frac{d\tau}{(1+\alpha\sin\tau)^3}\right]\frac{U_z^1}{\varphi_0}\right\}+ \\ +a(1+\alpha\sin\theta)\,a_2; \tag{1.3}$$

$$S=\frac{Eh}{2(1+\nu)}\frac{\alpha J_2(\alpha)}{J_3(\alpha)}\frac{1}{(1+\alpha\sin\theta)^2}\frac{U_z^1}{a\varphi_0}; \\ H=\frac{Eh^3}{12(1+\nu)}\frac{1}{(1+\alpha\sin\theta)^2}\frac{U_z^1}{a^2\varphi_0}. \tag{1.4}$$

It follows from the resulting relationships that in a tube with a closed profile the twisting moment may be neglected compared with the shear force. This is easy to see, if we substitute the stresses corresponding to both quantities

$$\sigma_S=\frac{S}{h}=\frac{E}{2(1+\nu)}\frac{\alpha J_2(\alpha)}{J_3(\alpha)}\frac{1}{(1+\alpha\sin\theta)^2}\frac{U_z^1}{a\varphi_0}, \\ \sigma_H=\frac{6H}{h^2}=\frac{E}{2(1+\nu)}\left(\frac{h}{a}\right)\frac{1}{(1+\alpha\sin\theta)^2}\frac{U_z^1}{a\varphi_0}. \tag{1.5}$$

Let us determine, from the shear force, the components of the principal vector and of the principal moment in the transverse section of the tube. We obtain (Chapter XV, 6.17)

$$B_\varphi=\oint S b^2\,d\theta=\frac{EJ}{a^2}\left\{\frac{1}{1+\nu}\frac{[J_2(\alpha)]^2}{J_3(\alpha)}\right\}\frac{U_z^1}{\varphi_0}; \\ F_z=\oint S(-\sin\theta)\,b\,d\theta=\frac{EJ}{a^2}\left\{\frac{1}{1+\nu}\frac{[J_2(\alpha)]^2}{J_3(\alpha)}\right\}\frac{U_z^1}{\varphi_0}; \\ F_\varphi=\oint S\cos\theta\;.\theta=0. \tag{1.6}$$

Here

$$J = \pi b^3 h \tag{1.7}$$

is the moment of inertia of the tube section with respect to its diameter. /360 We note that in computing the principal vector and the principal moment it is not necessary to take into account the twisting moment, because its contribution, as we can easily see, is quite insignificant.

Summarizing we may state that in a tube with a closed profile a membrane state of stress is established (see the introduction to Chapter XII).

2. Tube with a cut (trough).

Let us now assume that the tube has a cut at the section $\theta = \theta_0$. From equalities (Chapter XV, 3.4) we find

$$Q_{12}=S+\frac{2H}{R_2}=\frac{Ehc}{a^2\varphi_0}\frac{1}{(1+\alpha\sin\theta)^2}\left\{-\overline{U}_z^1+2(1-\nu)\frac{c}{a}\frac{\sin\theta}{1+\alpha\sin\theta}U_z^1\right\}. \tag{1.8}$$

We shall require that the cut be free of forces. This leads to the condition

$$\overline{U}_z^1=2(1-\nu)\frac{c}{a}\frac{\sin\theta_0}{1+\alpha\sin\theta_0}U_z^1. \tag{1.9}$$

In this case

$$S=-\frac{Eh^3}{6(1+\nu)}\frac{\sin\theta_0}{1+\alpha\sin\theta_0}\frac{1}{(1+\alpha\sin\theta)^2}\frac{U_z^1}{a^3\varphi_0}; \tag{1.10}$$
$$H=\frac{Eh^3}{12(1+\nu)}\frac{1}{(1+\alpha\sin\theta)^2}\frac{U_z^1}{a^2\varphi_0}.$$

It is easy to see that for a tube with an open profile the torsional stress will be characteristic

$$\sigma_H=\frac{6H}{h^2}=\frac{E}{2(1+\nu)}\frac{1}{(1+\alpha\sin\theta)^2}\left(\frac{h}{a}\right)\left(\frac{U_z^1}{a\varphi_0}\right) \tag{1.11}$$

and there is a state of stress associated with high moments in the shell (see introduction to Chapter XII).

Furthermore, by substituting $\overline{U}_z^1$ into expression (Chapter XV, 3.5) we obtain

$$v^c = a(1+\alpha\sin\theta)\left\{\int_{\theta_0-2\pi}^{\theta}\frac{\sin\tau\,d\tau}{(1+\alpha\sin\tau)^2} - \underline{-\frac{1}{3}\left(\frac{h}{a}\right)^2\frac{\sin\theta_0}{1+\alpha\sin\theta_0}\int_{\theta_0-2\pi}^{\theta}\frac{d\tau}{(1+\alpha\sin\tau)^3}}\right\}\frac{U_z^1}{\varphi_0} + a(1+\alpha\sin\theta)\,a_2.$$

The underlined term is very small. We drop it and obtain the following expression for v^c /361

$$v^c = a(1+\alpha\sin\theta)\int_{\theta_0-2\pi}^{\theta}\frac{\sin\tau\,d\tau}{(1+\alpha\sin\tau)^2}\frac{U_z^1}{\varphi_0} + a(1+\alpha\sin\theta)\,a_2. \tag{1.12}$$

Next we determine the components of the principal vector and of the principal moment and obtain

$$\begin{aligned} B_\varphi &= \oint\left(S+\frac{2H}{b}\right)b^2\,d\theta = \frac{EJ}{a^2}\left(\frac{h}{b}\right)^2\frac{1}{3(1+\nu)}\frac{J_2(\alpha)}{1+\alpha\sin\theta_0}\frac{U_z^1}{\varphi_0};\\ F_z &= \oint\left(S+\frac{2H}{b}\right)(-\sin\theta)\,b\,d\theta = \frac{EJ}{a^3}\left(\frac{h}{b}\right)^2\frac{1}{3(1+\nu)}\frac{J_2(\alpha)}{1+\alpha\sin\theta_0}\frac{U_z^1}{\varphi_0};\\ F_\rho &= \oint\left(S+\frac{2H}{b}\right)\cos\theta\,b\,d\theta = 0. \end{aligned} \tag{1.13}$$

Comparing the resulting expressions with analogous ones for the tube with a closed profile we see, first of all, that the closed tube is considerably more rigid. In order to obtain the same dislocation vertical displacements in the tube with a cut we must apply a twisting moment and a bending force, which are approximately smaller by a factor $(\frac{h}{b})^2$ than those applied to a closed profile. In this case the stress occurring in the tube is less by a factor $(\frac{h}{b})$. The principal difference between the tubes with a closed and an open profile as observed in the specific example is a general property, as we know.

We note in conclusion that during the deformation of the considered type the cross section remains circular, because

$$u_f^c = v_z^c = 0. \tag{1.14}$$

2. The First Symmetric Case

Let us assume that the tube is closed in the transverse direction. Then all of the quantities must be periodic functions of θ. Comparing equation (Chapter XV, 4.4) with (Chapter XV, 6.1) we obtain

$$\tilde{\theta} = \frac{2}{\varphi_0}\left(\overline{\Omega}_z^1 - i\Omega_z^1\right)X. \tag{2.1}$$

By using these expressions we can write the condition of periodicity for the vertical displacement (Chapter XV, 4.6_2) in the form (p = 0)

$$\overline{\Omega}_z^1 \oint \cos\theta \operatorname{Re} X\, d\theta + \Omega_z^1 \oint \cos\theta \operatorname{Im} X\, d\theta = 0. \tag{2.2}$$

From this we obtain

$$\overline{\Omega}_z^1 = -\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\Omega_z^1. \tag{2.3}$$

Now it follows from relationships (Chapter XV, 4.5) that /362

$$\begin{aligned}
\tilde{T}_1 &= \frac{2Ehc}{b\varphi_0}\Omega_z^1\left\{-\left(1 - i\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\right)\left(\frac{\alpha\cos\theta X}{1+\alpha\sin\theta} + i\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right)\right\};\\
\tilde{T}_2 &= \frac{2Ehc}{b\varphi_0}\Omega_z^1\left\{-\left(1 - i\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\right)\frac{dX}{d\theta}\right\}.
\end{aligned} \tag{2.4}$$

Separating the real and imaginary parts, we obtain

$$\begin{aligned}
T_1 &= \frac{2Ehc\Omega_z^1}{b\varphi_0}\left\{-\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Re} X - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\left(\frac{\alpha\cos\theta}{1+\alpha\sin\theta}\operatorname{Im} X + \right.\right.\\
&\qquad \left.\left. + \frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right)\right\};\\
T_2 &= \frac{2Ehc\Omega_z^1}{b\varphi_0}\left\{-\operatorname{Re}\frac{dX}{d\theta} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\frac{X}{d\theta}\right\};\\
M_1 &= \frac{2Ehc^2\Omega_z^1}{b\varphi_0}\left\{\operatorname{Im}\left(\frac{dX}{d\theta} + \nu\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right) + \frac{\nu}{2}\frac{\alpha\sin\theta}{1+\alpha\sin\theta} - \right.
\end{aligned} \tag{2.5}$$

$$-\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Re}\left(\frac{dX}{d\theta}+\nu\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right)\Big\};$$

$$M_z=\frac{2Ehc^2\Omega_z^1}{b\varphi_0}\left\{\operatorname{Im}\left(\nu\frac{dX}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right)+\underline{\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}}-\right.$$
$$\left.-\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Re}\left(\nu\frac{dX}{d\theta}+\frac{\alpha\cos\theta}{1+\alpha\sin\theta}X\right)\right\}.$$

The underlined terms represent the direct contribution made by the dislocation displacements and by functions of the static system to the expressions for the forces and moments. As we can see from the relationships which have been written out their role is small.

Let us determine the moments of the principal vector and of the principal moment applied to the cross section of the tube. We have

$$B_z=\oint T_z\sin\theta\,b^2\,d\theta=$$
$$=\frac{2Ehc\,b\,\Omega_z^1}{\varphi_0}\left\{-\operatorname{Re}\oint\sin\theta\frac{dX}{d\theta}d\theta-\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\oint\sin\theta\frac{dX}{d\theta}d\theta\right\};$$
$$B_\rho=\oint T_z(-\cos\theta)\,b^2\,d\theta=\qquad(2.6)$$
$$=\frac{2Ehc\,b\,\Omega_z^1}{\varphi_0}\left\{\operatorname{Re}\oint\cos\theta\frac{dX}{d\theta}d\theta+\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\oint\cos\theta\frac{dX}{d\theta}d\theta\right\};$$
$$F_\varphi=\oint T_z\,b\,d\theta=\frac{2Ehc\,\Omega_z^1}{\varphi_0}\left\{-\operatorname{Re}\oint\frac{dX}{d\theta}d\theta-\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\oint\frac{dX}{d\theta}d\theta\right\}.$$

If we apply to the first expression the operation of integration by parts, we obtain, by taking into account designations (Chapter XV, 6.25) and (Chapter XV, 6.26),

$$B_z=\frac{2\pi b\,Ehc\,\Omega_z^1}{\varphi_0}2\frac{(\operatorname{Re} I_1)^2+(\operatorname{Im} I_1)^2}{\operatorname{Re} I_1}=\frac{EJ}{a^2}\left\{\frac{\Lambda_1}{k^2\varphi_0}a\,\Omega_z^1\right\}.\qquad(2.7)$$

Since the function $\frac{dX}{d\theta}$ is even and periodic (Chapter XV, 6.6)

$$B_\rho=0,\ F_\varphi=0.\qquad(2.8)$$

In computing the principal moment we deliberately neglect the moment M_2. /363

It is easy to see that if this moment were taken into account, it would produce a small correction of the order $\frac{h}{a}$.

The maximum stress corresponding to the bending moment B_z (according to the elementary theory) is equal to

$$\sigma_0 = \frac{B_z}{\frac{J}{b}} = \frac{E\alpha\Lambda_1}{k^2\varphi_0}\Omega_z^1 = \frac{2Ec\Lambda_1}{b\varphi_0}\Omega_z^1. \tag{2.9}$$

If we now introduce the tangential and flexural stresses

$$\sigma_{T_1} = \frac{T_1}{h};\ \sigma_{T_2} = \frac{T_2}{h};\ \sigma_{M_1} = \frac{6M_1}{h^2};\ \sigma_{M_2} = \frac{6M_2}{h^2}, \tag{2.10}$$

by using equalities (2.5) and (2.9), we can represent them in the form

$$\begin{aligned}
\frac{\sigma_{T_1}}{\sigma_0} &= \frac{1}{\Lambda_1}\left\{-\frac{\alpha\cos\theta\,\mathrm{Re}\,X}{1+\alpha\sin\theta} - \frac{\mathrm{Im}\,I_1}{\mathrm{Re}\,I_1}\left(\frac{\alpha\cos\theta\,\mathrm{Im}\,X}{1+\alpha\sin\theta} + \frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right)\right\};\\
\frac{\sigma_{T_2}}{\sigma_0} &= \frac{1}{\Lambda_1}\left\{-\mathrm{Re}\,\frac{dX}{d\theta} - \frac{\mathrm{Im}\,I_1}{\mathrm{Re}\,I_1}\,\mathrm{Im}\,\frac{dX}{d\theta}\right\};\\
\frac{\sigma_{M_1}}{\sigma_0} &= \frac{6}{\mu\Lambda_1}\left\{\mathrm{Im}\left(\frac{dX}{d\theta} + \nu\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right) + \frac{\nu}{2}\frac{\alpha\sin\theta}{1+\alpha\sin\theta} - \right.\\
&\qquad \left. - \frac{\mathrm{Im}\,I_1}{\mathrm{Re}\,I_1}\,\mathrm{Re}\left(\frac{dX}{d\theta} + \nu\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)\right\};\\
\frac{\sigma_{M_2}}{\sigma_0} &= \frac{6}{\mu\Lambda_1}\left\{\mathrm{Im}\left(\nu\frac{dX}{d\theta} + \frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right) + \frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)} - \right.\\
&\qquad \left. - \frac{\mathrm{Im}\,I_1}{\mathrm{Re}\,I_1}\,\mathrm{Re}\left(\nu\frac{dX}{d\theta} + \frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)\right\}\\
&\qquad \left(\mu = \sqrt{12(1-\nu^2)}\right).
\end{aligned} \tag{2.11}$$

Furthermore, in view of (Chapter XV, 4.6) we obtain by dropping small terms

$$\begin{aligned}
u_\rho &= -(1+\alpha\sin\theta)\frac{\alpha\Omega_z^1}{\varphi_0};\\
u_z &= a_1 - \frac{2b\Omega_z^1}{\varphi_0}\int_{-\frac{\pi}{2}}^{\theta}\cos\theta\left(\mathrm{Im}\,X - \frac{\mathrm{Im}\,I_1}{\mathrm{Re}\,I_1}\,\mathrm{Re}\,X\right)d\theta.
\end{aligned} \tag{2.12}$$

The value of diameter compression is obtained from the first expression

$$\delta_\rho = u_\rho\left(-\frac{\pi}{2}\right) - u_\rho\left(\frac{\pi}{2}\right) = \frac{2b\Omega_z^1}{\varphi_0}. \tag{2.13}$$

Thus for a thin-walled shell the value of compression may be considered to be independent of the flexure radius a (ref. 361). From relationships (2.12)$_2$ /364

we can obtain an expression for the value of diameter expansion in the vertical direction

$$\delta_z = u_z(0) - u_z(\pi) = \omega\delta_\rho, \tag{2.14}$$

where

$$\omega = \int_0^{\frac{\pi}{2}} \cos\theta \left(\operatorname{Im} X - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1} \operatorname{Re} X \right) d\theta. \tag{2.15}$$

3. The Antisymmetric Case

Comparing equations (Chapter XV, 6.2 and 6.3) with (Chapter XV, 5.4), we obtain

$$\tilde{\chi} = -i\frac{4}{a\varphi_0}\left\{ Y - \begin{matrix} (U_y^1 + i\bar{U}_y^1) \\ (U_x^1 + i\bar{U}_x^1) \end{matrix} + Z \begin{matrix} a(\Omega_x^1 + i\bar{\Omega}_x^1) \\ a(\Omega_y^1 + i\bar{\Omega}_y^1) \end{matrix} \right\}. \tag{3.1}$$

We shall first determine the quantities $\bar{U}_x^1$, $\bar{U}_y^1$, $\bar{\Omega}_x^1$, $\bar{\Omega}_y^1$. Since we are considering a tube without a cut, the unknown displacements must be periodic functions of θ. On the other hand, the condition of periodicity, according to (Chapter XV, 5.6 and 5.7) is written in the form

$$\oint \frac{\cos\theta \operatorname{Re}\tilde{\chi}}{1+\alpha\sin\theta} d\theta = \oint \frac{(\sin\theta+\alpha)\operatorname{Re}\tilde{\chi}}{1+\alpha\sin\theta} d\theta = 0, \tag{3.2}$$

and if we take into account the preceding equality we obtain

$$\begin{matrix} -U_y^1 \\ U_x^1 \end{matrix} \frac{1}{2\pi}\oint \frac{\cos\theta \operatorname{Im} Y}{1+\alpha\sin\theta} d\theta + \begin{matrix} -\bar{U}_y^1 \\ \bar{U}_x^1 \end{matrix} \frac{1}{2\pi}\oint \frac{\cos\theta \operatorname{Re} Y}{1+\alpha\sin\theta} d\theta +$$
$$+ \begin{matrix} a\Omega_x^1 \\ a\Omega_y^1 \end{matrix} \frac{1}{2\pi}\oint \frac{\cos\theta \operatorname{Im} Z}{1+\alpha\sin\theta} d\theta + \begin{matrix} a\bar{\Omega}_x^1 \\ a\bar{\Omega}_y^1 \end{matrix} \frac{1}{2\pi}\oint \frac{\cos\theta \operatorname{Re} Z}{1+\alpha\sin\theta} d\theta = 0,$$
$$\begin{matrix} -U_y^1 \\ U_x^1 \end{matrix} \frac{1}{2\pi}\oint \frac{(\sin\theta+\alpha)\operatorname{Im} Y}{1+\alpha\sin\theta} d\theta + \begin{matrix} -\bar{U}_y^1 \\ \bar{U}_x^1 \end{matrix} \frac{1}{2\pi}\oint \frac{(\sin\theta+\alpha)\operatorname{Re} Y}{1+\alpha\sin\theta} d\theta +$$
$$+ \begin{matrix} a\Omega_x^1 \\ a\Omega_y^1 \end{matrix} \frac{1}{2\pi}\oint \frac{(\sin\theta+\alpha)\operatorname{Im} Z}{1+\alpha\sin\theta} d\theta + \begin{matrix} a\bar{\Omega}_x^1 \\ a\bar{\Omega}_y^1 \end{matrix} \frac{1}{2\pi}\oint \frac{(\sin\theta+\alpha)\operatorname{Re} Z}{1+\alpha\sin\theta} d\theta = 0.$$

Since, as we have mentioned in section 6, Chapter XV, Z is even and Y is odd, the underlined terms are equal to zero. Dropping them and utilizing designations (Chapter XV, 6.25), we obtain the unknown expressions

$$\begin{aligned} \overline{U}_y^1 &= -\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2} U_y^1; \\ \overline{U}_x^1 & \\ \overline{\Omega}_x^1 &= -\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3} \Omega_x^1. \\ \overline{\Omega}_y^1 & \end{aligned} \tag{3.3}$$

By using equalities (Chapter XV, 5.5) and (3.1) we find /365

$$\begin{aligned} \widetilde{T}_{1,1} &= \frac{4Ehc}{b a \varphi_0}\left\{\left(\frac{\alpha\cos\theta\, Y}{1+\alpha\sin\theta} + i\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)^2}\right)\begin{matrix} U_y^1\left(1 - i\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\right) \\ - U_x^1\left(1 - i\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\right)\end{matrix} + \right. \\ &\left. + \left(-\frac{\alpha\cos\theta\, Z}{1+\alpha\sin\theta} + i\frac{\alpha\cos\theta}{2(1+\alpha\sin\theta)^2}\right)\begin{matrix} a\Omega_x^1\left(1 - i\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\right) \\ a\Omega_y^1\left(1 - i\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\right)\end{matrix}\right\}; \\ \widetilde{T}_{2,1} &= \frac{4Ehc}{b a \varphi_0}\left\{\left(\frac{dY}{d\theta} + \frac{\alpha\cos\theta\, Y}{1+\alpha\sin\theta}\right)\begin{matrix} U_y^1\left(1 - i\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\right) \\ - U_x^1\left(1 - i\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\right)\end{matrix} - \right. \\ &\left. - \left(\frac{dZ}{d\theta} + \frac{\alpha\cos\theta\, Z}{1+\alpha\sin\theta}\right)\begin{matrix} a\Omega_x^1\left(1 - i\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\right) \\ a\Omega_y^1\left(1 - i\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\right)\end{matrix}\right\}; \\ \widetilde{S}_{,1} &= \frac{4Ehc}{b a \varphi_0}\frac{\alpha}{1+\alpha\sin\theta}\left\{Y\begin{matrix} U_y^1\left(1 - i\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\right) \\ U_x^1\left(1 - i\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\right)\end{matrix} + Z\begin{matrix} -a\Omega_x^1\left(1 - i\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\right) \\ a\Omega_y^1\left(1 - i\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\right)\end{matrix}\right\}. \end{aligned} \tag{3.4}$$

Let us compute the principal vector and the principal moment of forces applied to the transverse section of the tube by utilizing the real parts of the above expressions. As in the symmetric case, we do not take into account the effect of moment $M_{2,1}$ which introduces an insignificant correction

$$\begin{aligned} F_{\varphi,1} &= \oint T_{2,1} b\, d\theta = \frac{4Ehc}{a\varphi_0}\left\{\begin{matrix} U_y^1 \\ -U_x^1\end{matrix}\oint\left[\operatorname{Re}\left(\frac{dY}{d\theta} + \frac{\alpha\cos\theta\, Y}{1+\alpha\sin\theta}\right) + \right.\right. \\ &\left. + \frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Im}\left(\frac{dY}{d\theta} + \frac{\alpha\cos\theta\, Y}{1+\alpha\sin\theta}\right)\right] d\theta - \begin{matrix} a\Omega_x^1 \\ a\Omega_y^1\end{matrix}\oint\left[\operatorname{Re}\left(\frac{dZ}{d\theta} + \frac{\alpha\cos\theta\, Z}{1+\alpha\sin\theta}\right) + \right. \\ &\left.\left. + \frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\operatorname{Im}\left(\frac{dZ}{d\theta} + \frac{\alpha\cos\theta\, Z}{1+\alpha\sin\theta}\right)\right] d\theta\right\}; \\ B_{\varphi,1} &= \oint T_{2,1}(-\cos\theta) b^2 d\theta = \frac{4Ehcb}{a\varphi_0}\left\{\begin{matrix} -U_y^1 \\ U_x^1\end{matrix}\oint\left[\operatorname{Re}\left(\cos\theta\frac{dY}{d\theta} + \right.\right.\right. \end{aligned} \tag{3.5}$$

$$+\frac{\alpha\cos^2\theta\, Y}{1+\alpha\sin\theta}\Big)+\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Im}\Big(\cos\theta\frac{dY}{d\theta}+\frac{\alpha\cos^2\theta\, Y}{1+\alpha\sin\theta}\Big)\Big]d\theta+$$
$$+\begin{matrix}a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}\oint\Big[\operatorname{Re}\Big(\cos\theta\frac{dZ}{d\theta}+\frac{\alpha\cos^2\theta\, Z}{1+\alpha\sin\theta}\Big)+\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\operatorname{Im}\Big(\cos\theta\frac{dZ}{d\theta}+$$
$$+\frac{\alpha\cos^2\theta\, Z}{1+\alpha\sin\theta}\Big)\Big]d\theta\Big\};$$

$$B_{x,1}=\oint T_{2,1}\sin\theta\, b^2\, d\theta=\frac{4Ehc\,b}{a\varphi_0}\Big\{\begin{matrix}U_y^1\\ -U_x^1\end{matrix}\oint\Big[\operatorname{Re}\Big(\sin\theta\frac{dY}{d\theta}+$$
$$+\frac{\alpha\sin\theta\cos\theta\, Y}{1+\alpha\sin\theta}\Big)+\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Im}\Big(\sin\theta\frac{dY}{d\theta}+\frac{\alpha\sin\theta\cos\theta\, Y}{1+\alpha\sin\theta}\Big)\Big]d\theta-$$
$$-\begin{matrix}a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}\oint\Big[\operatorname{Re}\Big(\sin\theta\frac{dZ}{d\theta}+\frac{\alpha\sin\theta\cos\theta\, Z}{1+\alpha\sin\theta}\Big)+\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\operatorname{Im}\Big(\sin\theta\frac{dZ}{d\theta}+$$
$$+\alpha\frac{\sin\theta\cos\theta\, Z}{1+\alpha\sin\theta}\Big)\Big]d\theta\Big\};$$

$$B_{\varphi,1}=\oint S_{,1}\, b^2\, d\theta=\frac{4\alpha\, b\, Ehc}{a\varphi_0}\Big\{\begin{matrix}U_y^1\\ U_x^1\end{matrix}\oint\Big[\frac{\operatorname{Re} Y}{1+\alpha\sin\theta}+$$
$$+\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\frac{\operatorname{Im} Y}{1+\alpha\sin\theta}\Big]d\theta+\begin{matrix}-a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}\oint\Big[\frac{\operatorname{Re} Z}{1+\alpha\sin\theta}+$$
$$+\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\frac{\operatorname{Im} Z}{1+\alpha\sin\theta}\Big]d\theta\Big\};$$

$$F_{p,1}=\oint S_{,1}\cos\theta\, b\, d\theta=\frac{4\alpha\, Ehc}{a\varphi_0}\Big\{\begin{matrix}U_y^1\\ U_x^1\end{matrix}\oint\Big[\frac{\cos\theta\operatorname{Re} Y}{1+\alpha\sin\theta}+$$
$$+\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\frac{\cos\theta\operatorname{Im} Y}{1+\alpha\sin\theta}\Big]d\theta+\begin{matrix}-a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}\oint\Big[\frac{\cos\theta\operatorname{Re} Z}{1+\alpha\sin\theta}+$$
$$+\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\frac{\cos\theta\operatorname{Im} Z}{1+\alpha\sin\theta}\Big]d\theta\Big\};$$

$$F_{z,1}=\oint S_{,1}(-\sin\theta)\, b\, d\theta=-\frac{4\alpha\, Ehc}{a\varphi_0}\Big\{\begin{matrix}U_y^1\\ U_x^1\end{matrix}\oint\Big[\frac{\sin\theta\operatorname{Re} Y}{1+\alpha\sin\theta}+$$
$$+\frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\frac{\sin\theta\operatorname{Im} Y}{1+\alpha\sin\theta}\Big]d\theta+\begin{matrix}-a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}\oint\Big[\frac{\sin\theta\operatorname{Re} Z}{1+\alpha\sin\theta}+$$
$$+\frac{\operatorname{Im} I_3}{\operatorname{Re} I_3}\frac{\sin\theta\operatorname{Im} Z}{1+\alpha\sin\theta}\Big]d\theta\Big\}.$$

Integrating the underlined terms in parts and utilizing the designations of section 6, Chapter XV we obtain, by taking into account the even and odd properties of the basic functions (see also (Chapter XV, 6.35)), /366

$$\begin{aligned}aF_p&=\frac{EJ}{a^2}\Big\{\frac{2\Lambda_2}{k^2\varphi_0}\Big\}(U_y^1\sin\varphi+U_x^1\cos\varphi);\\ aF_\varphi&=\frac{FJ}{a^2}\Big\{\frac{2\Lambda_2}{k^2\varphi_0}\Big\}(U_y^1\cos\varphi-U_x^1\sin\varphi);\end{aligned}\qquad(3.6)$$

$$B_z=\frac{EJ}{a^2}\left\{\frac{2\Lambda_2}{k^2\varphi_0}\right\}(-U_y^1\cos\varphi+U_x^1\sin\varphi);$$
$$B_\varphi=\frac{EJ}{a^2}\left\{\frac{2\Lambda_3}{k^2\varphi_0}\right\}(-a\,\Omega_x^1\sin\varphi+a\,\Omega_y^1\cos\varphi); \qquad (3.7)$$
$$B_\rho=\frac{EJ}{a^2}\left\{\frac{2\Lambda_3}{k^2\varphi_0}\right\}(a\,\Omega_x^1\cos\varphi+a\,\Omega_y^1\sin\varphi);$$
$$F_z=0.$$

We further introduce the maximum stresses which, according to the elementary theory, correspond to the bending moments $B_{z,1}$ and $B_{\rho,1}$ /367

$$\sigma_{0z}=\frac{\frac{B_{z,1}}{J}}{b}=-\frac{E\alpha}{a}\frac{2\Lambda_2}{k^2\varphi_0}\frac{U_y^1}{-U_x^1}=-\frac{4Ec\Lambda_2}{ba\varphi_0}\frac{U_y^1}{-U_x^1},$$
$$\sigma_{0\rho}=\frac{\frac{B_{\rho,1}}{J}}{b}=\frac{E\alpha}{a}\frac{2\Lambda_3}{k^2\varphi_0}\frac{a\,\Omega_x^1}{a\,\Omega_y^1}=\frac{4Ec\Lambda_3}{ba\varphi_0}\frac{a\,\Omega_x^1}{a\,\Omega_y^1}. \qquad (3.8)$$

By separating the real and imaginary parts in expressions (3.4) and replacing Y and Z by expressions containing X and W from section 6, Chapter XV, we obtain, by using the designations introduced above,

$$\sigma_{T_1}=\sigma_{0z}\left\{-\frac{1}{\Lambda_2}\left[\frac{\alpha\cos\theta\,\mathrm{Re}\,X}{1+\alpha\sin\theta}+\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\left(\frac{\alpha\cos\theta\,\mathrm{Im}\,X}{1+\alpha\sin\theta}+\right.\right.\right.$$
$$\left.\left.\left.+\frac{\alpha(\sin\theta-\alpha\cos^2\theta)}{2(1+\alpha\sin\theta)^2}\right)\right]\frac{\cos\varphi}{\sin\varphi}\right\}+\sigma_{0\rho}\left\{-\frac{1}{\Lambda_3}\left[\frac{\alpha^2\cos\theta\,\mathrm{Re}\,W}{1+\alpha\sin\theta}+\right.\right.$$
$$\left.\left.+\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\left(\frac{\alpha^2\cos\theta\,\mathrm{Im}\,W}{1+\alpha\sin\theta}-\frac{\alpha^3}{2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}\right)\right]\frac{\cos\varphi}{\sin\varphi}\right\};$$
$$\sigma_{T_2}=\sigma_{0z}\left\{-\frac{1}{\Lambda_2}\left[\mathrm{Re}\left(\frac{dX}{d\theta}+\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)+\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\left(\mathrm{Im}\left(\frac{dX}{d\theta}+\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}\right)\right]\frac{\cos\varphi}{\sin\varphi}\right\}+\sigma_{0\rho}\left\{-\frac{1}{\Lambda_3}\left[\alpha\,\mathrm{Re}\left(\frac{dW}{d\theta}+\right.\right.\right.$$
$$\left.\left.\left.+\frac{\alpha\cos\theta\,W}{1+\alpha\sin\theta}\right)+\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\alpha\,\mathrm{Im}\left(\frac{dW}{d\theta}+\frac{\alpha\cos\theta\,W}{1+\alpha\sin\theta}\right)\right]\frac{\cos\varphi}{\sin\varphi}\right\}; \qquad (3.9)$$
$$\sigma_S=\sigma_{0z}\left\{\mp\frac{1}{\Lambda_2}\left[\mathrm{Re}\,X+\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\left(\mathrm{Im}\,X-\frac{\alpha\cos\theta}{2(1+\alpha\sin\theta)}\right)\right]\frac{\alpha}{1+\alpha\sin\theta}\frac{\sin\varphi}{\cos\varphi}\right\}+$$
$$+\sigma_{0\rho}\left\{\mp\frac{1}{\Lambda_3}\left[\alpha\,\mathrm{Re}\,W+\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\left(\alpha\,\mathrm{Im}\,W+\frac{1-\alpha^3}{2(1+\alpha\sin\theta)}\right)\right]\frac{\alpha}{1+\alpha\sin\theta}\frac{\sin\varphi}{\cos\varphi}\right\};$$
$$\sigma_{M_1}=\sigma_{0z}\left\{\frac{6}{\mu\Lambda_2}\left[\mathrm{Im}\left(\frac{dX}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)+\frac{\alpha\sin\theta}{2(1+\alpha\sin\theta)}+\right.\right.$$
$$\left.\left.+\frac{\nu\alpha}{2}\frac{\sin\theta-\alpha\cos^2\theta}{(1+\alpha\sin\theta)^2}-\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\mathrm{Re}\left(\frac{dX}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)\right]\frac{\cos\varphi}{\sin\varphi}\right\}+$$

$$+\sigma_{0\rho}\left\{\frac{6}{\mu\Lambda_3}\left[\alpha\,\mathrm{Im}\left(\frac{dW}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,W}{1+\alpha\sin\theta}\right)-\frac{\nu\alpha^3}{2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}-\right.\right.$$

$$\left.\left.-\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\alpha\,\mathrm{Re}\left(\frac{dW}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,W}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\};$$

$$\sigma_{M_z}=\sigma_{0z}\left\{\frac{6}{\mu\Lambda_2}\left[\mathrm{Im}\left(\nu\frac{dX}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)+\frac{\nu}{2}\frac{\alpha\sin\theta}{1+\alpha\sin\theta}+\right.\right.$$

$$+\frac{\alpha}{2}\frac{\sin\theta-\alpha\cos^2\theta}{(1+\alpha\sin\theta)^2}-\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\mathrm{Re}\left(\nu\frac{dX}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,X}{1+\alpha\sin\theta}\right)\left.\left.\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\}+$$

$$+\sigma_{0\rho}\left\{\frac{6}{\mu\Lambda_3}\left[\alpha\,\mathrm{Im}\left(\nu\frac{dW}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,W}{1+\alpha\sin\theta}\right)-\frac{\alpha^3}{2}\frac{\cos\theta}{(1+\alpha\sin\theta)^2}-\right.\right.$$

$$\left.\left.-\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\alpha\,\mathrm{Re}\left(\nu\frac{dW}{d\theta}+(1+\nu)\frac{\alpha\cos\theta\,W}{1+\alpha\sin\theta}\right)\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}\right\};$$

$$\sigma_H=\sigma_{0z}\left\{\mp\frac{6(1-\nu)}{\mu\Lambda_2}\left[\mathrm{Im}\,X-\frac{\alpha}{2}\frac{\cos\theta}{1+\alpha\sin\theta}-\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\mathrm{Re}\,X\right]\times\right.$$

$$\left.\times\frac{\alpha}{1+\alpha\sin\theta}\begin{matrix}\sin\varphi\\ \cos\varphi\end{matrix}\right\}+\sigma_{0\rho}\left\{\mp\frac{6(1-\nu)}{\mu\Lambda_3}\left[\alpha\,\mathrm{Im}\,W+\right.\right.$$

$$\left.\left.+\frac{1-\alpha^2}{2}\frac{1}{1+\alpha\sin\theta}-\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\alpha\,\mathrm{Re}\,W\right]\frac{\alpha}{1+\alpha\sin\theta}\begin{matrix}\sin\varphi\\ \cos\varphi\end{matrix}\right\}$$

$$\left(\mu=\sqrt{12(1-\nu^2)}\right).$$

Finally, from relationships (Chapter XV, 5.6-5.8) we find the following expressions for the displacements

$$u^{\mathrm{C}}_{z,1}=(1+\alpha\sin\theta)\left\{\left[\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\,\mathrm{Im}\,Y\,d\tau}{1+\alpha\sin\tau}-\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\,\mathrm{Re}\,Y\,d\tau}{1+\alpha\sin\tau}\right]\times\right.$$

$$\times\frac{4a}{\varphi_0}\begin{matrix}U^1_y\\ -U^1_x\end{matrix}-\left[\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\,\mathrm{Im}\,Z\,d\tau}{1+\alpha\sin\tau}-\frac{\mathrm{Im}\,I_3}{\mathrm{Re}\,I_3}\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\,\mathrm{Re}\,Z\,d\tau}{1+\alpha\sin\tau}\right]\left.\frac{4a}{\varphi_0}\begin{matrix}a\,\Omega^1_x\\ a\,\Omega^1_y\end{matrix}\right\}+$$

$$+a(1+\alpha\sin\theta)\begin{matrix}a_3\\ a_3\end{matrix}; \qquad (3.10)$$

$$u^{\mathrm{C}}_{\rho,1}=\left[\left(\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{(\sin\tau+\alpha)\,\mathrm{Im}\,Y}{1+\alpha\sin\tau}d\tau-\alpha\cos\theta\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\,\mathrm{Im}\,Y}{1+\alpha\sin\tau}d\tau\right)-\right.$$

$$\left.-\frac{\mathrm{Im}\,I_2}{\mathrm{Re}\,I_2}\left(\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{(\sin\tau+\alpha)\,\mathrm{Re}\,Y}{1+\alpha\sin\tau}d\tau-\alpha\cos\theta\int\limits_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\,\mathrm{Re}\,Y}{1+\alpha\sin\tau}d\tau\right)\right]\frac{4a}{\varphi_0}\begin{matrix}U^1_y\\ -U^1_x\end{matrix}-$$

$$-\left[\left(\int_{-\frac{\pi}{2}}^{\theta}\frac{(\sin\tau+\alpha)\operatorname{Im}Z}{1+\alpha\sin\tau}d\tau-\alpha\cos\theta\int_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\operatorname{Im}Z}{1+\alpha\sin\tau}d\tau\right)-\right.$$

$$\left.-\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\left(\int_{-\frac{\pi}{2}}^{\theta}\frac{(\sin\tau+\alpha)\operatorname{Re}Z}{1+\alpha\sin\tau}d\tau-\alpha\cos\theta\int_{-\frac{\pi}{2}}^{\theta}\frac{\cos\tau\operatorname{Re}Z}{1+\alpha\sin\tau}d\tau\right)\right]\frac{4\alpha}{\varphi_0}\begin{matrix}a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}-$$

$$-b\cos\theta\begin{matrix}a_3\\ a_5\end{matrix}+1\begin{matrix}a_4\\ a_6\end{matrix};$$

$$\pm v_{\cdot,1}^{C}=\frac{1}{\varphi_0}\left\{-1\begin{matrix}U_y^1\\ -U_x^1\end{matrix}+\alpha\cos\theta\begin{matrix}a\,\Omega_x^1\\ a\,\Omega_y^1\end{matrix}\right\}-u_{\rho,1}^{C}.$$

As in the symmetric case we can introduce quantities which establish respectively the horizontal compression and the vertical expansion of the diameter /369

$$\delta_\rho=u_{\rho,1}\left(-\frac{\pi}{2}\right)-u_{\rho,1}\left(\frac{\pi}{2}\right),\quad \delta_z=u_{z,1}(0)-u_{z,1}(\pi). \tag{3.1}$$

By using equalities (3.8) it is easy to obtain the following expressions for the introduced quantities

$$\delta_z=\frac{2\mu}{\Delta_3}\frac{b^2}{h}\frac{\sigma_{0z}}{E}\left\{\int_0^{\frac{\pi}{2}}\frac{\cos\tau\operatorname{Im}Y}{1+\alpha\sin\tau}d\tau-\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\int_0^{\frac{\pi}{2}}\frac{\cos\tau\operatorname{Re}Y}{1+\alpha\sin\tau}d\tau\right\};$$

$$\delta_\rho=\frac{\mu}{\Delta_3}\frac{b^2}{h}\frac{\sigma_{0z}}{E}\left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\sin\tau+\alpha)\operatorname{Im}Y}{1+\alpha\sin\tau}d\tau-\right.$$

$$\left.-\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\sin\tau+\alpha)\operatorname{Re}Y}{1+\alpha\sin\tau}d\tau\right\}+\frac{\mu}{\Delta_3}\frac{b^2}{h}\frac{\sigma_{0\rho}}{E}\times \tag{3.22}$$

$$\times\left\{-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\sin\tau+\alpha)\operatorname{Im}Z}{1+\alpha\sin\tau}d\tau+\frac{\operatorname{Im}I_2}{\operatorname{Re}I_2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\sin\tau+\alpha)\operatorname{Re}Z}{1+\alpha\sin\tau}d\tau\right\}.$$

$$(\mu=\sqrt{12(1-\nu^2)}).$$

4. Asymptotic Equations. Calculation Example

We shall obtain simple asymptotic expressions for the maximum stresses /370 which are analogous to those established in the preceding chapter.

First of all, we take into consideration the property of the function $X(\theta)$ to increase substantially when differentiated with respect to θ and replace relationships (2.11) and (3.9) with the following simpler relationships

$$\frac{\sigma_{T_1}}{\sigma_0} = \frac{1}{A_1}\left\{-\operatorname{Re}\frac{dX}{d\theta} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Im}\frac{dX}{d\theta}\right\}; \tag{4.1}$$

$$\frac{\sigma_{M_1}}{\sigma_0} = \frac{6}{\mu A_1}\left\{\operatorname{Im}\frac{dX}{d\theta} - \frac{\operatorname{Im} I_1}{\operatorname{Re} I_1}\operatorname{Re}\frac{dX}{d\theta}\right\}; \tag{4.2}$$

$$\sigma_{T_2} = \sigma_{0x}\left\{-\frac{1}{A_2}\left(\operatorname{Re}\frac{dX}{d\theta} + \frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Im}\frac{dX}{d\theta}\right)\right\}\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix} + \\ + \sigma_{0y}\left\{-\frac{\alpha}{A_2}\left(\operatorname{Re}\frac{dW}{d\theta} + \frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Im}\frac{dW}{d\theta}\right)\right\}\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}; \tag{4.3}$$

$$\sigma_{M_2} = \sigma_{0x}\left\{\frac{6}{\mu A_2}\left(\operatorname{Im}\frac{dX}{d\theta} - \frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Re}\frac{dX}{d\theta}\right)\right\}\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix} + \\ + \sigma_{0y}\left\{\frac{6\alpha}{\mu A_2}\left(\operatorname{Im}\frac{dW}{d\theta} - \frac{\operatorname{Im} I_2}{\operatorname{Re} I_2}\operatorname{Re}\frac{dW}{d\theta}\right)\right\}\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix} \tag{4.4}$$

$$\left(\mu = \sqrt{12(1-\nu^2)}\right).$$

According to (Chapter XV, 6.25)

$$I_1 = \frac{1}{2\pi}\oint \cos\theta X d\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos\theta X d\vartheta.$$

Carrying out transformations analogous to those in section 1, Chapter XVI, we obtain

$$\operatorname{Re} I_1 = \frac{1}{2}; \quad \operatorname{Im} I_1 = 0. \tag{4.5}$$

From this, by means of equalities (Chapter XV, 6.34 and 6.26) we find

$$\frac{\operatorname{Im} I_1}{\operatorname{Re} I_1} = 0; \quad \frac{\operatorname{Im} I_2}{\operatorname{Re} I_2} = -\frac{\alpha}{\sqrt{1-\alpha^2}}; \quad \frac{\operatorname{Im} I_3}{\operatorname{Re} I_3} = \frac{\sqrt{1-\alpha^2}}{\alpha}; \tag{4.6}$$

$$A_1 = 1; \quad A_2 = \frac{1}{1-\alpha^2}; \quad A_3 = 1. \tag{4.7}$$

Having these expressions at our command, it is now easy to obtain (see section 1, Chapter XVI) the following equations for the maximum stresses

$$\frac{\sigma_{M_2}^{(0)}}{\sigma_0} = 1.86(1-\nu^2)^{-\frac{1}{6}}\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}; \tag{4.8}$$

$$\frac{\sigma_{T_z}(0)}{\sigma_0} = \mp 0.86\,(1-\nu^2)^{\frac{1}{3}}\,\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}\,\varphi(\theta_*)\,\omega'(\theta_*), \tag{4.9}$$

where

$$\alpha = \frac{b}{a};\ \beta = \frac{b}{h};$$
$$\theta_* \approx \omega_* = \mp \frac{1.225}{\lambda};\ \sigma_0 = \frac{B_z}{\frac{J}{b}} = \frac{B_z}{\pi b^2 h}. \tag{4.10}$$

With regard to the antisymmetric cases, we can introduce the designations

$$\{\mathrm{I}\} = \mathrm{Re}\,\frac{dX}{d\theta} - \frac{\alpha}{\sqrt{1-\alpha^2}}\,\mathrm{Im}\,\frac{dX}{d\theta},$$
$$\{\mathrm{II}\} = \mathrm{Im}\,\frac{dX}{d\theta} + \frac{\alpha}{\sqrt{1-\alpha^2}}\,\mathrm{Re}\,\frac{dX}{d\theta}, \tag{4.11}$$

and convert expressions (4.3)-(4.4) into the following form

$$\sigma_{T_z} = \left[-(1-\alpha^2)\{\mathrm{I}\}\sigma_{0z} - \sqrt{1-\alpha^2}\,\{\mathrm{II}\}\sigma_{0\rho}\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix};$$
$$\sigma_{M_1} = \frac{6}{\sqrt{12(1-\nu^2)}}\left[(1-\alpha^2)\{\mathrm{II}\}\sigma_{0z} - \sqrt{1-\alpha^2}\,\{\mathrm{I}\}\sigma_{0\rho}\right]\begin{matrix}\cos\varphi\\ \sin\varphi\end{matrix}. \tag{4.12}$$

According to (Chapter XV, 8.47)

$$\{\mathrm{I}\} = -\frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)\left\{\mathrm{Re}\,E'(y) + \frac{\alpha}{\sqrt{1-\alpha^2}}\,\mathrm{Im}\,E'(y)\right\};$$
$$\{\mathrm{II}\} = \frac{\lambda^2}{2}\varphi(\theta)\omega'(\theta)\left\{\mathrm{Im}\,E'(y) - \frac{\alpha}{\sqrt{1-\alpha^2}}\,\mathrm{Re}\,E'(y)\right\}. \tag{4.13}$$

The values of θ, for which the above expressions achieve extremal values, are determined by the rapidly varying terms contained in the figure brackets. The extremal values of expressions in the brackets

$$A_1(\alpha) = \mathrm{extr}\left\{\mathrm{Re}\,E'(y) + \frac{\alpha}{\sqrt{1-\alpha^2}}\,\mathrm{Im}\,E'(y)\right\},$$
$$A_2(\alpha) = \mathrm{extr}\left\{\mathrm{Im}\,E'(y) - \frac{\alpha}{\sqrt{1-\alpha^2}}\,\mathrm{Re}\,E'(y)\right\} \tag{4.14}$$

are shown in figure 51 and more accurately in table 10.

In particular, we can see the variation in the extremal values of the stresses as a function of α from the above. There is also a provision for the nonsymmetric nature of the stress diagrams resulting from the fact that α is different from 0.

By substituting expressions (4.13)-(4.14) into (4.12) we obtain the following asymptotic equations

$$\frac{\sigma_{T_1}}{\sigma_{0z}} = 1.145\,(1-\nu^2)^{\frac{1}{3}}\,(1-\alpha^2)\,\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\,\omega'(\theta_*)\,A_1(\alpha); \tag{4.15}$$

$$\frac{\sigma_{M_1}}{\sigma_{0z}} = 1.99\,(1-\nu^2)^{-\frac{1}{6}}\,(1-\alpha^2)\,\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\,\omega'(\theta_*)\,A_2(\alpha); \tag{4.16}$$

$$\frac{\sigma_{T_1}}{\sigma_{0\rho}} = -1.145\,(1-\nu^2)^{\frac{1}{3}}\sqrt{1-\alpha^2}\,\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\,\omega'(\theta_*)\,A_2(\alpha); \tag{4.17}$$

$$\frac{\sigma_{M_1}}{\sigma_{0\rho}} = 1.99\,(1-\nu^2)^{-\frac{1}{6}}\sqrt{1-\alpha^2}\,\alpha^{\frac{2}{3}}\beta^{\frac{2}{3}}\varphi(\theta_*)\,\omega'(\theta_*)\,A_1(\alpha), \tag{4.18}$$

where

$$\alpha = \frac{b}{a};\quad \beta = \frac{b}{h};\quad \sigma_{0z} = \frac{B_{z,1}}{\pi b^2 h};\quad \sigma_{0\rho} = \frac{B_{\rho,1}}{\pi b^2 h}, \tag{4.19}$$

while the values

$$\theta_i^* \approx \omega_i^* = \frac{y_i^*}{\lambda}\,(i=1,\ 2) \tag{4.20}$$

are determined from figure 51 or from table 10.

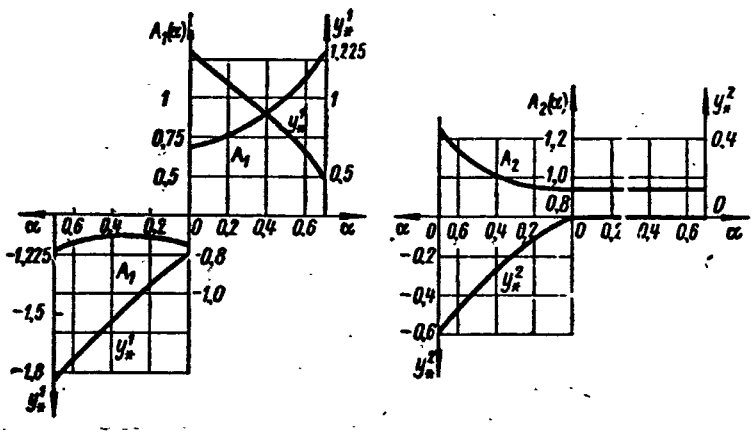

Figure 51

The asymptotic equations which have been presented make it possible /372
to estimate rapidly the stresses and assign the tube (thickness) parameters.

The method of determining stresses in a tube based on equations presented in the present chapter do not differ from the corresponding procedures of the preceding chapters. It is even simpler, because we have to deal only with a particular solution determined by the function $X(\theta)$. Therefore we shall limit ourselves to the presentation of computation results (ref. 363) for a thin-walled semiring ($\varphi_0 = \pi$) twisted at its ends by moments (fig. 52) $B_\varphi = B_y = B$. The tube parameters are $\lambda = 3.5$; $\alpha = \frac{1}{3}$. Figure 52 shows the longitudinal $\left(\frac{\sigma^+_{2,1}}{\sigma_{0\rho}}\right)$ and transverse stresses $\left(\frac{\sigma^+_{1,1}}{\sigma_{0\rho}}\right)$ in the external fibers of the tube (in the section $\varphi = 0$). The circles and squares designate the corresponding experimental values taken from reference 373.

The presented uniform solution of the general Saint-Venant problem refines the existing particular solutions and extends the scope of their applica- /373
tion. In particular it is valid for any values of α (of course, when $\alpha < 1$). We can also use it to clarify the effect of the parameter α on the state of stress (when $2k^2 = \sqrt{12(1-\nu^2)}\,\alpha\beta$ is fixed). The utilization of simplified equations (4.1)-(4.4), to say nothing of the even simpler asymptotic equations, makes the proposed exact method no less awkward than the existing approximate methods.

The reader has undoubtedly noticed that in the present chapter we have limited ourselves to the determination of stresses and have said nothing concerning the determination of tube rigidity. This is not accidental. The existing experimental material (for example, reference 373) shows that while the design stresses are close to their experimental values (fig. 52), the rigidity of the tube depends substantially on the real boundary conditions at its ends.

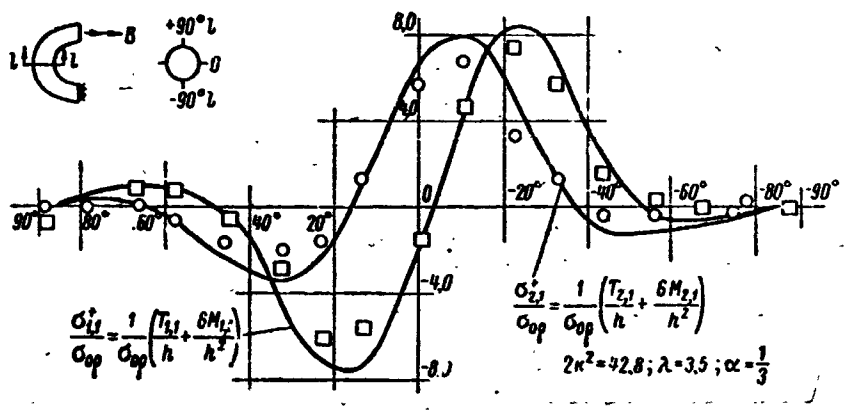

Figure 52

The solution presented in the present chapter also serves as a good basis for constructing solutions which take into account both the nature of end constraints and the nonlinear effects associated with the action of surface loading (pressure).

TABLE 1

$$\omega(\theta, \alpha) = \operatorname{sign}\theta\left[\frac{3}{2}\int_0^{|\theta|}\sqrt{\frac{|\sin\theta|}{1+\alpha\sin\theta}}\,d|\theta|\right]^{1/3}$$

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
90	1,4780	1,4440	1,4130	1,3850	1,3600	1,3360
80	1,3310	1,3020	1,2760	1,2520	1,2300	1,2100
70	1,1780	1,1550	1,1330	1,1140	1,0950	1,0780
60	1,0200	1,0020	0,9847	0,9691	0,9546	0,9411
50	0,8568	0,8136	0,8311	0,8196	0,8087	0,7984
40	0,6900	0,6812	0,6728	0,6649	0,6575	0,6503
30	0,5202	0,5150	0,5101	0,5054	0,5009	0,4965
20	0,3481	0,3457	0,3434	0,3411	0,3390	0,3369
10	0,1744	0,1738	0,1732	0,1726	0,1721	0,1715
0	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
−10	−0,1744	−0,1750	−0,1756	−0,1763	−0,1769	−0,1776
−20	−0,3481	−0,3505	−0,3531	−0,3558	−0,3585	−0,3613
−30	−0,5202	−0,5257	−0,5317	−0,5375	−0,5437	−0,5507
−40	−0,6900	−0,6997	−0,7100	−0,7208	−0,7323	−0,7454
−50	−0,8568	−0,8714	−0,8873	−0,9042	−0,9230	−0,9440
−60	−1,0200	−1,0400	−1,0620	−1,0870	−1,1140	−1,1450
−70	−1,1780	−1,2050	−1,2340	−1,2660	−1,3030	−1,3460
−80	−1,3310	−1,3640	−1,4020	−1,4410	−1,4880	−1,5450
−90	−1,4780	−1,5170	−1,5600	−1,6100	−1,6670	−1,7390

TABLE 2

$$\omega'(\theta, \alpha) = \sqrt{\frac{\sin\theta}{(1+\alpha\sin\theta)\,\omega(\theta, \alpha)}}$$

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
90	0,8226	0,7934	0,7680	0,7453	0,7247	0,7064
80	0,8602	0,8332	0,8030	0,7792	0,7579	0,7385
70	0,8932	0,8624	0,8356	0,8112	0,7898	0,7701
60	0,9214	0,8918	0,8658	0,8422	0,8208	0,8014
50	0,9456	0,9184	0,8940	0,8718	0,8515	0,8329
40	0,9652	0,9416	0,9201	0,9003	0,8819	0,8649
30	0,9804	0,9616	0,9440	0,9275	0,9120	0,8976
20	0,9912	0,9781	0,9655	0,9536	0,9421	0,9311
10	0,9975	0,9909	0,9843	0,9779	0,9713	0,9652
0	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
−10	0,9975	1,0047	1,0120	1,0190	1,0270	1,0346
−20	0,9912	1,0051	1,0200	1,0350	1,0514	1,0686
−30	0,9804	1,0006	1,0200	1,0460	1,0720	1,1000
−40	0,9652	0,9909	1,0190	1,0510	1,0870	1,1270

NOTE: Commas in tables represent decimal points.

TABLE 2 (Continued)

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
—50	0,9456	0,9758	1,0096	1.0490	1,0940	1,1470
—60	0,9214	0,9548	0,9931	1,0380	1,0910	1,1550
—70	0,8932	0,9276	0,9684	1.0170	1,0750	1,1480
—80	0,8602	0,8949	0,9352	0,9849	1,0449	1,1210
—90	0,8226	0,8558	0,8951	0,9420	0,9999	1,0729

TABLE 3. TABLE OF AIRY FUNCTIONS h_1

y	Re h_1	Im h_1	Re h_1'	Im h_1'
0	—0,000000	—1,074000	0,678300	0,391600
0,1	—0,039000	—1,007000	0,678200	0,386500
0,2	—0,077000	—0,938800	0,677300	0,371900
0,3	—0,113100	—0,871100	0,674900	0,349400
0,4	—0,146600	—0,803900	0,670300	0,320100
0,5	—0,177000	—0,737200	0,663000	0,285500
0,6	—0,203600	—0,671400	0,652400	0,246800
0,7	—0,226200	—0,606800	0,638400	0,205300
0,8	—0,244600	—0,543800	0,620700	0,162300
0,9	—0,258700	—0,482800	0,599300	0,118700
1,0	—0,268400	—0,424100	0,574200	0,075700
1,1	—0,273800	—0,368100	0,545700	0,034200
1,2	—0,275300	—0,315000	0,514100	—0,005000
1,3	—0,272900	—0,265300	0,479800	—0,041200
1,4	—0,267200	—0,219200	0,443300	—0,073900
1,5	—0,258300	—0,176700	0,405200	—0,102500
1,6	—0,246800	—0,138100	0,366000	—0,126800
1,7	—0,233100	—0,103500	0,326400	—0,146600
1,8	—0,217800	—0,072850	0,287000	—0,162000
1,9	—0,200800	—0,046100	0,248300	—0,172900
2,0	—0,183200	—0,023160	0,210800	—0,179600
2,1	—0,165100	—0,003880	0,175100	—0,182300
2,2	—0,146800	0,011940	0,141600	—0,181300
2,3	—0,128900	0,024530	0,110600	—0,177200
2,4	—0,111500	0,034160	0,082410	—0,170200
2,5	—0,094930	0,041110	0,057160	—0,160900
2.6	—0,079380	0,045690	0,034970	—0,149800
2,7	—0,065020	0,018210	0,015870	—0,137300
2,8	—0,051950	0,048970	—0,000170	—0,123900
2,9	—0,040250	0,048280	—0,013260	—0,110000
3,0	—0,029950	0,046410	—0,023580	—0,096050
3,1	—0,021030	0,043650	—0,031310	—0,082290
3,2	—0,013470	0,040230	—0,036700	—0,069070
3,3	—0,007192	0,036380	—0,040020	—0,056620
3,4	—0,002114	0,032280	—0,041540	—0,045120
3,5	0,001867	0,028120	—0,041550	—0,034700
3,6	0,004865	0,024010	—0,040330	—0,025460
3,7	0,006999	0,020080	—0,038140	—0,017420
3,8	0,008389	0,016410	—0,035230	—0,010590

TABLE 3 (Continued)

y	Re h_1	Im h_1	Re h_1'	Im h_1'
3,9	0,009155	0,013050	—0,031830	—0,004929
4,0	0,009412	0,010050	—0,028150	—0,000379
4,1	0,009265	0,007429	—0,024360	0,003146
4,2	0,008813	0,005182	—0,020500	0,005748
4,3	0,008143	0,003305	—0,016990	0,007537
4,4	0,007329	0,001776	—0,013620	0,008629
4,5	0,006436	0,000571	—0,010550	0,009139
4,6	0,005517	—0,000346	—0,007836	0,009179
4,7	0,004613	—0,001009	—0,005482	0,008854
4,8	0,003755	—0,001454	—0,003498	0,008261
4,9	0,002967	—0,001720	—0,001872	0,007484
5,0	0,002261	—0,001840	—0,000582	0,006597
5,1	0,001649	—0,001847	0,000401	0,005663
5,2	0,001129	—0,001769	0,001112	0,004729
5,3	0,000702	—0,001632	0,001588	0,003834
5,4	0,000360	—0,001453	0,001868	0,003006
5,5	0,000098	—0,001263	0,001989	0,002264
5,6	—0,000096	—0,001064	0,001987	0,001618
5,7	—0,000229	—0,000870	0,001892	0,001072
5,8	—0,000313	—0,000688	0,001734	0,000625
5,9	—0,000358	—0,000524	0,001536	0,000272
6,0	—0,000371	—0,000381	0,001318	0,000004

y	Re h_2	Im h_2	Re h_2'	Im h_2'
0	0,0000	1,0740	0,6783	—0,3916
0,1	0,0390	1,1420	0,6784	—0,3860
0,2	0,0768	1,2100	0,6793	—0,3683
0,3	0,1122	1,2780	0,6817	—0,3372
0,4	0,1437	1,3470	0,6862	—0,2912
0,5	0,1699	1,4150	0,6933	—0,2290
0,6	0,1889	1,4850	0,7033	—0,1492
0,7	0,1991	1,5560	0,7159	—0,0503
0,8	0,1983	1,6290	0,7309	0,0692
0,9	0,1845	1,7020	0,7473	0,2108
1,0	0,1553	1,7780	0,7635	0,3762
1,1	0,1084	1,8550	0,7775	0,5670
1,2	0,0410	1,9330	0,7862	0,7849
1,3	—0,0495	2,0120	0,7859	1,0320
1,4	—0,1663	2,0900	0,7715	1,3090
1,5	—0,3123	2,1660	0,7371	1,6170
1,6	—0,4908	2,2370	0,6752	1,9580
1,7	—0,7051	2,2990	0,5768	2,3330
1,8	—0,9585	2,3500	0,4317	2,7400
1,9	—1,2540	2,3840	0,2274	3,1780
2,0	—1,5950	2,3930	—0,0499	3,6440
2,1	—1,9840	2,3710	—0,4162	4,1330
2,2	—2,4220	2,3070	—0,8893	4,6370
2,3	—2,9110	2,1890	—1,4890	5,1440
2,4	—3,4510	2,0040	—2,2360	5,6380
2,5	—4,0380	1,7360	—3,1530	6,0930

TABLE 3 (Continued

y	Re h_2	Im h_2	Re h_2'	Im h_2'
2,6	—4,6680	1,3670	— 4,2620	6,4950
2,7	—5,3340	0,8764	— 5,5870	6,7950
2,8	—6,0220	0,2416	— 7,1490	6,9520
2,9	—6,7170	— 0,5619	— 8,9650	6,9100
3,0	—7,3950	— 1,5600	—11,0500	6,6010
3,1	—8,0260	— 2,7810	—13,4000	5,9440
3,2	—8,5690	— 4,2500	—16,0200	4,8420
3,3	—8,9760	— 5,9920	—18,8700	3,1840
3,4	—9,1830	— 8,0310	—21,9200	0,8420
3,5	—9,1170	—10,3800	—25,0900	— 2,3270
3,6	—8,6850	—13,0500	—28,2600	— 6,4780
3,7	—7,7830	—16,0300	—31,2800	— 11,7800
3,8	—6,2860	—19,2900	—33,9400	— 18,4000
3,9	—4,0540	—22,7900	—35,9500	— 26,4900
4,0	—0,9330	—26,4500	—36,9700	— 36,2200
4,1	3,2480	—30,1400	—36,5300	— 47,6800
4,2	8,6640	—33,6900	—34,1000	— 60,9400
4,3	15,4900	—36,8700	—29,0200	— 75,9600
4,4	23,9100	—39,3800	—20,5000	— 92,5700
4,5	34,0500	—40,8300	— 7,6600	—110,5000
4,6	46,0300	—40,7400	10,4900	—129,1000
4,7	59,8700	—38,5200	35,0500	—147,6000
4,8	75,5100	—33,4700	67,1500	—164,8000
4,9	92,7400	—24,8000	107,9000	—179,1000
5,0	111,1600	—11,5700	158,3000	—188,3000
5,1	130,1000	7,2200	219,3000	—169,7000
5,2	148,7000	32,6500	291,1000	—179,7000
5,3	165,6000	65,8100	373,8000	—154,2000
5,4	178,9000	107,7000	466,1000	—108,1000
5,5	186,3000	159,3000	566,0000	— 35,8000
5,6	184,9000	221,0000	669,5000	69,4000
5,7	171,1000	293,1000	770,7000	214,2000
5,8	140,5000	374,8000	861,2000	405,9000
5,9	88,2000	464,6000	929,2000	651,1000
6,0	8,3000	559,4000	959,4000	955,7000

TABLE 4

y	Re $E(y)$	Re $E'(y)$	Im $E(y)$	Im $E'(y)$
0,00	—1,288	0,000	0,000	0,939
0,05	—1,287	0,050	0,047	0,937
0,10	—1,283	0,100	0,094	0,932
0,15	—1,277	0,149	0,140	0,924
0,20	—1,268	0,197	0,186	0,913
0,25	—1,257	0,245	0,231	0,899
0,30	—1,244	0,292	0,276	0,882
0,35	—1,228	0,337	0,320	0,862
0,40	—1,210	0,380	0,362	0,839
0,45	—1,190	0,422	0,403	0,813

TABLE 4 (Continued)

y	Re $E(y)$	Re $E'(y)$	Im $E(y)$	Im $E'(y)$
0,50	−1,168	0,462	0,443	0,785
0,55	−1,144	0,500	0,482	0,755
0,60	−1,118	0,535	0,519	0,723
0,65	−1,090	0,568	0,554	0,688
0,70	−1,061	0,599	0,588	0,652
0,75	−1,030	0,627	0,619	0,614
0,80	−0,998	0,652	0,649	0,574
0,85	−0,965	0,675	0,677	0,534
0,90	−0,931	0,695	0,702	0,493
0,95	−0,896	0,712	0,726	0,450
1,00	−0,860	0,726	0,747	0,407
1,05	−0,823	0,737	0,767	0,364
1,10	−0,786	0,745	0,784	0,321
1,15	−0,749	0,750	0,799	0,278
1,20	−0,711	0,753	0,812	0,235
1,25	−0,673	0,753	0,822	0,193
1,30	−0,636	0,750	0,830	0,151
1,35	−0,599	0,745	0,836	0,110
1,40	−0,562	0,737	0,841	0,070
1,45	−0,525	0,727	0,844	0,031
1,50	−0,489	0,715	0,846	−0,007
1,55	−0,454	0,700	0,845	−0,043
1,60	−0,419	0,683	0,842	−0,078
1,65	−0,385	0,664	0,837	−0,111
1,70	−0,352	0,644	0,831	−0,142
1,75	−0,320	0,623	0,823	−0,171
1,80	−0,290	0,601	0,814	−0,198
1,85	−0,261	0,578	0,804	−0,222
1,90	−0,232	0,554	0,793	−0,244
1,95	−0,204	0,529	0,780	−0 264
2,00	−0,178	0,503	0,766	−0,282
2,05	−0,153	0,476	0,751	−0,298
2,10	−0,130	0,449	0,736	−0,313
2,15	−0,108	0,421	0,720	−0,326
2,20	−0,088	0,394	0,703	−0,337
2,25	−0,069	0,366	0,686	−0,346
2,30	−0,051	0,339	0,669	−0,353
2,35	−0,035	0,312	0,651	−0,358
2,40	−0,020	0,286	0,633	−0,361
2,45	−0,006	0,260	0,615	−0,362
2,50	0,006	0,235	0,597	−0,362
2,55	0,017	0,211	0,579	−0,361
2,60	0,027	0,188	0,561	−0,358
2,65	0,036	0,165	0,543	−0,354
2,70	0,044	0,144	0,525	−0,349
2,75	0,050	0,123	0,508	−0,342
2,80	0,056	0,104	0,491	−0,335
2,85	0,061	0,086	0,475	−0,327
2,90	0,065	0,069	0,459	−0,318
2,95	0,068	0,053	0,443	−0,308
3,00	0,070	0,038	0,428	−0,298
3,05	0,071	0,024	0,413	−0,287
3,10	0,072	0,011	0,399	−0,276

TABLE 4 (Continued)

y	Re $E(y)$	Re $E'(y)$	Im $E(y)$	Im $E'(y)$
3,15	0,073	0,000	0,386	—0,264
3,20	0,073	—0,010	0,373	—0,253
3,25	0,072	—0,019	0,360	—0,241
3,30	0,071	—0,027	0,348	—0,230
3,35	0,069	—0,034	0,337	—0,218
3,40	0,067	—0,040	0,327	—0,206
3,45	0,065	—0,045	0,317	—0,195
3,50	0,063	—0,050	0,307	—0,184
3,55	0,060	—0,053	0,298	—0,173
3,60	0,058	—0,056	0,290	—0,163
3,65	0,056	—0,058	0,282	—0,152
3,70	0,053	—0,059	0,275	—0,142
3,75	0,050	—0,059	0,268	—0,132
3,80	0,047	—0,059	0,261	—0,123
3,85	0,044	—0,059	0,255	—0,114
3,90	0,041	—0,058	0,250	—0,106
3,95	0,038	—0,057	0,246	—0,098
4,00	0,035	—0,056	0,242	—0,090
4,05	0,032	—0,054	0,238	—0,083
4,10	0,029	—0,052	0,234	—0,077
4,15	0,027	—0,050	0,230	—0,071
4,20	0,024	—0,048	0,226	—0,066
4,25	0,022	—0,046	0,223	—0,061
4,30	0,020	—0,043	0,220	—0,057
4,35	0,018	—0,040	0,218	—0,053
4,40	0,016	—0,038	0,215	—0,049
4,45	0,014	—0,035	0,213	—0,046
4,50	0,012	—0,032	0,210	—0,043
4,55	0,011	—0,030	0,208	—0,040
4,60	0,009	—0,027	0,206	—0,038
4,65	0,008	—0,025	0,205	—0,036
4,70	0,007	—0,022	0,203	—0,034
4,75	0,006	—0,020	0,201	—0,033
4,80	0,005	—0,018	0,200	—0,031
4,85	0,004	—0,016	0,198	—0,030
4,90	0,003	—0,014	0,197	—0,029
4,95	0,003	—0,012	0,195	—0,029
5,00	0,002	—0,010	0,191	—0,028

TABLE 5

	L_1					
$\theta°$	α					
	0	0,1	0,2	0,3	0,4	0,5
90	—0,3730	—0,3707	—0,3686	—0,3668	—0,3650	—0,3635
80	—0,3170	—0,3144	—0,3115	—0,3093	—0,3074	—0,3055
70	—0,2671	—0,2636	—0,2611	—0,2581	—0,2560	—0,2539
60	—0,2219	—0,2182	—0,2151	—0,2121	—0,2095	—0,2073
50	—0,1805	—0,1765	—0,1729	—0,1696	—0,1669	—0,1640
40	—0,1421	—0,1374	—0,1338	—0,1297	—0,1264	—0,1236
30	—0,1046	—0,1001	—0,0959	—0,0919	—0,0882	—0,0848
20	—0,0694	—0,0643	—0,0598	—0,0557	—0,0515	—0,0475
10	—0,0340	—0,0289	—0,0243	—0,0209	—0,0144	—0,0105

TABLE 5 (Continued)

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
0	0,0000	0,0050	0,0100	0,0150	0,0200	0,0250
−10	0,0340	0,0392	0,0452	0,0491	0,0550	0,0600
−20	0,0694	0,0740	0,0790	0,0844	0,0902	0,0966
−30	0,1046	0,1096	0,1144	0,1203	0,1270	0,1336
−40	0,1421	0,1463	0,1514	0,1572	0,1638	0,1713
−50	0,1805	0,1849	0,1899	0,1956	0,2022	0,2099
−60	0,2219	0,2262	0,2310	0,2362	0,2429	0,2510
−70	0,2671	0,2706	0,2751	0,2803	0,2864	0,2943
−80	0,3170	0,3200	0,3233	0,3284	0,3342	0,3408
−90	0,3730	0,3756	0,3788	0,3825	0,3873	0,3927
			L_1'			
90	−0,3443	−0,3473	−0,3498	−0,3520	−0,3542	−0,3557
80	−0,3015	−0,3028	−0,3064	−0,3085	−0,3105	−0,3123
70	−0,2700	−0,2725	−0,2745	−0,2770	−0,2785	−0,2804
60	−0,2471	−0,2493	−0,2510	−0,2531	−0,2550	−0,2568
50	−0,2293	−0,2315	−0,2332	−0,2354	−0,2371	−0,2392
40	−0,2166	−0,2184	−0,2199	−0,2222	−0,2243	−0,2259
30	−0,2076	−0,2087	−0,2105	−0,2127	−0,2150	−0,2168
20	−0,2013	−0,2026	−0,2041	−0,2060	−0,2080	−0,2102
10	−0,1976	−0,1983	−0,1995	−0,2014	−0,2042	−0,2069
0	−0,1964	−0,1968	−0,1978	−0,1996	−0,2022	−0,2054
−10	−0,1976	−0,1975	−0,1984	−0,1996	−0,2023	−0,2055
−20	−0,2013	−0,2006	−0,2008	−0,2018	−0,2038	−0,2070
−30	−0,2076	−0,2073	−0,2051	−0,2087	−0,2091	−0,2139
−40	−0,2166	−0,2160	−0,2160	−0,2158	−0,2164	−0,2199
−50	−0,2293	−0,2281	−0,2273	−0,2262	−0,2266	−0,2284
−60	−0,2471	−0,2449	−0,2427	−0,2416	−0,2401	−0,2399
−70	−0,2700	−0,2680	−0,2652	−0,2622	−0,2594	−0,2565
−80	−0,3015	−0,2986	−0,2960	−0,2912	−0,2861	−0,2815
−90	−0,3440	−0,3412	−0,3370	−0,3321	−0,3259	−0,3185

TABLE 6

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
			L_2			
90	0,1270	0,1293	0,1314	0,1332	0,1350	0,1364
80	0,1027	0,1052	0,1081	0,1103	0,1123	0,1141
70	0,0830	0,0864	0,0887	0,0919	0,0941	0,0962
60	0,0667	0,0705	0,0736	0,0765	0,0792	0,0816
50	0,0526	0,0568	0,0603	0,0636	0,0665	0,0693
40	0,0403	0,0447	0,0476	0,0522	0,0555	0,0587
30	0,0293	0,0338	0,0383	0,0420	0,0457	0,0492
20	0,0193	0,0239	0,0287	0,0326	0,0371	0,0406
10	0,0099	0,0148	0,0195	0,0230	0,0278	0,0325

TABLE 6 (Continued)

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
0	0,0000	0,0050	0,0100	0,0150	0,0200	0,0250
−10	−0,0099	−0,0048	0,0012	0,0059	0,0112	0,0172
−20	−0,0193	−0,0142	−0,0090	−0,0040	0,0020	0,0084
−30	−0,0293	−0,0244	−0,0198	−0,0136	−0,0071	−0,0005
−40	−0,0403	−0,0358	−0,0307	−0,0247	−0,0184	−0,0110
−50	−0,0526	−0,0482	−0,0434	−0,0375	−0,0309	−0,0231
−60	−0,0667	−0,0626	−0,0578	−0,0525	−0,0458	−0,0378
−70	−0,0830	−0,0795	−0,0752	−0,0699	−0,0636	−0,0558
−80	−0,1027	−0,0997	−0,0962	−0,0912	−0,0855	−0,0786
−90	−0,1270	−0,1244	−0,1212	−0,1176	−0,1128	−0,1074
			L_2'			
90	0,1557	0,1528	0,1502	0,1480	0,1459	0,1442
80	0,1247	0,1227	0,1197	0,1176	0,1156	0,1137
70	0,1026	0,1000	0,0971	0,0956	0,0941	0,0922
60	0,0864	0,0840	0,0823	0,0803	0,0783	0,0765
50	0,0749	0,0728	0,0710	0,0689	0,0670	0,0656
40	0,0666	0,0648	0,0631	0,0613	0,0595	0,0577
30	0,0603	0,0593	0,0580	0,0565	0,0549	0,0533
20	0,0565	0,0556	0,0540	0,0520	0,0497	0,0471
10	0,0543	0,0536	0,0523	0,0504	0,0479	0,0450
0	0,0536	0,0532	0,0521	0,0503	0,0478	0,0446
−10	0,0543	0,0543	0,0535	0,0520	0,0495	0,0460
−20	0,0565	0,0569	0,0565	0,0553	0,0530	0,0494
−30	0,0603	0,0610	0,0612	0,0603	0,0586	0,0547
−40	0,0666	0,0672	0,0670	0,0675	0,0664	0,0629
−50	0,0749	0,0762	0,0770	0,0781	0,0779	0,0761
−60	0,0864	0,0884	0,0903	0,0917	0,0932	0,0933
−70	0,1026	0,1046	0,1072	0,1104	0,1132	0,1159
−80	0,1247	0,1273	0,1300	0,1347	0,1393	0,1446
−90	0,1557	0,1590	0,1630	0,1678	0,1742	0,1815

TABLE 7

$$\varphi(\theta) = \frac{1}{\gamma' \omega'(\theta)(1 + \alpha \sin\theta)}$$

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
90	1,1025	1,0704	1,0417	1,0160	0,9928	0,9714
80	1,0782	1,0482	1,0200	0,9953	0,9730	0,9526
70	1,0581	1,0295	1,0037	0,9806	0,9593	0,9399
60	1,0418	1,0158	0,9922	0,9709	0,9513	0,9332
50	1,0285	1,0057	0,9848	0,9658	0,9481	0,9317
40	1,0179	0,9990	0,9814	0,9650	0,9498	0,9354
30	1,0099	0,9952	0,9814	0,9682	0,9559	0,9441
20	1,0044	0,9942	0,9846	0,9752	0,9663	0,9577
10	1,0013	0,9960	0,9909	0,9859	0,9812	0,9764

TABLE 7 (Continued)

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
0	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
—10	1,0013	1,0064	1,0119	1,0175	1,0229	1,0288
—20	1,0044	1,0150	1,0258	1,0377	1,0496	1,0625
—30	1,0099	1,0266	1,0426	1,0606	1,0798	1,1010
—40	1,0179	1,0384	1,0612	1,0856	1,1128	1,1435
—50	1,0285	1,0534	1,0815	1,1126	1,1481	1,1888
—60	1,0418	1,0708	1,1036	1,1409	1,1843	1,2358
—70	1,0581	1,0909	1,1278	1,1703	1,2208	1,2817
—80	1,0782	1,1133	1,1539	1,2003	1,2568	1,3257
—90	1,1025	1,1395	1,1818	1,2315	1,2912	1,3659

TABLE 8

$\varphi'(\theta)$

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
90	0,1534	0,1471	0,1415	0,1366	0,1323	0,1284
80	0,1268	0,1180	0,1081	0,1010	0,0949	0,0895
70	0,1043	0,0905	0,0793	0,0697	0,0618	0,0550
60	0,0849	0,0676	0,0537	0,0419	0,0318	0,0233
50	0,0680	0,0477	0,0308	0,0164	0,0041	—0,0066
40	0,0527	0,0296	0,0098	—0,0073	—0,0224	—0,0355
30	0,0386	0,0131	—0,0095	—0,0297	—0,0478	—0,0638
20	0,0252	—0,0020	—0,0275	—0,0505	—0,0721	—0,0920
10	0,0125	—0,0166	—0,0437	—0,0706	—0,0972	—0,1212
0	0,0000	—0,0300	—0,0600	—0,0900	—0,1200	—0,1500
—10	—0,0125	—0,0428	—0,0750	—0,1070	—0,1432	—0,1781
—20	—0,0252	—0,0553	—0,0881	—0,1235	—0,1630	—0,2064
—30	—0,0386	—0,0671	—0,0998	—0,1377	—0,1815	—0,2318
—40	—0,0527	—0,0796	—0,1112	—0,1496	—0,1962	—0,2528
—50	—0,0680	—0,0923	—0,1218	—0,1589	—0,2058	—0,2666
—60	—0,0849	—0,1059	—0,1320	—0,1656	—0,2098	—0,2694
—70	—0,1043	—0,1209	—0,1423	—0,1704	—0,2077	—0,2603
—80	—0,1268	—0,1389	—0,1540	—0,1745	—0,1993	—0,2388
—90	—0,1534	—0,1607	—0,1695	—0,1801	—0,1936	—0,2103

TABLE 9

$\varphi(\theta)\omega'(\theta)$

θ°	α					
	0	0,1	0,2	0,3	0,4	0,5
90	0,9069	0,8492	0,8000	0,7572	0,7195	0,6862
80	0,9275	0,8709	0,8191	0,7755	0,7374	0,7035
70	0,9451	0,8878	0,8387	0,7955	0,7576	0,7238

TABLE 9 (Continued)

δ°	α					
	0	0,1	0,2	0,3	0,4	0,5
60	0,9599	0,9059	0,8590	0,8177	0,7808	0,7479
50	0,9725	0,9236	0,8804	0,8420	0,8073	0,7760
40	0,9825	0,9406	0,9030	0,8688	0,8376	0,8090
30	0,9901	0,9570	0,9264	0,8980	0,8718	0,8474
20	0,9956	0,9724	0,9506	0,9300	0,9104	0,8917
10	0,9988	0,9869	0,9754	0,9641	0,9539	0,9424
0	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
—10	0,9988	1,0111	1,0240	1,0368	1,0505	1,0644
—20	0,9956	1,0202	1,0463	1,0740	1,1035	1,1354
—30	0,9901	1,0272	1,0655	1,1094	1,1575	1,2111
—40	0,9825	1,0290	1,0814	1,1410	1,2096	1,2887
—50	0,9725	1,0279	1,0919	1,1671	1,2560	1,3636
—60	0,9599	1,0224	1,0960	1,1842	1,2921	1,4273
—70	0,9451	1,0119	1,0922	1,1902	1,3124	1,4714
—80	0,9275	0,9963	1,0791	1,1822	1,3132	1,4861
—90	0,9069	0,9752	1,0578	1,1601	1,2911	1,4642

TABLE 10

α	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7
y_1^*	1,225	1,150	1,050	1,000	0,900	0,825	0,700	0,600
	—1,225	—1,300	—1,400	—1,450	—1,550	—1,650	—1,750	—1,850
$A_1(\alpha)$	0,753	0,778	0,811	0,854	0,910	0,983	1,088	1,243
	—0,753	—0,735	—0,723	—0,717	—0,719	—0,728	—0,751	—0,796
y_2^*	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,000	—0,050	—0,150	—0,250	—0,300	—0,400	—0,500	—0,600
$A_2(\alpha)$	0,939	0,939	0,939	0,939	0,939	0,939	0,939	0,939
	0,939	0,942	0,954	0,976	1,009	1,058	1,132	1,246

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